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Three-Dimensional Lattice Approach to Fractional Generalization of Continuum Gradient Elasticity

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Abstract: A relationship between discrete and continuous fractional-order nonlocal elasticity theory is discussed. As a discrete system we consider three-dimensional lattice with long-range interactions that are described by fractional-order lattice operators. We prove that the continuous limit of suggested three-dimensional lattice equations gives continuum differential equations with the Riesz derivatives of non-integer orders. The proposed lattice models give a new microstructural basis for elasticity of materials with power-law type of non-locality. Moreover these lattice models allow us to have a unified microscopic description for fractional and usual (non-fractional) gradient elasticity continuum.

Keywords: Fractional dynamics, lattice, gradient elasticity, fractional derivative, long-range interaction.

1 Introduction

Deformations of elastic materials can be described by lattice approach [1,2], and continuum approach [3,4]. Continuum approach can be considered as a continuous limit of lattice approach, where the length-scales of continuum element are much larger than the distances between the lattice particles. Continuum models of elastic materials with microstructure have been suggested by Mindlin in [5]. In the Mindlin's models, two scale-types of physical quantities are used to characterize elastic materials at the micro and macro scales. The quantities of microstructured materials are considered for both of these scales. The Mindlin's continuum models of the materials with microstructures differ by equations that describe connections of the microscopic quantities with macroscopic quantities. The main Midlin's continuum models of gradient elasticity [5] are the first-gradient and the second-gradient models. The first Midlin's model is characterized by assumption that the microscopic deformation gradient is the first gradient of the macroscopic strain. The second Midlin's model is characterized by assumption that the microscopic deformation gradient is defined as the second gradient of the macroscopic displacement. Despite these difference these two gradient models have the same displacements equations [5].

In strict sense, the models of gradient elasticity cannot be considered as real nonlocal models. It is caused by the fact that the equations of these models include a finite number of integer-order derivatives of with respect to coordinates. To describe weak nonlocality we should use infinite series with integer-order derivatives. It is difficult to solve this problem for general infinite series of integer-order derivatives. We suggest solution of this problem for power-law type of weak nonlocality by using derivatives of non-integer orders. This possibility is based on the fact that the derivatives of non-integer orders are actually equivalent to infinite number of derivatives of integer orders (see Lemma 15.3 in [6]).

Derivatives and integrals of non-integer orders [6,7,8] have a long history [9,10,11] and a wide application in physics and mechanics [12,13,14,15,16]. The derivatives of non-integer orders allow us to formulate generalization of models of elastic continuum with weak nonlocality of power-law type. First time the derivatives fractional order with respect to space coordinates were used in the elasticity theory by Gubenko [17,18] in 1957. Recently the derivatives of non-integer orders have been used to describe continua with power-law type of non-locality (for example, see [19,20,21] and [22,23]). Fractional generalizations of integral non-local models of elasticity are considered in [34,35,36,24,37]. In these models the integration of non-integer orders is used to describe a strong nonlocality of continuum. In our consideration we will focus on continuum models with weak nonlocality and corresponding lattice models.

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Differential equations with fractional derivatives are powerful tools to describe continua with nonlocal properties of power-law type. In papers [30,29,32,33] we have shown that the differential equations with fractional derivatives can be obtained from models of physical lattices with long-range interactions. The one-dimensional lattice models for fractional gradient elasticity and the corresponding continuum equations have been suggested in [22,23,24]. All suggested lattice models of fractional gradient elasticity are one-dimensional models only. In this paper, we use the lattice fractional calculus suggested in [32,33] to propose three-dimensional models of lattices with long-range interactions and continua with power-law nonlocality. We propose three-dimensional lattice models for fractional gradient elasticity of the Riesz type and the corresponding models of fractional nonlocal continuum. To give these three-dimensional generalizations, we use a lattice fractional calculus based on the fractional-order derivatives of Riesz type. In this paper, we apply this new mathematical tool to describe physical lattices with long-range interactions and corresponding fractional elasticity equations for nonlocal continuum with power-law nonlocality. A new relationship between discrete and continuous fractional-order nonlocal dynamical systems is discussed. We consider a lattice model with long-range interactions for the Mindlin continuum model of first gradient elasticity for isotropic materials and its generalizations for fractional order nonlocalities.

2 Lattice with Long-Range Interactions

For simplification we consider unbounded three-dimensional lattices. For unbounded lattices we can use three non-coplanar vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, that are the shortest vectors by which a lattice can be displaced and be brought back into itself. Sites of this lattice can be characterized by the number vector $\mathbf{n} = (n_1, n_2, n_3)$, where n_j $(j = 1,2,3)$ are integer. For simplification, we consider a lattice with mutually perpendicular primitive lattice vectors $\mathbf{a}_j$, $(j = 1,2,3)$, i.e. a primitive orthorhombic Bravais lattice. Let us choose directions of the Cartesian axes to coincide with the vectors $\mathbf{a}_j$. Then $\mathbf{a}_j = a_j\,\mathbf{e}_j$, where $a_j = |\mathbf{a}_j| > 0$ and $\mathbf{e}_j$ are the basis vectors of the Cartesian coordinate system, and the vector $\mathbf{n}$ can be represented as $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$.

Let us consider a coordinate origin at one of the lattice sites. The positions of site with $\mathbf{n} = (n_1, n_2, n_3)$ are described by the vector $\mathbf{r}(\mathbf{n}) = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$. We assume that the equilibrium positions of particles coincide with the lattice sites $\mathbf{r}(\mathbf{n})$. If particles are displaced relative to their equilibrium positions then the coordinates of the lattice particles differ form coordinates $\mathbf{r}(\mathbf{n})$ of lattice sites. For this case, we describe the coordinates of $\mathbf{n}$-particle by the vector field $\mathbf{u}(\mathbf{n},t) = \sum_{j=1}^{3} u_j(\mathbf{n},t)\,\mathbf{e}_j$ that is the displacement of this particle. Here $u_j(\mathbf{n},t) = u_j(n_1,n_2,n_3,t)$ are components of the displacement vector for lattice particle that is defined by the vector $\mathbf{n} = (n_1, n_2, n_3)$.

To describe long-range interactions in the lattices, we will define fractional-order lattice operators. Let us give a definition [32,33] of lattice partial derivative of arbitrary positive real order α_j in the direction $\mathbf{e}_j = \mathbf{a}_j/|\mathbf{a}_j|$ in the lattice.

Definition. *Lattice fractional partial derivatives are the operators* $\mathbb{D}_L^+\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ *and* $\mathbb{D}_L^-\begin{bmatrix}\alpha_j\\ j\end{bmatrix}$ *such that*

$$\mathbb{D}_L^+\begin{bmatrix}\alpha_j\\ j\end{bmatrix} u(\mathbf{m}) = \left(\mathbb{D}_L^+\begin{bmatrix}\alpha_j\\ j\end{bmatrix} u\right)(\mathbf{n}) = \frac{1}{a_j^{\alpha_j}} \sum_{m_j=-\infty}^{+\infty} K_{\alpha_j}^+(n_j - m_j)\,u(\mathbf{m}), \quad (j = 1,2,3), \tag{1}$$

$$\mathbb{D}_L^-\begin{bmatrix}\alpha_j\\ j\end{bmatrix} u(\mathbf{m}) = \left(\mathbb{D}_L^-\begin{bmatrix}\alpha_j\\ j\end{bmatrix} u\right)(\mathbf{n}) = \frac{1}{a_j^{\alpha_j}} \sum_{m_j=-\infty}^{+\infty} K_{\alpha_j}^-(n_j - m_j)\,u(\mathbf{m}), \quad (j = 1,2,3), \tag{2}$$

where $n_i = m_i$ *for* $i \neq j$ *(i.e.* $\mathbf{n} = \mathbf{m} + (n_j - m_j)\mathbf{e}_j$*),* $\mathbf{m} \in \mathbb{Z}^3$, $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, *and the interaction kernels* $K_{\alpha_j}^{\pm}(n_j - m_j)$ *are defined by the equations*

$$K_{\alpha_j}^+(n_j - m_j) = \frac{\pi^{\alpha_j}}{\alpha_j + 1}\, {}_1F_2\left(\frac{\alpha_j+1}{2}; \frac{1}{2}, \frac{\alpha_j+3}{2}; -\frac{\pi^2 (n_j - m_j)^2}{4}\right), \quad \alpha_j > 0, \tag{3}$$

$$K_{\alpha_j}^-(n - m) = -\frac{\pi^{\alpha_j+1}(n-m)}{\alpha_j + 2}\, {}_1F_2\left(\frac{\alpha_j+2}{2}; \frac{3}{2}, \frac{\alpha_j+4}{2}; -\frac{\pi^2 (n-m)^2}{4}\right), \quad \alpha_j > 0, \tag{4}$$

where ${}_1F_2$ *is the Gauss hypergeometric function [25]. The parameters* $\alpha_j > 0$ *will be called the orders of the lattice derivatives.*

Let us explain the reasons for definition the interaction kernels $K_{\alpha}^{\pm}(n - m)$ in the forms (3), (4), and describe some properties of these kernels.

