Applied Mathematics & Information Sciences **5**(1) (2011), 74-84
– An International Journal

Perron's theorem for q–delay difference equations

J. O. Alzabut[1] and T. Abdeljawad[2]

[1]Department of Mathematics and Physical Sciences, Prince Sultan University

P.O.Box 66388 Riyadh 11586, Saudi Arabia

Email Address: jalzabut@psu.edu.sa

[2]Department of Mathematics and Computer Science, Çankaya University

06530 Ankara, Turkey

Email Address: thabet@cankaya.edu.tr

Received Oct. 27, 2009; Revised Feb. 15, 2010; Accepted Mar. 4, 2010

In this paper, we prove that if a linear q–delay difference equation satisfies Perron's condition then its trivial solution is uniformly asymptotically stable.

Keywords: Perron, q–calculus, q–difference equation, uniform asymptotic stability.

1 Introduction

Studies on linear q–difference equations have started at the beginning of the last century in intensive works by Jackson [1], Carmichael [2], Mason [3], Adams [4], Trjitzinsky [5] and some other authors such as Poincare, Picard and Ramanunjan. However, from the 1930s up to the beginning of the 1980s, the theory of linear q–difference equations has lagged noticeably behind the sister theories of linear difference and differential equations. Since the 1980s, an extensive and somewhat surprising interest in the subject reappeared in many areas of mathematics, physics and applications including mainly new difference calculus and orthogonal polynomials, q–combinatorics, q–arithmetics, integrable systems and variational q–calculus, see the recent papers [6–11].

The asymptotic properties of q–difference equations have been rarely considered in the literature. There are few recent results dealing with subjects like spectral analysis, oscillation behavior of solutions, factorization method and symmetries of these type of equations, we name the papers [12–18]. The stability behavior of q–delay difference equations, in particular, is our main concern in this paper. We start by mentioning a few background details that serve to motivate the results of this paper.

It is well known in the theory of ordinary differential equations (see eg. [19, page 120]) that if for every continuous function $f(t)$ bounded on $[0,\infty)$, the solution of the equation $x'(t) = A(t)x(t) + f(t),\ t \geq 0$ satisfying $x(0) = 0$ is bounded on $[0,\infty)$, then the trivial solution of the corresponding homogeneous equation $x'(t) = A(t)x(t),\ t \geq 0$ is uniformly asymptotically stable. Later on, this result, which is known as Perron Theorem [20], was extended in [19, page 371] to delay differential equations. Indeed, it was shown that if for every continuous function $f(t)$ bounded on $[0,\infty)$, the solution of the equation

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t),\ t \geq 0,\ \tau > 0$$

satisfying $x(t) = 0$ for $t \in [-\tau, 0]$, is bounded on $[0,\infty)$, then the trivial solution of the equation

$$x'(t) = A(t)x(t) + B(t)x(t-\tau),\ t \geq 0 \tag{1.1}$$

is uniformly asymptotically stable. Perron's theorem for impulsive delay differential equations has been considered in the paper [21]. A discrete analogue of the above result has been published in [22]. In particular, Perron theorem has been proved for equations of form

$$\Delta x(n) = A(n)x(n) + B(n+1)x(n-j+1),\ n \geq 0,\ j \in \{2,3,4,\ldots\}, \tag{1.2}$$

where $\Delta x(t)$ denotes the forward difference $x(t+1) - x(t)$. Recently, we proved this result for a type of impulsive delay difference equations [23]. For more related results, see the papers [24–26]. To the best of authors' knowledge, however, there are a few results concerning stability of q–delay difference equations [27, 28]. Motivated by this, we contribute to the theory of q–difference equations by proving Perron's theorem for a type of q–delay difference equations of the form

$$D_q x(t) = A(t)x(t) + qB(qt)x(q^{-\alpha_0+1}t),\ t \in q^{\mathbb{Z}},$$

where $q^{\mathbb{Z}} = \{q^i : i \in \mathbb{Z}\}$ with $q > 1$ and $\alpha_0 \in \mathbb{N}$.

2 Adjoint equation and solutions representations

We introduce some preliminary notations that would help in understanding later analysis. For the function $f : q^{\mathbb{Z}} \to \mathbb{R}$, the expression

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.1}$$

is called the q–derivative (or Jackson derivative [29]) of function f. Together with the definition of q–derivative, arises naturally that of the q–integral of a given function. In view of Definition 1.71 in [30, page 26], one can designate the indefinite q-integral by

$$\int f(t)_q t = F(t) + C, \tag{2.2}$$

where F is a pre–antiderivative of f and C is an arbitrary constant. The definite q–integral turns out to be defined as follows

$$\int_a^b f(t)_q t = (q-1)\sum_{i=\alpha}^{\beta-1} q^i f(q^i) = (q-1)\sum_{t=a}^{q^{-1}b} t f(t), \tag{2.3}$$

where $a = q^\alpha$ and $b = q^\beta$. We note that (2.3) is the q–analogue of formula (ii) of Theorem 1.79 in [30, page 29].

We are now in a position to define the q–analogue of some well known rules of calculus. The q–derivative of the composition of functions f and g is given by

$$D_q(f\circ g)(t) = (D_q f)(g(t))D_q g(t), \tag{2.4}$$

where $g(t) = ct,\ c\in\mathbb{R}$. The q–derivative of the product of functions f and g is interpreted as

$$D_q(fg)(t) = f(qt)D_q g(t) + D_q f(t)g(t). \tag{2.5}$$

The fundamental theorem of calculus for q–difference operator turns out to be defined as follows

$$\int_a^b D_q f(t)_q t = f(b) - f(a), \tag{2.6}$$

where $a, b \in q^{\mathbb{Z}}$. By means of (2.4), one can write the Newton–Libniz formula for q–difference operator in this form

$$D_q \int_{k(t)}^{h(t)} f(x)_q x = f(h(t))D_q h(t) - f(k(t))D_q k(t), \tag{2.7}$$

where $h(t) = q^{\alpha_0} t$ and $k(t) = qt$.

We shall prove Perron's theorem for the q–delay difference equation

$$D_q x(t) = A(t)x(t) + qB(qt)x(q^{-\alpha_0+1}t),\ t\in q^{\mathbb{Z}},\ \alpha_0\in\mathbb{N}, \tag{2.8}$$

where it is assumed that $A, B : q^{\mathbb{Z}} \to \mathbb{R}^{m\times m}$ are bounded matrices. Equation (2.8) is being designated to be the q–version of equations (1.1) and (1.2).

By a solution of (2.8), we mean a function x which is defined for all $t \in [q^{-\alpha_0+1}t_0, \infty)_{q^{\mathbb{Z}}}$ and satisfies (2.8) for $t \in [t_0, \infty)_{q^{\mathbb{Z}}} = \{q^i : i \geq \beta\},\ \beta\in\mathbb{Z}$. It is easy to see that for any given $t_0 \geq a = q^{\alpha_1}$ and initial condition of the form

$$x(t) = \phi(t), \quad t\in [q^{-\alpha_0+1}t_0, t_0]_{q^{\mathbb{Z}}} \tag{2.9}$$

(2.8) has a unique solution $x(t)$ which is defined for $t \in [q^{-\alpha_0+1}t_0, \infty)_{q^{\mathbb{Z}}}$ and satisfies the initial condition (2.9).

We shall start by constructing the adjoint equation of (2.8) with respect to a function resembles the one obtained in [19, page 359]. It turns out that the *q*–analogue of this function has the form

$$< y(t), x(t) >= y^T(t)x(t) + \int_{qt}^{q^{\alpha_0}t} y^T(\alpha)B(\alpha)x(q^{-\alpha_0}\alpha)_q\alpha, \tag{2.10}$$

where $''T''$ denotes the transposition.

Define the equation

$$D_q y(t) = -A^T(t)y(qt) - q^{\alpha_0}B^T(q^{\alpha_0}t)y(q^{\alpha_0}t). \tag{2.11}$$

Lemma 2.1. *Let $x(t)$ be any solution of (2.8) and $y(t)$ be any solution of (2.11), then*

$$< y(t), x(t) >= c = \text{constant}. \tag{2.12}$$

Proof. Clearly, it suffices to show that $D_q < y(t), x(t) >= 0$. Then

$$D_q < y(t), x(t) >= D_q(y^T(t)x(t)) + D_q\Big[\int_{qt}^{q^{\alpha_0}t} y^T(\alpha)B(\alpha)x(q^{-\alpha_0}\alpha)_q\alpha\Big].$$