The kernels $K^{\pm}_{\alpha_j}(n_j)$ are real-valued functions of integer variable $n_j \in \mathbb{Z}$. The kernel $K^{+}_{\alpha_j}(n_j)$ is even function and $K^{-}_{\alpha_j}(n_j)$ is odd function, i.e., $K^{+}_{\alpha_j}(-n_j) = +K^{+}_{\alpha_j}(n_j)$, and $K^{-}_{\alpha_j}(-n_j) = -K^{-}_{\alpha_j}(n_j)$ hold for all $n_j \in \mathbb{Z}$ and $j = 1,2,3$.

The Fourier series transforms $\hat{K}^{+}_{\alpha_j}(k_j)$ of the kernels $K^{+}_{\alpha_j}(n_j)$ in the form

$$\hat{K}^{+}_{\alpha_j}(k_j) = \sum_{n_j=-\infty}^{+\infty} e^{-ik_jn_j} K^{+}_{\alpha_j}(n_j) = 2\sum_{n_j=1}^{\infty} K^{+}_{\alpha_j}(n_j)\cos(k_jn_j) + K^{+}_{\alpha_j}(0) \tag{5}$$

satisfy the condition

$$\hat{K}^{+}_{\alpha_j}(k_j) = |k_j|^{\alpha_j}, \quad (\alpha_j > 0). \tag{6}$$

The Fourier series transforms $\hat{K}^{-}_{\alpha_j}(k_j)$ of the kernels $K^{-}_{\alpha_j}(n_j)$ in the form

$$\hat{K}^{-}_{\alpha_j}(k_j) = \sum_{n_j=-\infty}^{+\infty} e^{-ik_jn_j} K^{-}_{\alpha_j}(n_j) = -2i\sum_{n_j=1}^{\infty} K^{-}_{\alpha_j}(n_j)\sin(k_jn_j) \tag{7}$$

satisfy the condition

$$\hat{K}^{-}_{\alpha_j}(k_j) = i\,\mathrm{sgn}(k_j)\,|k_j|^{\alpha_j}, \quad (\alpha_j > 0). \tag{8}$$

Note that we use the minus sign in the exponents of (5) and (7) instead of plus in order to have the plus sign for plane waves and for the Fourier series.

The form (3) of the interaction term $K^{+}_{\alpha_j}(n_j - m_j)$ is completely determined by the requirement (6) If we use an inverse relation to (5) with $\hat{K}^{+}_{\alpha_j}(k_j) = |k_j|^{\alpha_j}$ that has the form

$$K^{+}_{\alpha_j}(n_j) = \frac{1}{\pi}\int_0^{\pi} k_j^{\alpha_j}\cos(n_jk_j)\,dk, \quad (\alpha_j \in \mathbb{R}, \quad \alpha_j > 0), \tag{9}$$

then we get equation (3) for the interaction kernel $K^{+}_{\alpha_j}(n_j - m_j)$. The form (4) of the interaction term $K^{-}_{\alpha_j}(n_j - m_j)$ is completely determined by (6) If we use an inverse relation to (7) with $\hat{K}^{-}_{\alpha_j}(k_j) = i\,\mathrm{sgn}(k_j)\,|k_j|^{\alpha_j}$ that has the form

$$K^{-}_{\alpha_j}(n_j) = -\frac{1}{\pi}\int_0^{\pi} k_j^{\alpha_j}\sin(n_jk_j)\,dk_j \quad (\alpha_j \in \mathbb{R}_+, \quad \alpha_j > 0), \tag{10}$$

then we get equation (4) for the interaction kernel $K^{-}_{\alpha_j}(n_j - m_j)$. Note that $K^{-}_{\alpha_j}(0) = 0$.

The interactions with (3) and (4) for integer and non-integer orders α_j can be interpreted as a long-range interactions of n-particle with all other particles.

Let us give the exact forms of the kernels $\hat{K}^{\pm}_{\alpha}(k)$ for integer positive $\alpha \in \mathbb{N}$. Equations (3) and (4) for the case $\alpha \in \mathbb{N}$ can be simplified by using Equation 2.5.3.5 of [26]. For integer values $\alpha = 1,2,3,4$, we get the kernels $K^{+}_{\alpha}(n)$ with $n \neq 0$ in the form

$$K^{+}_{1}(n) = -\frac{1-(-1)^n}{\pi n^2}, \quad K^{+}_{2}(n) = \frac{2(-1)^n}{n^2}, \tag{11}$$

$$K^{+}_{3}(n) = \frac{3\pi(-1)^n}{n^2} + \frac{6(1-(-1)^n)}{\pi n^4}, \quad K^{+}_{4}(n) = \frac{4\pi^2(-1)^n}{n^2} - \frac{24(-1)^n}{n^4}, \tag{12}$$

where $n \neq 0$, $n \in \mathbb{Z}$, and $K^{+}_{m}(0) = \pi^m/(m+1)$ for all $m \in \mathbb{N}$. For $\alpha = 1,2,3,4$, the kernels $K^{-}_{\alpha}(n)$ with $n \neq 0$ are

$$K^{-}_{1}(n) = \frac{(-1)^n}{n}, \quad K^{-}_{2}(n) = \frac{(-1)^n\pi}{n} + \frac{2(1-(-1)^n)}{\pi n^3}, \tag{13}$$

$$K^{-}_{3}(n) = \frac{(-1)^n\pi^2}{n} - \frac{6(-1)^n}{n^3}, \quad K^{-}_{4}(n) = \frac{(-1)^n\pi^3}{n} - \frac{12(-1)^n\pi}{n^3} - \frac{24(1-(-1)^n)}{\pi n^5}, \tag{14}$$

where $n \neq 0$, $n \in \mathbb{Z}$, and $K^{-}_{m}(0) = 0$ for all $m \in \mathbb{N}$. Note that $(1-(-1)^n) = 2$ for odd n, and $(1-(-1)^n) = 0$ for even n. These kernels can allow us to consider lattice models for usual (non-fractional) gradient elasticity [27,28].

In the definition of lattice fractional derivatives (1) and (2) the value $j \in \{1,2,3\}$ characterizes the component n_j of the lattice vector $\mathbf{n}$ with respect to which this derivative is taken. It is similar to the variable x_j in the usual partial derivatives for the space $\mathbb{R}^3$. The lattice operators $\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ are analogous to the partial derivatives of order α with respect to coordinates x_j for a continuum model.

To describe isotropic physical lattices we should use the lattice operators ${}^R\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ and ${}^R_B\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ with orders $\alpha_j = \alpha$ for all $j = 1,2,3$.

For simplification, we use the combination of the repeated fractional-order lattice operators

$$ {}^R\mathbb{D}^{\pm,\pm}_L \begin{bmatrix} \alpha_i\, \beta_j \\ i\; j \end{bmatrix} = {}^R\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_i \\ i \end{bmatrix} {}^R\mathbb{D}^{\pm}_L \begin{bmatrix} \beta_j \\ j \end{bmatrix}, \tag{15} $$

where i, j take values from the set $\{1;2;3\}$. The action of the operator (15) on the lattice fields $u_k(\mathbf{m},t)$ is

$$ {}^R\mathbb{D}^{\pm,\pm}_L \begin{bmatrix} \alpha_i\, \beta_j \\ i\; j \end{bmatrix} u_k(\mathbf{m},t) = \sum_{m_i=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} K^{\pm}_{\alpha_i}(n_i - m_i) K^{\pm}_{\beta_j}(n_j - m_j)\, u_k(\mathbf{m},t), \tag{16} $$

where $i,j,k \in \{1,2,3\}$. Analogously, we can define the repeated fractional-order lattice operators ${}^R\mathbb{D}^{\pm,\pm,\pm}_L \begin{bmatrix} \alpha_i\, \beta_j\, \gamma_l \\ i\; j\; l \end{bmatrix}$, ${}^R\mathbb{D}^{\pm,\pm,\mp}_L \begin{bmatrix} \alpha_i\, \beta_j\, \gamma_l \\ i\; j\; l \end{bmatrix}$, and other.

3 Three-Dimensional Lattice Models for Fractional Gradient Elasticity

The gradient terms are used to take into account so-called weak nonlocality. In order to describe a weak nonlocality of power-law type, we should use terms with the fractional gradients and fractional Laplace operators. The one-dimensional lattice models for fractional elasticity and the correspondent continuum equations have been suggested in [22,23,24]. To generalize the one-dimensional lattice models of fractional elasticity for three-dimensional lattices we can apply the fractional-order lattice operators of the Riesz type. For simplification we will consider a primitive orthorhombic Bravais lattice with long-range interactions, where $\mathbf{a}_i = a_i\, \mathbf{e}_i$, and $\mathbf{e}_i$ is the basis of the Cartesian coordinate system.