By virtue of relations (2.5) and (2.7), we have

$$\begin{aligned} D_q < y(t), x(t) > &= y^T(qt)D_q x(t) + D_q y^T(t)x(t) + y^T(q^{\alpha_0}t)B(q^{\alpha_0}t)x(t)D_q q^{\alpha_0}t \\ &- y^T(qt)B(qt)x(q^{-\alpha_0+1}t)D_q qt. \end{aligned}$$

In view of equations (2.8) and (2.11), we obtain

$$\begin{aligned} D_q < y(t), x(t) > &= y^T(qt)\big[A(t)x(t) + qB(qt)x(q^{-\alpha_0+1}t)\big] \\ &- \big[y^T(qt)A(t) + q^{\alpha_0}y^T(q^{\alpha_0}t)B(q^{\alpha_0}t)\big]x(t) \\ &+ q^{\alpha_0}y^T(q^{\alpha_0}t)B(q^{\alpha_0}t)x(t) \\ &- qy^T(qt)B(qt)x(q^{-\alpha_0+1}t) = 0. \end{aligned}$$

Thus, $< y(t), x(t) >= c =$ constant. The proof is complete.

By virtue of Lemma 2.1, we may say that equation (2.11) is an adjoint of (2.8). It is easy to verify also that the adjoint of (2.11) is (2.8), that is, they are mutually adjoint of each other.

Definition 2.1. A matrix solution $X(t, s)$ of (2.8) satisfying $X(t, t) = I$ and $X(t, s) = 0$ for $t < s$ is called a fundamental matrix of (2.8).

Definition 2.2. A matrix solution $Y(t, s)$ of (2.11) satisfying $Y(t, t) = I$ and $Y(t, s) = 0$ for $t > s$ is called a fundamental matrix of (2.11).

It is to be noted that the construction of function (2.10) is of special interest in itself. We shall use function (2.10) to derive the solutions representations of equations (2.8) and (2.11).

In view of relation (2.12), we observe that

$$< y(t), x(t) >=< y(t_0), x(t_0) > . \tag{2.13}$$

Looking at function (2.10), if we replace $x(s)$ by $X(s, t_0)$ and $y(s)$ by $Y(s, t)$ in (2.13) and use the properties of the fundamental matrices, we get the identity

$$X(t, t_0) = Y^T(t_0, t). \tag{2.14}$$

Furthermore, replacing $y(s)$ by $Y(s, t)$ in (2.13) and using identity (2.14) and the properties of the fundamental matrix $Y(t, s)$, we have the following result.

Lemma 2.2. *Let $X(t, s)$ be a fundamental matrix of (2.8) and $t_0 = q^\beta \geq a = q^{\alpha_1}$ ($1 \leq \alpha_1 \leq \beta$). If $x(t)$ is a solution of (2.8), then*

$$x(t) = X(t, t_0)x(t_0) + \int_{qt_0}^{q^{\alpha_0}t_0} X(t, \alpha)B(\alpha)x(q^{-\alpha_0}\alpha)_q\alpha. \tag{2.15}$$

One can also obtain the solutions representation of equation (2.11) in like manner. Indeed, upon replacing $x(s)$ by $X(s, t)$ in relation (2.13), we can derive the solutions representation of the adjoint equation (2.11). Namely,

Lemma 2.3. *Let $Y(t, s)$ is a fundamental matrix of (2.11) and $t_0 = q^\beta \geq a = q^{\alpha_1}$ ($1 \leq \alpha_1 \leq \beta$). If $y(t)$ is a solution of (2.11), then*

$$y(t) = Y(t, t_0)y(t_0) + \int_{qt_0}^{q^{\alpha_0}t_0} Y(t, q^{-\alpha_0}\alpha)B^T(\alpha)y(\alpha)_q\alpha. \tag{2.16}$$

Consider the equation

$$D_q x(t) = A(t)x(t) + qB(qt)x(q^{-\alpha_0+1}t) + f(t), \ t \in q^{\mathbb{Z}}, \tag{2.17}$$

where $f : q^{\mathbb{Z}} \to \mathbb{R}^m$. Then the solutions representation of (2.17) is given by the following result.

Lemma 2.4. *Let $X(t, s)$ be a fundamental matrix of (2.8) and $t_0 = q^\beta \geq a = q^{\alpha_1}$ ($1 \leq \alpha_1 \leq \beta$). If $x(t)$ is a solution of (2.17), then*

$$x(t) = X(t, t_0)x(t_0) + \int_{qt_0}^{q^{\alpha_0}t_0} X(t, \alpha)B(\alpha)x(q^{-\alpha_0}\alpha)_q\alpha + \int_{t_0}^{t} X(t, q\alpha)f(\alpha)_q\alpha. \tag{2.18}$$

The proof of the above statement is straightforward and can be achieved by direct substitution and by using the relation

$$D_q \int_a^t f(t, \tau)_q\tau = \int_a^t D_q^t f(t, \tau)_q\tau + f(qt, t).$$

3 Perron's theorem

Perron condition for equation (2.8) is formulated as follows.

Definition 3.1. Equation (2.8) is said to verify Perron's condition if for every bounded function $f(t)$ on $[a,\infty)_{q^{\mathbb{Z}}}$, the solution of (2.17) with $x(t) = 0$ for $t \in [q^{-\alpha_0+1}a, a]$ is bounded on $[a,\infty)_{q^{\mathbb{Z}}}$.

Lemma 3.1. *If equation (2.8) verifies Perron's condition, then there exists a constant C such that*

$$\int_a^t \|X(t,q\alpha)\|_q\alpha < C \quad \textit{for} \quad t \geq a,\ t \in q^{\mathbb{Z}}, \tag{3.1}$$

where $\|\cdot\|$ *denotes any convenient matrix norm.*

Proof. By virtue of Lemma 2.4, the solution of (2.17) satisfying (2.9) with $\phi(t) = 0$ has the form

$$x(t) = \int_a^t X(t,q\alpha)f(\alpha)_q\alpha.$$

Let $\mathcal{B}$ denote the set of all bounded functions f on $[a,\infty)_{q^{\mathbb{Z}}}$ supplied by the norm $\|f\|_\infty = \sup_{t\in[a,\infty)_{q^{\mathbb{Z}}}} \|f(t)\|$. Clearly, $\mathcal{B}$ is a Banach space.

For each $t \in [a,\infty)_{q^{\mathbb{Z}}}$, define a sequence of linear operators $U_t : \mathcal{B} \to \mathbb{R}^m$ by

$$U_t(f) = \int_a^t X(t,q\alpha)f(\alpha)_q\alpha.$$

By using the estimate $\|U_t(f)\| \leq \int_a^t \|X(t,q\alpha)\|_q\alpha\|f\|_\infty$, it follows that the operators U_t are bounded. By virtue of Perron condition, we deduce that for each $f \in \mathcal{B}$ we can find $c_f > 0$ such that $\sup_{t\in[a,\infty)_{q^{\mathbb{Z}}}} \|U_t(f)\| \leq c_f$. Hence, by using the Banach–Steinhaus Theorem, there exists a constant $L > 0$ such that

$$\sup_t \|U_t(f)\| \leq L\|f\|_\infty, \quad \text{for all} \quad f \in \mathcal{B}. \tag{3.2}$$

For fixed $t \in [a,\infty)_{q^{\mathbb{Z}}}$, let $x_{rk}(1 \leq r,k \leq m)$ be the elements of the matrix $X(t,q\alpha)$ where $a \leq \alpha < t,\ \alpha \in q^{\mathbb{Z}}$. Let e_p denote the canonical basis having the unity at the p-th place and zero otherwise. Let f_α^r be the element of $\mathcal{B}$ with its $\alpha-$component the vector V_r of $\mathbb{R}^m$ and zeros otherwise, where $V_r = \sum_{k=1}^m \text{sign} x_{rk}e_k$. The vector $Xf_\alpha^r(\alpha)$ will have its $r-th$ component equal to $\sum_{k=1}^m |x_{rk}|$.

From (3.2), we can write

$$\|\int_a^t X(t,q\alpha)f_\alpha^r(\alpha)_q\alpha\| \leq M_2,$$

where $M_2 = L\sup_r \|V_r\|$. Hence

$$\int_a^t \sum_{r=1}^m |x_{rk}(t,q\alpha)|_q\alpha \leq M_2.$$

Since the above relation is true for every r, we take the summation $\sum_{k=1}^{m}$ of both sides to deduce that there exists C such that (3.1) holds. The proof is finished.