For microstructural models of the three-dimensional fractional gradient elasticity of anisotropic continua, we use the lattice equations

$$ M \frac{d^2 u_i(\mathbf{n},t)}{dt^2} = \sum_{j,l=1}^{3} A^L_{ijkl}\, {}^R\mathbb{D}^{-,-}_L \begin{bmatrix} 1\, 1 \\ j\, l \end{bmatrix} u_k(\mathbf{m},t) + $$
$$ + \sum_{j,m,l=1}^{3} B^L_{ijkl}\, {}^R\mathbb{D}^{-,+,-}_L \begin{bmatrix} 1\, \alpha\, 1 \\ j\, m\, l \end{bmatrix} u_k(\mathbf{m},t) + F_i(\mathbf{n},t), \tag{17} $$

where $u_k(\mathbf{m},t) = u_k(m_1,m_2,m_3,t)$ is the displacement for the lattice, and A^L_{ijkl} and B^L_{ijkl} are the lattice coupling constants. We assume that the fourth-order tensors A^L_{ijkl} and B^L_{ijkl} have the same type of symmetry as the fourth-order elastic stiffness tensor C_{ijkl}:

$$ A^L_{ijkl} = A^L_{jikl} = A^L_{ijlk} = A^L_{klij}, \quad B^L_{ijkl} = B^L_{jikl} = B^L_{ijlk} = B^L_{klij}. \tag{18} $$

For primitive orthorhombic Bravais lattice, we have nine coupling constants A^L_{ijkl} and nine gradient coupling constants B^L_{ijkl}.

To describe anisotropic long-range interaction in lattices, we should use the lattice operators ${}^R\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ and ${}^R_B\mathbb{D}^{\pm}_L \begin{bmatrix} \alpha_j \\ j \end{bmatrix}$ with unequal orders α_j at least for one $j = 1,2,3$.

In the Mindlin continuum models of elastic materials with microstructure [5], two different types of quantities are used for the micro and macro scales. These models of gradient elasticity differ in the assumed relation between the microscopic deformation and the macroscopic displacement. At the same time, despite the theoretical differences between these models, the equations for displacements of these models are identical [5]. In order to derive a fractional generalization of the Mindlin gradient models [5], and a corresponding there-dimensional lattice model, we assume that lattice is characterized by the mutually perpendicular vectors $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{a}_3$ with equal length $a_1 = a_2 = a_3 = a$. As lattice equations for the Mindlin gradient elasticity we can consider the equation

$$ M \frac{d^2 u_i(\mathbf{n},t)}{dt^2} = A^L_0(\alpha) \sum_{j=1}^{3} {}^R\mathbb{D}^{+}_L \begin{bmatrix} 2\alpha \\ j \end{bmatrix} \ddot{u}_i(\mathbf{m},t) - A^L_1(\alpha) \sum_{j:\, j\neq i}^{R} \mathbb{D}^{-,-}_L \begin{bmatrix} \alpha\;\alpha \\ j\;\; i \end{bmatrix} u_i(\mathbf{m},t) - $$

$$-A_2^L(\alpha)\, {}^R\mathbb{D}_L^+ \begin{bmatrix} 2\alpha \\ i \end{bmatrix} u_i(\mathbf{m},t) - A_3^L(\alpha) \sum_{j\neq i}^{3} {}^R\mathbb{D}_L^+ \begin{bmatrix} 2\alpha \\ j \end{bmatrix} u_i(\mathbf{m},t) -$$

$$-B_1^L(\alpha) \sum_{j:\, j\neq i}^{3} \left({}^R\mathbb{D}_L^{-,-} \begin{bmatrix} 3\alpha\ \alpha \\ j\ \ i \end{bmatrix} u_j(\mathbf{m},t) + {}^R\mathbb{D}_L^{-,-} \begin{bmatrix} \alpha\ 3\alpha \\ j\ \ i \end{bmatrix} \right) u_j(\mathbf{m},t) -$$

$$-B_2^L(\alpha) \sum_{j:\, j\neq i}^{3} {}^R\mathbb{D}_L^{+,+} \begin{bmatrix} 2\alpha\ 2\alpha \\ j\ \ i \end{bmatrix} u_i(\mathbf{m},t) - B_3^L(\alpha)\, {}^R\mathbb{D}_L^+ \begin{bmatrix} 4\alpha \\ i \end{bmatrix} u_i(\mathbf{m},t) -$$

$$-B_4^L(\alpha) \sum_{\substack{k,j \\ k\neq j;k\neq i;j\neq i}}^{3} {}^R\mathbb{D}_L^{-,-,+} \begin{bmatrix} \alpha\ \alpha\ 2\alpha \\ j\ \ i\ \ k \end{bmatrix} u_j(\mathbf{m},t) - B_5^L(\alpha) \sum_{\substack{k,j \\ k\neq j}}^{3} {}^R\mathbb{D}_L^{+,+} \begin{bmatrix} 2\alpha\ 2\alpha \\ j\ \ k \end{bmatrix} u_i(\mathbf{m},t) -$$

$$-B_6^L(\alpha) \sum_{j=1}^{3} {}^R\mathbb{D}_L^+ \begin{bmatrix} 4\alpha \\ j \end{bmatrix} u_i(\mathbf{m},t) + F_i(\mathbf{n},t), \qquad (19)$$

where $A_0^L(\alpha)$, $A_1^L(\alpha)$, $A_2^L(\alpha)$, $A_3^L(\alpha)$, and $B_1^L(\alpha)$, ..., $B_6^L(\alpha)$ are corresponding coupling constants of the lattice long-range interactions.

In the lattice model (19) all lattice operators have fractional orders. For wide class of nonlocal elastic material the short-range and long-range particle interactions are present at the same time. This means that the lattice equations should include the lattice operators of integer and non-integer orders. For this class of materials, we can use the lattice equation in the form

$$M\,\ddot{u}_i(\mathbf{n},t) = A_0^L \sum_{j=1}^{3} {}^R\mathbb{D}_L^+ \begin{bmatrix} 2 \\ j \end{bmatrix} \ddot{u}_i(\mathbf{m},t) +$$

$$+A_1^L \sum_{j=1}^{3} {}^R\mathbb{D}_L^{-,-} \begin{bmatrix} 1\,1 \\ j\,i \end{bmatrix} u_j(\mathbf{m},t) + A_2^L \sum_{j=1}^{3} {}^R\mathbb{D}_L^+ \begin{bmatrix} 2 \\ j \end{bmatrix} u_i(\mathbf{m},t) -$$

$$+B_1^L \sum_{j,m,i}^{3} {}^R\mathbb{D}_L^{-,+,-} \begin{bmatrix} 1\,\alpha\,1 \\ j\,m\,i \end{bmatrix} u_j(\mathbf{m},t) + B_2^L \sum_{j,m,i}^{3} {}^R\mathbb{D}_L^{-,+,-} \begin{bmatrix} 1\,\alpha\,1 \\ j\,m\,j \end{bmatrix} u_i(\mathbf{m},t) + F_i(\mathbf{n},t), \qquad (20)$$

where the displacement for the lattice is $u_i(\mathbf{m},t) = u_i(m_1,m_2,m_3,t)$, and A_0^L, A_1^L, A_2^L, B_1^L, B_2^L are the coupling constants of the lattice long-range interactions.

These three-dimensional lattice models in the continuum limit give fractional generalization of the Mindlin model of the first gradient elasticity and allow us to have a microstructural basis for continua with weak nonlocality.

4 Continuum Fractional Derivatives of the Riesz Type

Let us give definitions of continuum derivatives of non-integer orders that allow us to describe materials with nonlocality of power-law type.