Lemma 3.2. *If equation (2.8) verifies Perron's condition, then there exists a constant $M > 0$ such that*

$$\|X(t,s)\| < M \quad for \quad t \geq s \geq a.$$

Proof. Having taken into account that $Y^T(r,t)$ satisfies equation (2.11), we integrate both sides with respect to r from s to t $(s \geq a)$ to get

$$Y^T(t,t) - Y^T(s,t) = -\int_s^t Y^T(qr,t)A(r)_q r - \int_s^t q^{\alpha_0} Y^T(q^{\alpha_0} r,t)B(q^{\alpha_0} r)_q r.$$

It follows that

$$Y^T(s,t) \quad = \quad I + \int_s^t Y^T(qr,t)A(r)_q r + q^{\alpha_0} \int_s^t Y^T(q^{\alpha_0} r,t)B(q^{\alpha_0} r)_q r.$$

Changing the variable $qu = q^{\alpha_0} r$, we obtain

$$Y^T(s,t) \quad = \quad I + \int_s^t Y^T(qr,t)A(r)_q r + q \int_{q^{\alpha_0 - 1} s}^{q^{\alpha_0 - 1} t} Y^T(qr,t)B(qr)_q r.$$

Using the relation $Y^T(s,t) = X(t,s)$ and that $Y(r,t) = 0$ for $r > t$, we have

$$X(t,s) \quad = \quad I + \int_s^t X(t,qr)A(r)_q r + q \int_{q^{\alpha_0 - 1} s}^{t} X(t,qr)B(qr)_q r.$$

Taking the norm of both sides yields

$$\|X(t,s)\| \quad \leq \quad 1 + \gamma(1+q) \int_s^t \|X(t,qr)\|_q r,$$

where $\gamma = \max\{\sup_{t \geq a} \|A(t)\|, \sup_{t \geq a} \|B(t)\|\}$. Employing inequality (3.1) results in the desired conclusion.

Lemma 3.3. *If equation (2.8) verifies Perron's condition, then its zero solution is uniformly stable.*

Proof. Let $x(t;t_0,\phi)$ be the solution of (2.8) satisfying (2.9). In view of Lemma 2.2, the solution has the form

$$x(t;t_0,\phi) \quad = \quad X(t,t_0)x(t_0) + \int_{\sigma(t_0)}^{\tau^{-1}(t_0)} X(t,s)B(s)x(q^{-\alpha} s)_q s, \;\; t \geq t_0.$$

Changing the variable $u = q^{-\alpha_0} s$, we get

$$x(t;t_0,\phi) \quad = \quad X(t,t_0)x(t_0) + q^{\alpha_0} \int_{q^{-\alpha+1} t_0}^{t_0} X(t,q^{\alpha_0} s)B(q^{\alpha_0} s)\phi(s)_q s, \;\; t \geq t_0.$$

By virtue of Lemma 3.2, we obtain

$$\|x(t;t_0,\phi)\| \leq M_1\|\phi\|_0,$$

where $M_1 = M(1+\gamma\alpha_0 q^{\alpha_0})$ and $\|\phi\|_0 = \sup_{t\in[q^{-\alpha+1}t_0,t_0]}\|x(t)\|$. Thus, the trivial solution is uniformly stable.

Theorem 3.1. *If equation (2.8) verifies Perron condition, then its zero solution is uniformly asymptotically stable.*

Proof. In view of Lemma 3.3, one can deduce that it remains to prove that

$$\lim_{t\to\infty} x(t;t_0,\phi) = 0 \tag{3.3}$$

uniformly with respect to t_0 and ϕ.

For our purpose, let $\lambda \geq t_0$, then the solution has the form

$$x(t;t_0,\phi) = X(t,\lambda)x(\lambda;t_0,\phi) + \int_{q^{-\alpha+1}\lambda}^{\lambda} X(t,q^{\alpha_0}s)B(q^{\alpha_0}s)x(s;t_0,\phi)_q s.$$

Integrating both sides with respect to λ from t_0 to t, we have

$$\begin{aligned}\int_{t_0}^{t} x(t;t_0,\phi)_q\lambda &= \int_{t_0}^{t} X(t,\lambda)x(\lambda;t_0,\phi)_q\lambda \\ &+ \int_{t_0}^{t}\int_{q^{-\alpha+1}\lambda}^{\lambda} X(t,q^{\alpha_0}s)B(q^{\alpha_0}s)x(s;t_0,\phi)_q s_q\lambda\end{aligned}$$

or

$$\begin{aligned}(n-n_0-1)x(t;t_0,\phi) &= \sum_{m_0=n_0}^{n-1} X(q^n,q^{m_0})x(q^{m_0};q^{n_0},\phi) \\ &+ \sum_{m_0=n_0}^{n-1}\sum_{k=m_0-\alpha_0+1}^{m_0-1} X(q^n,q^{k+\alpha_0})B(q^{k+\alpha_0})x(q^k;q^{n_0},\phi),\end{aligned}$$