The continuum derivative of non-integer order α is defined [6,7] by the equation

$$\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r}) = \frac{1}{d_1(m,\alpha)} \int_{\mathbb{R}^1} \frac{1}{|z_i|^{\alpha+1}} (\Delta_{z_i}^m u)(\mathbf{r})\, dz_i, \quad (0<\alpha<m), \qquad (21)$$

where $(\Delta_{z_i}^m u)(\mathbf{r})$ is a finite difference of order m of a function $u(\mathbf{r})$ with the vector step $\mathbf{z}_i = z_i\,\mathbf{e}_i \in \mathbb{R}^3$ for the point $\mathbf{r}\in\mathbb{R}^3$. The centered difference

$$(\Delta_{z_i}^m u)(\mathbf{z}_i) = \sum_{n=0}^{m} (-1)^n \frac{m!}{n!\,(m-n)!}\, u(\mathbf{r}-(m/2-n)\,z_i\,\mathbf{e}_i). \qquad (22)$$

The constant $d_1(m,\alpha)$ is defined by

$$d_1(m,\alpha) = \frac{\pi^{3/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma((1+\alpha)/2)\sin(\pi\alpha/2)},$$

where

$$A_m(\alpha) = 2 \sum_{j=0}^{[m/2]} (-1)^{j-1} \frac{m!}{j!(m-j)!} (m/2-j)^{\alpha}$$

for the centered difference (22). The constants $d_1(m,\alpha)$ is different from zero for all $\alpha > 0$ in the case of an even m and centered difference $(\Delta_i^m u)$ (see Theorem 26.1 in [6]). Note that the integral (21) does not depend on the choice of $m > \alpha$. Therefore, we can always choose an even number m so that it is greater than parameter α, and we can use the centered difference (22) for all positive real values of α.

The continuum fractional derivatives $\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r})$ can be considered as the Riesz derivative of the function $u(\mathbf{r})$ with respect to one component $x_i \in \mathbb{R}^1$ of the vector $\mathbf{r} \in \mathbb{R}^3$, i.e. the operator $\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ is a partial fractional derivative of Riesz type. An important property of the Riesz fractional derivatives is the Fourier transform $\mathscr{F}$ of this operators in the form

$$\mathscr{F}\left(\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r})\right)(\mathbf{k}) = |k_i|^{\alpha} (\mathscr{F}u)(\mathbf{k}). \tag{23}$$

The property (23) is valid for functions $u(\mathbf{r})$ from the space $C^{\infty}(\mathbb{R}^1)$ of infinitely differentiable functions on $\mathbb{R}^1$ with compact support. It also holds for the Lizorkin space (see Section 8.1 in [6]).

Using $(-i)^{2m} = (-1)^m$, the Riesz fractional derivatives for even $\alpha = 2m$, where $m \in \mathbb{N}$, are connected with the usual partial derivatives of integer orders $2m$ by the relation

$$\mathbb{D}_C^+ \begin{bmatrix} 2m \\ i \end{bmatrix} u(\mathbf{r}) = (-1)^m \frac{\partial^{2m} u(\mathbf{r})}{\partial x_i^{2m}}. \tag{24}$$

The fractional derivatives $\mathbb{D}_C^+ \begin{bmatrix} 2m \\ i \end{bmatrix}$ for even orders α are local operators. Note that the Riesz derivative $\mathbb{D}_C^+ \begin{bmatrix} 1 \\ i \end{bmatrix}$ cannot be considered as a derivative of first order with respect to x_i, i.e., $\mathbb{D}_C^+ \begin{bmatrix} 1 \\ i \end{bmatrix} u(\mathbf{r}) \neq \partial u(\mathbf{r})/\partial x_i$. All Riesz derivatives for odd orders $\alpha = 2m+1$, where $m \in \mathbb{N}$, are non-local operators that cannot be considered as usual derivatives $\partial^{2m+1}/\partial x^{2m+1}$.

We can define a continuum fractional integral $\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ of the Riesz type as the Riesz potential of order α with respect to x_i by the equation

$$\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r}) = \int_{\mathbb{R}^1} R_{\alpha}(x_i - z_i)\, u(\mathbf{r} + (z_i - x_i)\,\mathbf{e}_i)\, dz_i, \quad (\alpha > 0), \tag{25}$$

where $\mathbf{e}_i$ is the basis of the Cartesian coordinate system, the function $R_{\alpha}(\mathbf{r})$ is the Riesz kernel. that is defined by

$$R_{\alpha}(\mathbf{r}) = \begin{cases} \gamma_3^{-1}(\alpha) |\mathbf{r}|^{\alpha-n} & \alpha \neq 3+2n, \quad n \in \mathbb{N}, \\ -\gamma_3^{-1}(\alpha) |\mathbf{r}|^{\alpha-3} \ln |\mathbf{r}| & \alpha = 3+2n, \quad n \in \mathbb{N}. \end{cases} \tag{26}$$

The constant $\gamma_3(\alpha)$ has the form

$$\gamma_3(\alpha) = \begin{cases} 2^{\alpha} \pi^{3/2} \Gamma(\alpha/2)/\Gamma((3-\alpha)/2) & \alpha \neq 3+2n, \\ (-1)^{(3-\alpha)/2} 2^{\alpha-1} \pi^{3/2}\, \Gamma(\alpha/2)\, \Gamma(1+[\alpha-3]/2) & \alpha = 3+2n, \end{cases} \tag{27}$$

where $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+$. An important property of this Riesz fractional integration is the Fourier transform $\mathscr{F}$ of this operators in the form

$$\mathscr{F}\left(\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r})\right)(\mathbf{k}) = |k_i|^{-\alpha} (\mathscr{F}u)(\mathbf{k}). \tag{28}$$

Note the distinction between the continuum fractional integral $\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ and the Riesz potential consists in the use of $|k_i|^{-\alpha}$ instead of $|\mathbf{k}|^{-\alpha}$. The continuum integral $\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ is an integration of $u(\mathbf{r})$ with respect to one variable x_i instead of all variables x_1, x_2, x_3 in the Riesz potential.

If $u(\mathbf{r})$ as a function of x_i belongs to the Lizorkin space, then we have [6] the semi-group property

$$\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} \mathbb{I}_C^+ \begin{bmatrix} \beta \\ i \end{bmatrix} u(\mathbf{r}) = \mathbb{I}_C^+ \begin{bmatrix} \alpha+\beta \\ i \end{bmatrix} u(\mathbf{r}), \tag{29}$$

where $\alpha > 0$ and $\beta > 0$, and the continuum fractional derivative $\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ yields an operator inverse to the continuum fractional integration $\mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix}$ as

$$\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} \mathbb{I}_C^+ \begin{bmatrix} \alpha \\ i \end{bmatrix} u(\mathbf{r}) = u(\mathbf{r}), \quad (\alpha > 0). \tag{30}$$

Note that the property (30) is also valid for the continuum fractional integration in the frame of L_p-spaces $L_p(\mathbb{R}^1)$ for $1 \leqslant p < 1/\alpha$ (see Theorem 26.3 in [6]).

The continuum fractional derivative $\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ j \end{bmatrix}$ with $\alpha = 1$ cannot be considered as usual derivative of first order with respect to x_j. Therefore we will define new continuum fractional derivative $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ j \end{bmatrix}$ of the Riesz type by the equation

$$\mathbb{D}_C^- \begin{bmatrix} \alpha \\ j \end{bmatrix} = \begin{cases} \dfrac{\partial}{\partial x_j} \mathbb{D}_C^+ \begin{bmatrix} \alpha - 1 \\ j \end{bmatrix} & \alpha > 1 \\ \dfrac{\partial}{\partial x_j} & \alpha = 1 \\ \dfrac{\partial}{\partial x_j} \mathbb{I}_C^+ \begin{bmatrix} 1 - \alpha \\ j \end{bmatrix} & 0 < \alpha < 1. \end{cases} \tag{31}$$

For $0 < \alpha < 1$ the operator $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$ is analogous to the conjugate Riesz derivative [31]. Therefore, the operator $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$ for all positive values α also can be called the conjugate derivative of the Riesz type.

The Fourier integral transform $\mathscr{F}$ of the fractional derivative (31) is given by

$$\mathscr{F}\left(\mathbb{D}_C^- \begin{bmatrix} \alpha \\ j \end{bmatrix} u(\mathbf{r})\right)(\mathbf{k}) = i k_j \, |k_j|^{\alpha-1} (\mathscr{F}u)(\mathbf{k}) = i \, \mathrm{sgn}(k_j) \, |k_j|^{\alpha} (\mathscr{F}u)(\mathbf{k}). \tag{32}$$

For the odd integer values of α, equations (24) and (31) give the relation

$$\mathbb{D}_C^- \begin{bmatrix} 2m+1 \\ i \end{bmatrix} u(\mathbf{r}) = (-1)^j \frac{\partial^{2m+1} u(\mathbf{r})}{\partial x_i^{2m+1}}, \quad (m \in \mathbb{N}). \tag{33}$$

Equation (33) means that the fractional derivatives $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ i \end{bmatrix}$ of the odd orders α are local operators represented by the usual derivatives of integer orders.

Note that the continuum derivative $\mathbb{D}_C^- \begin{bmatrix} 2 \\ j \end{bmatrix}$ cannot be considered as a local derivative of second order with respect to x_j. The derivatives $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ j \end{bmatrix}$ for even orders $\alpha = 2m$, where $m \in \mathbb{N}$, are non-local operators that cannot be considered as usual derivatives $\partial^{2m}/\partial x^{2m}$.