where $t=q^n,\ t_0=q^{n_0},\ \lambda=q^{m_0}$ and $s=q^k$. Interchanging the order of summations to get

$$\begin{aligned}(n-n_0-1)x(q^n;q^{n_0},\phi) &= \sum_{m_0=n_0}^{n-1} X(q^n,q^{m_0})x(q^{m_0};q^{n_0},\phi) \\ &+ \sum_{k=n_0-\alpha_0+1}^{n_0-1}\sum_{m_0=n_0}^{k+\alpha_0-1} X(q^n,q^{k+\alpha_0})B(q^{k+\alpha_0})x(q^k;q^{n_0},\phi) \\ &+ \sum_{k=n_0}^{n-\alpha_0}\sum_{m_0=k+1}^{k+\alpha_0-1} X(q^n,q^{k+\alpha_0})B(q^{k+\alpha_0})x(q^k;q^{n_0},\phi) \\ &+ \sum_{k=n-\alpha_0+1}^{n-2}\sum_{m_0=k+1}^{n-1} X(q^n,q^{k+\alpha_0})B(q^{k+\alpha_0})x(q^k;q^{n_0},\phi).\end{aligned}$$

Using that $X(s,t)=0$ for $s<t$, the last term of the above equation vanishes. Taking the norm for both sides and using Lemma 3.1, Lemma 3.2 and Lemma 3.3, we obtain

$$\begin{aligned}(n-n_0-1)\|x(q^n;q^{n_0},\phi)\| &\leq M_1C\|\phi\|_0+\gamma\|\phi\|_0M\alpha_0^2M_1q^{\alpha_0}\\ &+\gamma\|\phi\|_0M_1(\alpha_0-1)q^{\alpha_0}\sum_{k=n_0+\alpha_0-1}^{n-1}\|X(q^n,q^{k+\alpha_0})\|.\end{aligned}$$

It follows that

$$(n-n_0-1)\|x(q^n;q^{n_0},\phi)\|\leq M_2\|\phi\|_0,$$

where

$$M_2=M_1C+\gamma MM_1\alpha_0^2q^{\alpha_0}+\gamma M_1(\alpha_0-1)Cq^{\alpha_0}.$$

Thus,

$$\|x(q^n;q^{n_0},\phi)\|\leq\frac{M_2}{(n-n_0-1)}\|\phi\|_0.$$

Letting $t\to\infty$ which takes place as $n\to\infty$, we get the desired conclusion (3.3). The solution is uniformly asymptotically stable.

References

[1] H. F. Jackson, *q-Difference equations*, Am. J. Math. **32** (1910), 305–314.

[2] R. D. Carmichael, *The general theory of linear q–difference equations*, Am. J. Math. **34** (1912), 147–168.

[3] T. E. Mason, *On properties of the solution of linear q–difference equations with entire function coefficients*, Am. J. Math. **37** (1915), 439–444.

[4] C. R. Adams, *On the linear ordinary q–difference equation*, Am. Math. Ser. II, **30** (1929), 195–205.

[5] W. J. Trjitzinsky, *Analytic theory of linear q–difference equations*, Acta Mathematica, (1933).

[6] G. Bangerezako,*The factorization method for the Askey-Wilson polynomials*, J. Comput. Appl. Math. **107** (2) (1999), 219–232.

[7] G. Bangerezako, *Variational q–calculus*, J. Math. Anal. Appl. **289** (2) (2004), 650–665.

[8] G. Bangerezako, *Variational calculus on q–nonuniform lattices*, J. Math. Anal. Appl. **306** (1) (2005), 161–179.

[9] A. Fitouhi and N. Bettaibi, *Applications of the Mellin transform in quantum calculus*, J. Math. Anal. Appl. **328** (1) (2007), 518–534.

[10] A. Fitouhi and L. Dhaoudi, *On a q–Paley-Wiener theorem*, J. Math. Anal. Appl. **294** (1) (2004), 17–23.

[11] M. E. H. Ismail and M. Rahman, *Inverse operators, q–fractional Integrals and q–Bernoulli polynomials*, J. Approx. Theory **114** (2) (2002), 269–307.

[12] M. Bohner and M. Adivar, *Spectrum and principal vectors of q–difference equations*, Indian J. Math. **48** (1) (2006), 17–33.