Using equations (24) and (33), we can state that the partial derivatives of integer orders are obtained from the fractional derivatives of the Riesz type $\mathbb{D}_C^{\pm} \begin{bmatrix} \alpha \\ j \end{bmatrix}$ for odd values $\alpha = 2m+1 > 0$ by $\mathbb{D}_C^- \begin{bmatrix} \alpha \\ j \end{bmatrix}$, and for even values $\alpha = 2m > 0$, where $m \in \mathbb{N}$, by $\mathbb{D}_C^+ \begin{bmatrix} \alpha \\ j \end{bmatrix}$ only. The continuum derivatives of the Riesz type $\mathbb{D}_C^- \begin{bmatrix} 2m \\ j \end{bmatrix}$ and $\mathbb{D}_C^+ \begin{bmatrix} 2m+1 \\ j \end{bmatrix}$ are nonlocal differential operators of integer orders.

5 From Lattice Models to Continuum Models

Using the methods suggested in [30,29], we can define the operation that transforms a lattice field $u(\mathbf{n})$ into a field $u(\mathbf{r})$ of continuum, For this transformations, we will consider the lattice scalar field $u(\mathbf{n})$ as Fourier series coefficients of some function $\hat{u}(\mathbf{k})$ for $k_j \in [-k_{j0}/2, k_{j0}/2]$, where $j = 1,2,3$. As a next step we use the continuous limit $\mathbf{k}_0 \to \infty$ to obtain $\tilde{u}(\mathbf{k})$. Finally we apply the inverse Fourier integral transformation to obtain the continuum scalar field $u(\mathbf{r})$. Let us describe these steps with details:

Step 1: The discrete Fourier series transform $u(\mathbf{n}) \to \mathscr{F}_{\Delta}\{u(\mathbf{n})\} = \hat{u}(\mathbf{k})$ of the lattice scalar field $u(\mathbf{n})$ is defined by

$$\hat{u}(\mathbf{k}) = \mathscr{F}_{\Delta}\{u(\mathbf{n})\} = \sum_{n_1,n_2,n_3=-\infty}^{+\infty} u(\mathbf{n}) \, e^{-i(\mathbf{k},\mathbf{r}(\mathbf{n}))}, \tag{34}$$

where $\mathbf{r}(\mathbf{n}) = \sum_{j=1}^{3} n_j \, \mathbf{a}_j$, and $a_j = 2\pi/k_{j0}$ is distance between lattice particle in the direction $\mathbf{a}_j$.

Step 2: The passage to the limit $\hat{u}(\mathbf{k}) \to \mathrm{Lim}\{\hat{u}(\mathbf{k})\} = \tilde{u}(\mathbf{k})$, where we use $a_j \to 0$ (or $k_{j0} \to \infty$), allows us to derive the function $\tilde{u}(\mathbf{k})$ from $\hat{u}(\mathbf{k})$. By definition $\tilde{u}(\mathbf{k})$ is the Fourier integral transform of the continuum field $u(\mathbf{r})$, and the function $\hat{u}(k)$ is the Fourier series transform of the lattice field $u(\mathbf{n})$, where

$$u(\mathbf{n}) = \frac{(2\pi)^3}{k_{10}k_{20}k_{30}} u(\mathbf{r}(\mathbf{n})),$$

and $\mathbf{r}(\mathbf{n}) = \sum_{j=1}^{3} n_j a_j = \sum_{j=1}^{3} 2\pi n_j/k_{j0} \to \mathbf{r}$.

Step 3: The inverse Fourier integral transform $\tilde{u}(\mathbf{k}) \to \mathscr{F}^{-1}\{\tilde{u}(\mathbf{k})\} = u(\mathbf{r})$ is defined by

$$u(\mathbf{r}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} dk_1 dh_2 dk_3 \, e^{i\sum_{j=1}^{3} k_j x_j} \tilde{u}(\mathbf{k}) = \mathscr{F}^{-1}\{\tilde{u}(\mathbf{k})\}. \tag{35}$$

The combination $\mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_\Delta$ of the operations $\mathscr{F}^{-1}$, Lim, and $\mathscr{F}_\Delta$ define the lattice-continuum transform operation

$$\mathscr{T}_{L\to C} = \mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_\Delta \tag{36}$$

that maps lattice models into the continuum models [29,30].

The lattice-continuum transform operation $\mathscr{T}_{L\to C}$ as the combination of three operations $\mathscr{F}^{-1} \circ \mathrm{Limit} \circ \mathscr{F}_\Delta$ can be applied not only for lattice fields but also for lattice operators. The operation $\mathscr{T}_{L\to C}$ allows us to map of lattice derivatives $\mathbb{D}_L^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$ into continuum derivatives $\mathbb{D}_C^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$.

The functions $\hat{K}_\alpha^{\pm}(k_i)$, are defined by the discrete Fourier series transform $\mathscr{F}_\Delta$ of the kernels of lattice operators, and the functions $\tilde{K}_\alpha^{\pm}(k_i)$ are defined by the Fourier integral transforms $\mathscr{F}$ of the correspondent continuum derivatives. The equation that defines $\hat{K}_\alpha^{\pm}(k_i)$ has the form

$$\mathscr{F}_\Delta\left(\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ i\end{bmatrix} u(\mathbf{m})\right) = \frac{1}{a_i^\alpha} \hat{K}_\alpha^{\pm}(k_i a_i)\, \hat{u}(\mathbf{k}), \tag{37}$$

where $\hat{u}(\mathbf{k}) = \mathscr{F}_\Delta\{u(\mathbf{m})\}$, and $\mathscr{F}_\Delta$ is an operator notation for the discrete Fourier series transform. The equation that defines $\tilde{K}_\alpha^{\pm}(k_i)$ is

$$\mathscr{F}\left(\mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ i\end{bmatrix} u(\mathbf{r})\right) = \tilde{K}_\alpha^{\pm}(k_i)\, \tilde{u}(\mathbf{k}), \tag{38}$$

where $\tilde{u}(\mathbf{k}) = \mathscr{F}\{u(\mathbf{r})\}$, and $\mathscr{F}_\Delta$ is an operator notation for the Fourier transform. In general, the order of the partial derivative $\mathbb{D}_C^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$ is defined by the order of lattice operator $\mathbb{D}_L^{\pm}\left[\begin{smallmatrix}\alpha\\ i\end{smallmatrix}\right]$. This order can be integer and non-integer positive real number.

We can formulate and prove statement about relation between the lattice and continuum fractional derivatives of non-integer orders.

Proposition *The lattice-continuum transform operation $\mathscr{T}_{L\to C}$ maps the lattice fractional derivatives*

$$\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a_i^\alpha} \sum_{m_j=-\infty}^{+\infty} K_\alpha^{\pm}(n_j - m_j)\, u(\mathbf{m}), \tag{39}$$

where $K_\alpha^{\pm}(n_j - m_j)$ are defined by (3),(4), into the continuum fractional derivatives of order α with respect to coordinate x_i by

$$\mathscr{T}_{L\to C}\left(\mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\right) = \mathbb{D}_C^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}), \tag{40}$$

where $u(\mathbf{r}) = \mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_\Delta\,(u(\mathbf{m}))$.

Proof. The multiplication equation (39) by $\exp(-ik_j n_j a_j)$, and the sum over n_j from $-\infty$ to $+\infty$ give

$$\sum_{n_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} \mathbb{D}_L^{\pm}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m}) = \frac{1}{a_j} \sum_{n_j=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_\alpha^{\pm}(n_j - m_j)\, u(\mathbf{m}). \tag{41}$$

Using (34), the right-hand side of (41) gives

$$\sum_{n_j=-\infty}^{+\infty} \sum_{m_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_\alpha^{\pm}(n_j - m_j)\, u(\mathbf{m}) =$$

$$= \sum_{n_j=-\infty}^{+\infty} e^{-ik_j n_j a_j} K_\alpha^{\pm}(n_j - m_j) \sum_{m_j=-\infty}^{+\infty} u(\mathbf{m}) =$$

$$= \sum_{n'_j=-\infty}^{+\infty} e^{-ik_j n'_j a_j} K^{\pm}_{\alpha}(n'_j) \sum_{m_j=-\infty}^{+\infty} u(\mathbf{m}) e^{-ik_j m_j a_j} = \hat{K}^{\pm}_{\alpha}(k_j a_j)\, \hat{u}(\mathbf{k}), \tag{42}$$

where $n'_j = n_j - m_j$.

As a result, equation (41) has the form

$$\mathscr{F}_{\Delta}\Big(\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\Big) = \frac{1}{a_j^{\alpha}} \hat{K}^{\pm}_{\alpha}(k_j a_j)\, \hat{u}(\mathbf{k}), \tag{43}$$

where $\mathscr{F}_{\Delta}$ is an operator notation for the Fourier series transform.

Then we use

$$\hat{K}^{+}_{\alpha}(a_j k_j) = |a_j k_j|^{\alpha}, \tag{44}$$

$$\hat{K}^{-}_{\alpha}(a_j k_j) = i\, \mathrm{sgn}(k_j)\, |a_j k_j|^{\alpha}, \tag{45}$$

and, the limit $a_j \to 0$ gives

$$\tilde{K}^{+}_{\alpha}(k_j) = \lim_{a_j \to 0} \frac{1}{a_j^{\alpha}} \hat{K}^{+}_{\alpha}(k_j a_j) = |k_j|^{\alpha}, \tag{46}$$

$$\tilde{K}^{-}_{\alpha}(k_j) = \lim_{a_j \to 0} \frac{1}{a_j^{\alpha}} \hat{K}^{-}_{\alpha}(k_j a_j) = i k_j\, |k_j|^{\alpha-1}. \tag{47}$$

As a result, equation (43) in the limit $a_j \to 0$ gives

$$\mathrm{Lim} \circ \mathscr{F}_{\Delta}\Big(\mathbb{D}^{\pm}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\Big) = \tilde{K}^{\pm}_{\alpha}(k_j)\, \tilde{u}(\mathbf{k}), \tag{48}$$

where

$$\tilde{K}^{+}_{\alpha}(k_j) = |k_j|^{\alpha}, \quad \tilde{K}^{-}_{\alpha}(k_j) = i k_j\, |k_j|^{\alpha-1}, \quad \tilde{u}(\mathbf{k}) = \mathrm{Lim}\, \hat{u}(\mathbf{k}).$$

The inverse Fourier transform of (48) is

$$\mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_{\Delta}\Big(\mathbb{D}^{+}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\Big) = \mathbb{D}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}), \quad (\alpha > 0), \tag{49}$$

$$\mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_{\Delta}\Big(\mathbb{D}^{-}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\Big) = \frac{\partial}{\partial x_j}\Big(\mathbb{D}^{+}_{C}\begin{bmatrix}\alpha-1\\ j\end{bmatrix} u(\mathbf{r})\Big), \quad (\alpha > 1), \tag{50}$$

$$\mathscr{F}^{-1} \circ \mathrm{Lim} \circ \mathscr{F}_{\Delta}\Big(\mathbb{D}^{-}_{L}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{m})\Big) = \frac{\partial}{\partial x_j} \mathbb{I}^{+}_{C}\begin{bmatrix}1-\alpha\\ j\end{bmatrix} u(\mathbf{r}), \quad (0 < \alpha < 1). \tag{51}$$

Here the fractional derivative and fractional integral are

$$\mathbb{D}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}) = \mathscr{F}^{-1}\{|k_j|^{\alpha} \tilde{u}(\mathbf{k})\}, \quad \mathbb{I}^{+}_{C}\begin{bmatrix}\alpha\\ j\end{bmatrix} u(\mathbf{r}) = \mathscr{F}^{-1}\{|k_j|^{-\alpha} \tilde{u}(\mathbf{k})\}, \tag{52}$$

where we use the connection between the continuum derivative (and the continuum integral) of the Riesz type of the order α and the corresponding Fourier integral transforms.

As a result, we prove that lattice fractional derivatives are transformed (40) into continuum fractional derivatives of the Riesz type.

This ends the proof.

Using the Proposition (40), and the independence of n_i and n_j for $i \neq j$, it is easy to prove that the continuum limits for the lattice mixed partial derivatives (15) and (16) have the form

$$\mathscr{T}_{L \to C}\left(\mathbb{D}^{\pm,\pm}_{L}\begin{bmatrix}\alpha_1\, \alpha_2\\ i\ j\end{bmatrix} u(\mathbf{m})\right) = \mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha_1\\ i\end{bmatrix} \mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha_2\\ j\end{bmatrix} u(\mathbf{r}), \quad (i \neq j), \tag{53}$$

$$\mathscr{T}_{L \to C}\left(\mathbb{D}^{\pm,\mp}_{L}\begin{bmatrix}\alpha_1\, \alpha_2\\ i\ j\end{bmatrix} u(\mathbf{m})\right) = \mathbb{D}^{\pm}_{C}\begin{bmatrix}\alpha_1\\ i\end{bmatrix} \mathbb{D}^{\mp}_{C}\begin{bmatrix}\alpha_2\\ j\end{bmatrix} u(\mathbf{r}), \quad (i \neq j). \tag{54}$$

We have similar relations for other mixed lattice fractional derivatives.

6 Three-Dimensional Continuum Models of Fractional Gradient Elasticity

The proved Proposition allows us to get continuum models from the suggested lattice models by using the lattice-continuum transform operation (36).

In the continuum limit ($a_j \to 0$), the lattice equations (17) give the continuum equations for the fractional gradient elasticity in the form

$$\rho \frac{\partial^2 u_i(\mathbf{r},t)}{\partial t^2} = \sum_{j,l=1}^{3} A^C_{ijkl} \, {}^R\mathbb{D}^{+}_C \begin{bmatrix} 1\,1 \\ j\,l \end{bmatrix} u_k(\mathbf{r},t) + \sum_{j,m,l=1}^{3} B^C_{ijkl} \, {}^R\mathbb{D}^{-,+,-}_C \begin{bmatrix} 1\,\alpha\,1 \\ j\,m\,l \end{bmatrix} u_k(\mathbf{r},t) + f_i(\mathbf{r},t), \tag{55}$$

where $u_i(\mathbf{r},t)$ are the components of the displacement vector field for continuum, and A^C_{ijkl} and B^C_{ijkl} are the coupling constants for the non-local continuum. We note that the continuum operators, which are used in equation (55), can be represented by

$${}^R\mathbb{D}^{+}_C \begin{bmatrix} 1\,1 \\ j\,l \end{bmatrix} = \frac{\partial^2}{\partial x_j \partial x_l}, \tag{56}$$

$${}^R\mathbb{D}^{-,+,-}_C \begin{bmatrix} 1\,\alpha\,1 \\ j\,m\,l \end{bmatrix} = {}^R\mathbb{D}^{-}_C \begin{bmatrix} 1 \\ j \end{bmatrix} {}^R\mathbb{D}^{+}_C \begin{bmatrix} \alpha \\ m \end{bmatrix} {}^R\mathbb{D}^{-}_C \begin{bmatrix} 1 \\ l \end{bmatrix} = \frac{\partial}{\partial x_j} \, {}^R\mathbb{D}^{+}_C \begin{bmatrix} \alpha \\ m \end{bmatrix} \frac{\partial}{\partial x_l}. \tag{57}$$

The coupling constants of continuum are defined by the lattice coupling constants A^L_{ijkl} and B^L_{ijkl} by the relations

$$A^C_{ijkl} = \frac{a_l\, a_j\, \rho}{M} A^L_{ijkl}, \quad B^C_{ijkl} = \frac{a_l\, a_j \left(\sum_{m=1}^{3} a_m^{2\alpha}\right) \rho}{M} B^L_{ijkl}. \tag{58}$$

In the case $a_1 = a_2 = a_3 = a$, we get the fourth-order elastic stiffness tensor C_{ijkl} in the form

$$C_{ijkl} = A^C_{ijkl} = \frac{a^2 \rho}{M} A^L_{ijkl}. \tag{59}$$

If $B^L_{ijkl} = g_B A^L_{ijkl}$, then the scale parameter l_s^2 is $l_s^2 = 3a^{2\alpha} g_B$, and we have $B^C_{ijkl} = l^2_\alpha C_{ijkl}$. For isotropic materials, C_{ijkl} are expressed in terms of the Lame constants λ and μ by

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + \mu\, (\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}). \tag{60}$$

Note that x_k, a_k, l^2_α are dimensionless values.