[13] M. Bohner and M. Adivar, *Spectral analysis of* q*–difference equations with spectral singularities*, Math. Comput. Modelling **43** (7–8) (2006), 695-703.

[14] M. Bohner and M. Unal, *Kneser's theorem in q–calculus*, J. Phys. A: Math. Gen. **38** (2005), 6729–6739.

[15] G. Bangerezako, *The factorization method for the general second order q–difference equation and the Laguerre–Han polynomials on the general q–lattice*, J. Phys. A: Math. Gen. **36** (2003), 765–773.

[16] J. P. Bézivin, *Sur les* é*quations functionnelles aux q–diff*é*rences*, Aequatones Math. **43** (1993), 159–176.

[17] D. Levi, J. Negro and M. A. del Olmo, *Discrete q–derivatives and symmetries of q–difference equations*, J. Phys. A:Math. Gen. **37** (2004), 3459–3473.

[18] K. Victor and C. Pokman, *Quantum calculus*. Universitext. Springer-Verlag, New York, 2002.

[19] A. Halanay, *Differential Equations: Stability, Oscillation, Time Lags*, Academic Press Inc., 1966.

[20] O. Perron, *Die Stabilitatsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.

[21] M. U. Akhmet, J. Alzabut and A. Zafer, *Perron's theorem for linear impulsive differential equations with distributed delay*, J. Comput. Appl. Math. **193** (1) (2006), 204–218.

[22] J. O. Alzabut and T. Abdeljawad, *Perron Type Criterion for Linear Difference Equations with Distributed Delay*, Discrete Dyn. Nat. Soc., 2007. doi:1155/10840.

[23] J. O. Alzabut and T. Abdeljawad, *Asymptotic behavior for a class of linear impulsive difference equations: Perron–like theorem*, Dyn. Syst. Appl., submitted.

[24] A. Anokhin , L. Berezansky and E. Braverman, *Exponential stability of linear delay impulsive differential equations*, J. Math. Anal. Appl. **193** (1995), 923–941.

[25] V. A. Tyshkevich, *A perturbations-accumulation problem for linear differential equation with time lag*, Differ. Equ. **14** (1978), 177–186.

[26] N. V. Azbelev, L. M. Berezanskii, P. M. Simonov and A. V. Chistyakov, *Stability of linear systems with time lag*, Differ. Equ. **23** (1987), 493–500.

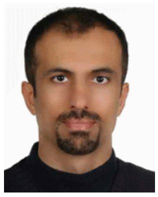
[27] A. Murat and R. Youssef, *Stability and periodicity in dynamic delay equations*, Comput. Math. Appl. **58** (2) (2009), 264–272.

[28] D. R. Anderson, R. J. Krueger and A. C. Peterson, *Delay dynamic equations with stability*, Adv. Difference Equ. Volume 2006, Art. ID 94051, 1–19.

[29] G. Bangerezako, *An Introduction to q–difference equations*: http//www.math.ucl.ac.be/membres/bangerezako/qbook712.pdf .

[30] M. Bohner and A. Peterson, *Dynamic Equations on Time Scale: An Introduction with Applications*, Boston MA, Birkhauser, 2001.

Jehad is Palestinian. He was born in 1972 in Jeddah, Saudi Arabia. He has received his Ph.D. degree in 2004 from Middle East Technical University in Ankara, Turkey. He began his career as a part time teaching assistant in Department of Mathematics at Middle East Technical University. He served as a lecturer and an assistant professor of applied mathematics in Çankaya University during the period 2002–2007. Currently, he is an associate professor of applied mathematics at Prince Sultan University in Riyadh, Saudi Arabia. Dr. Jehad's research interests are focused on studying the qualitative properties of solutions (oscillation, periodicity, stability and controllability) of differential equations, difference equations and equations involving impulses, delay and fractional derivatives. Biological models which are governed by such type of equations are also among his field.



Thabet Abdeljawad is Palestinian. He was born in 1971 in West Bank where he has grown up until he finished his high school in 1990. He has received his Ph.D. degree in 2000 from Middle East Technical University in Ankara, Turkey. He has been working as an assistant professor in Department of Mathematics and Computer Sciences at Çankaya since 2001. Dr. Thabet has different research fields of interest: functional analysis; topology and fixed point theory; fractional dynamical systems; variational calculus with delay and fractional variational calculus and stability of dynamical systems.