If $\alpha = 2$, then equation (55) gives the well-known continuum equation of gradient elasticity

$$\rho\, \ddot{u}_i(\mathbf{r},t) = \sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2 u_k(\mathbf{r},t)}{\partial x_j\, \partial x_l} \pm l^2_\alpha \sum_{j,k,l,m=1}^{3} C_{ijkl} \frac{\partial^4 u_k(\mathbf{r},t)}{\partial x_j\, \partial x_m^2\, \partial x_l} + f_i(\mathbf{r},t). \tag{61}$$

Let us give the stress-strain constitutive relation for fractional gradient elasticity (55). Equation (55) can be represented in the form

$$\rho\, \ddot{u}_i(\mathbf{r},t) = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(\mathbf{r},t)}{\partial x_j} + f_i(\mathbf{r},t), \tag{62}$$

where $\sigma_{ij}(\mathbf{r},t)$ is the stress tensor that is connected with the strain tensor $\varepsilon_{kl}(\mathbf{r},t)$ by the constitutive relation

$$\sigma_{ij}(\mathbf{r},t) = \sum_{k,l=1}^{3} A^C_{ijkl}\, \varepsilon_{kl}(\mathbf{r},t) + \sum_{k,l,m=1}^{3} B^C_{ijkl} \, {}^R\mathbb{D}^{+}_C \begin{bmatrix} \alpha \\ m \end{bmatrix} \varepsilon_{kl}(\mathbf{r},t). \tag{63}$$

where $\varepsilon_{kl}(\mathbf{r},t) = 1/2\,(\partial u_k(\mathbf{r},t)/\partial x_l + \partial u_l(\mathbf{r},t)/\partial x_k)$. If we use (59) and assume that $B^C_{ijkl} = \pm l^2_\alpha A^C_{ijkl}$, then relation (63) can be rewritten as

$$\sigma_{ij}(\mathbf{r},t) = \sum_{k,l=1}^{3} C_{ijkl} \left(1 \pm l^2_\alpha \, {}^R\Delta^{\alpha,+}_C\right) \varepsilon_{kl}, \tag{64}$$

where ${}^{R}\Delta_{C}^{\alpha,+}$ is the fractional Laplacian of the Riesz type of the form

$$ {}^{R}\Delta_{C}^{\alpha,+}=\sum_{m=1}^{3} {}^{R}\mathbb{D}_{C}^{+}\begin{bmatrix}\alpha\\ m\end{bmatrix}. \quad (65)$$

Equation (64) gives the constitutive relation for fractional gradient elasticity. For $\alpha=2$, relation (64) has the form

$$\sigma_{ij}(\mathbf{r},t)=\sum_{k,l=1}^{3} C_{ijkl}\left(1\mp l_2^2\,\Delta\right)\varepsilon_{kl}(\mathbf{r},t). \quad (66)$$

This is the well-known stress-strain constitutive relation for gradient elasticity. If we consider the case with $u_x(\mathbf{r},t)=u(x,t)$, $f_x(\mathbf{r},t)=f(x,t)$, where the other components, u_y, u_z, f_y, f_z, are equal to zero, then we get the one-dimensional fractional elasticity models suggested in [22,23]. The lattice models (20) and (17) are three-dimensional generalizations of the one-dimensional lattice models proposed in [22,23]. In addition, the equation (17) of lattice with long-range interactions allows us to derive the stress-strain constitutive relations for fractional nonlocal elasticity by using usual law (62).

The continuum limit for lattice equations (20) gives the continuum equations of the fractional gradient elasticity in the form

$$\rho\,\ddot{u}_i(\mathbf{r},t)-A_0^C\sum_{j=1}^{3}\frac{\partial^2\ddot{u}_i(\mathbf{r},t)}{\partial x_j^2}=A_1^C\sum_{j=1}^{3}\frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_j\partial x_i}+A_2^C\sum_{j=1}^{3}\frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2}+$$

$$+B_1^C\sum_{j,m=1}^{3}\frac{\partial}{\partial x_j}{}^{R}\mathbb{D}_{C}^{+}\begin{bmatrix}\alpha\\ m\end{bmatrix}\frac{\partial u_j(\mathbf{r},t)}{\partial x_i}+B_2^C\sum_{j,m=1}^{3}\frac{\partial}{\partial x_j}{}^{R}\mathbb{D}_{C}^{+}\begin{bmatrix}\alpha\\ m\end{bmatrix}\frac{\partial u_i(\mathbf{r},t)}{\partial x_j}+f_i(\mathbf{r},t), \quad (67)$$

where the constants for continuum are defined by

$$A_i^C=\frac{a^2\rho}{M}A_i^L \quad (i=0,1,2), \quad B_j^C=\frac{a^{2+\alpha}\rho}{M}B_j^L \quad (j=1,2). \quad (68)$$

The Lame constants λ and μ are defined by the lattice coupling constants

$$\mu=\frac{a^2\rho}{M}A_2^L, \quad \lambda=\frac{a^2\rho}{M}(A_1^L-A_2^L). \quad (69)$$

The three additional parameters l_1, $l_2(\alpha)$, $l_3(\alpha)$ of the Mindlin model are

$$l_1^2=\frac{A_0^L a^2}{M}, \quad l_2^2(\alpha)=\frac{a^\alpha\,|B_1^L|}{|A_1^L|}, \quad l_3^2(\alpha)=\frac{a^\alpha\,|B_2^L|}{|A_2^L|}. \quad (70)$$

Note that x_k, a, l_1^2, $l_2^2(\alpha)$, $l_3^2(\alpha)$ are dimensionless values. Equations (67) can be considered as the fractional Mindlin equations.

The three-dimensional lattice model (19) in the continuum limit gives the fractional generalization of Mindlin model of the first gradient elasticity, if the Lame constants λ and μ are defined by the lattice coupling constants

$$\frac{\mu_\alpha}{\rho}=\frac{a^{2\alpha}A_3^L(\alpha)}{M}, \quad \frac{\lambda_\alpha}{\rho}=\frac{a^{2\alpha}}{M}\left(A_1^L(\alpha)-A_3^L(\alpha)\right), \quad (71)$$

and the three additional parameters l_1, l_2, l_3 of the Mindlin model are

$$l_1^2(\alpha)=\frac{a^{2\alpha}A_0^L(\alpha)}{M}, \quad l_2^2(\alpha)=\frac{a^{2\alpha}B_1^L(\alpha)}{A_1^L(\alpha)}, \quad l_3^2(\alpha)=\frac{B_5^L(\alpha)}{A_3^L(\alpha)}, \quad (72)$$

where the coupling constants are not independent

$$A_2^L(\alpha)=A_1^L(\alpha)+A_3^L(\alpha), \quad B_1^L(\alpha)=B_2^L(\alpha)=B_3^L(\alpha)=B_4^L(\alpha), \quad B_5^L(\alpha)=B_6^L(\alpha). \quad (73)$$

In the continuum limit ($a\to 0$), we obtain the equations for fractional non-local continuum model that is a generalization of the Mindlin first gradient elasticity. These equations have the form

$$\rho\,\frac{\partial^2 u_i}{\partial t^2}=\rho\,l_1^2(\alpha)\sum_{j=1}^{3}{}^{R}\mathbb{D}_{C}^{+}\begin{bmatrix}2\alpha\\ j\end{bmatrix}\ddot{u}_i(\mathbf{r},t)+$$

$$+(\lambda_\alpha+\mu_\alpha)\left(\sum_{j:\, j\neq i}^{3} {}^{R}\mathbb{D}_C^{-,-}\begin{bmatrix}\alpha\ \alpha\\ j\ i\end{bmatrix} u_j(\mathbf{r},t)+{}^{R}\mathbb{D}_C^{+}\begin{bmatrix}2\alpha\\ i\end{bmatrix} u_i(\mathbf{r},t)\right)+\mu_\alpha \sum_{j=1}^{3} {}^{R}\mathbb{D}_C^{+}\begin{bmatrix}2\alpha\\ i\end{bmatrix} u_i(\mathbf{r},t)-$$

$$-(\lambda_\alpha+\mu_\alpha)\, l_2^2(\alpha) \sum_{j:\, j\neq i}^{3}\left({}^{R}\mathbb{D}_C^{-,-}\begin{bmatrix}3\alpha\ \alpha\\ j\ i\end{bmatrix} u_j(\mathbf{r},t)+{}^{R}\mathbb{D}_C^{-,-}\begin{bmatrix}\alpha\ 3\alpha\\ j\ i\end{bmatrix} u_j(\mathbf{r},t)\right)-$$

$$-(\lambda_\alpha+\mu_\alpha)\, l_2^2(\alpha) \sum_{j:\, j\neq i}^{3} {}^{R}\mathbb{D}_C^{+,+}\begin{bmatrix}2\alpha\ 2\alpha\\ j\ i\end{bmatrix} u_i(\mathbf{r},t)-$$

$$-(\lambda_\alpha+\mu_\alpha)\, l_2^2(\alpha)\left(\sum_{\substack{k,j:\\ j\neq i;\, j\neq k;\, k\neq i}}^{3} {}^{R}\mathbb{D}_C^{+,-,-}\begin{bmatrix}2\alpha\ \alpha\ \alpha\\ k\ j\ i\end{bmatrix} u_i(\mathbf{r},t)+{}^{R}\mathbb{D}_C^{+}\begin{bmatrix}4\alpha\\ i\end{bmatrix} u_i(\mathbf{r},t)\right)-$$

$$-\mu_\alpha\, l_3^2(\alpha)\left(\sum_{\substack{k,l\\ k\neq l}}^{3} {}^{R}\mathbb{D}_C^{+,+}\begin{bmatrix}2\alpha\ 2\alpha\\ k\ j\end{bmatrix} u_i(\mathbf{r},t)+\sum_{j=1}^{3} {}^{R}\mathbb{D}_C^{+}\begin{bmatrix}4\alpha\\ i\end{bmatrix} u_i(\mathbf{r},t)\right)+f_i(\mathbf{r},t), \tag{74}$$

where $u_i(\mathbf{r},t)$ are components of the displacement field for the continuum, and $f_i(\mathbf{r},t)$ are the components of the body force.

For $\alpha=1$, equations (74) give the differential equations for gradient elasticity

$$\rho\,\frac{\partial^2 u_i(\mathbf{r},t)}{\partial t^2}=\rho\, l_1^2 \sum_{j=1}^{3}\frac{\partial^2 \ddot{u}_i(\mathbf{r},t)}{\partial x_j^2}+$$

$$+(\lambda+\mu)\left(\sum_{j:\, j\neq i}^{3}\frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_j\,\partial x_i}+\frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_i^2}\right)+\mu\sum_{j=1}^{3}\frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2}-$$

$$-(\lambda+\mu)\, l_2^2 \sum_{j:\, j\neq i}^{3}\left(\frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_j\,\partial x_i^3}+\frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_j^3\,\partial x_i}+\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_j^2\,\partial x_i^2}\right)-$$

$$-(\lambda+\mu)\, l_2^2\left(\sum_{\substack{k,j:\\ j\neq i;\, j\neq k;\, k\neq i}}^{3}\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\,\partial x_j\,\partial x_i}+\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_i^4}\right)-$$

$$-\mu\, l_3^2\left(\sum_{\substack{k,l\\ k\neq l}}^{3}\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\,\partial x_j^2}+\sum_{j=1}^{3}\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_j^4}\right)+f_i(\mathbf{r},t), \tag{75}$$

where $\lambda=\lambda_1$, $\mu=\mu_1$, and $l_j=l_j(1)$, where $j=1,2,3$. In equations (75) the derivatives of integer orders with respect to the same spatial coordinates are clearly marked. Equations (75) can be written as the Mindlin equations for displacements components in the form

$$\rho\,\ddot{u}_i(\mathbf{r},t)-\rho\, l_1^2\sum_{j=1}^{3}\frac{\partial^2 \ddot{u}_i(\mathbf{r},t)}{\partial x_j^2}=(\lambda+\mu)\sum_{j=1}^{3}\frac{\partial^2 u_j(\mathbf{r},t)}{\partial x_i\partial x_j}+\mu\sum_{j=1}^{3}\frac{\partial^2 u_i(\mathbf{r},t)}{\partial x_j^2}-$$

$$-(\lambda+\mu)\, l_2^2\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^4 u_j(\mathbf{r},t)}{\partial x_k^2\partial x_i\partial x_j}-\mu\, l_3^2\sum_{k=1}^{3}\sum_{j=1}^{3}\frac{\partial^4 u_i(\mathbf{r},t)}{\partial x_k^2\,\partial x_j^2}+f_i(\mathbf{r},t), \tag{76}$$

where $f_i(\mathbf{r},t)$ are the components of the body force, $u_i(\mathbf{r},t)$ are components of the displacement field for the continuum, and

$$l_2^2=\frac{4\lambda_1+4\lambda_2+3\lambda_3+2\lambda_4+3\lambda_5}{2\,(\lambda+\mu)},\qquad l_3^2=\frac{\lambda_3+2\lambda_4+\lambda_5}{2\,\mu}. \tag{77}$$

As a result, continuum equations (76) have two Lame constants and three additional parameters l_1, l_2, l_3. Note that equations (76) for Mindlin gradient elasticity model can be obtained [5] by using the expressions of the kinetic density

$$T=\frac{1}{2}\rho\,\partial_t u_i\,\partial_t u_i+\frac{1}{2}\rho\, l_1^2\,\dot{u}_{i,j}\,\dot{u}_{i,j}, \tag{78}$$

the density of the deformation energy in the form

$$U=\frac{1}{2}\lambda\,\varepsilon_{ii}\varepsilon_{jj}+\mu\,\varepsilon_{ij}\varepsilon_{ij}+\lambda_1\,\varepsilon_{ik,i}\varepsilon_{jj,k}+\lambda_2\,\varepsilon_{kk,i}\varepsilon_{jj,i}+\lambda_3\,\varepsilon_{ik,i}\varepsilon_{jk,j}+\lambda_4\,\varepsilon_{jk,i}\varepsilon_{jk,i}+\lambda_5\,\varepsilon_{jk,i}\varepsilon_{ij,k}, \tag{79}$$

where λ and μ are the usual Lame constants and the various λ_i $(i=1,...,5)$ are 5 additional constitutive coefficients, ρ is the mass density, u_k is the displacement, ε_{ij} is the strain, and $\varepsilon_{ij}=(1/2)(u_{i,j}+u_{j,i})$.

If the lattice equations (19) would be written only through even lattice fractional-order operators ${}^R\mathbb{D}^+_L\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$, then the correspondent continuum equations contain the continuum fractional derivatives ${}^R\mathbb{D}^+_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$, of orders 1 and 3 that are non-local operators. In this case, we cannot get the usual Mindlin model with derivatives of integer orders. Therefore, we suggest the equations of lattice model that contain two type of lattice fractional derivatives ${}^R\mathbb{D}^\pm_L\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$, in the suggested form (19). It is obvious that we would like to have such a fractional generalization of partial differential equations that yield the original equations in the limit case, when the orders of fractional derivatives become equal to initial integer values. This desirable correspondence and the property of the continuum fractional derivatives ${}^R\mathbb{D}^\pm_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ to be the local operators of integer orders α only if we use ${}^R\mathbb{D}^-_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for the odd values of α, and if we use ${}^R\mathbb{D}^+_C\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ for the even values of α, allow us to consider equations in the form (19) with the fractional-order lattice operators ${}^R\mathbb{D}^\pm_L\left[\begin{smallmatrix}\alpha\\ j\end{smallmatrix}\right]$ as basic equations of lattices with long-range interactions.

7 Conclusions

Elasticity of weak nonlocal continuum is discussed in this paper. Three-dimensional lattice models with long-range interactions are suggested for fractional gradient elasticity. These lattice models give new microstructural basis of unified description of gradient nonlocal continuum models. The suggested type of long-range interactions can be considered for integer and non-integer (fractional) orders of non-locality. It allows us to get lattice models for the local and nonlocal elasticity theories of continuum mechanics.

For clarity, we select the main differences between this paper and the numerical approach for fractional differential equations with the Riesz derivatives [38,39], the finite difference methods [40,41], the tool of the discrete fractional calculus [42]-[48].

1) The discrete models, which are proposed in this paper, are microstructural models of physical lattices. These models and the corresponding equations are not discretization of continuous models and the fractional differential equations of nonlocal continuum.

2) The suggested lattice models correspond to the continuum models exactly. They are not asymptotically equivalent, i.e. they are not an approximation. Equations of lattice models exactly correspond to fractional differential equations without any approximation. (For details about exact and asymptotic connections of lattice and continuum models see [32]).

3) The numerical methods for fractional partial differential equations with Riesz space fractional derivatives, which are considered in [38,39], replace the Riesz fractional derivatives by the finite differences with power-law weights (the finite-difference approximation). The same type of replacements is used in the finite difference methods [40,41], The discrete fractional calculus, which are used in [42]-[46], are also based on the finite differences with power-law weights (the finite fractional differences). Our approach is based on special type of infinite fractional differences that describe long-range interactions in physical lattices.

In general, the finite differences correspond to models with nearest-neighbor and next-nearest-neighbor interactions [27,28]. In this paper, we suggest physical lattice models with long-range interactions of power-law type. The long-range type of interactions and the corresponding discretizations are very important in fractional nonlocal models. Nonlocal continuum theory [50,51] is based on the assumption that the forces between particles are a long-range type, thus reflecting the long-range character of interatomic and intermolecular forces. We assume that fractional finite differences cannot completely reflect all characteristic properties of the fractional-order derivatives.

It is well-known that the fractional derivative of non-integer order can be represented in the form of an infinite series of derivatives of integer orders (for example, see Lemma 15.3 in [6]). The cutting of this series can be considered only as an approximation. Similarly, the long-range lattice interactions and the infinite fractional differences can be represented only as a infinite sum of special finite differences with power-law weights.

Derivatives and integrals of non-integer orders describe nonlocality of power-law type at macroscopic scale. The long-range interactions, which represented by infinite differences, describe nonlocality at micro and nanoscales. Therefore, the suggested lattice models with long-range interactions more correctly describe the continuum media with nonlocality of power-law type.

For discrete maps with power-law memory, which are equivalent to the fractional differential equations with the periodic sequence of delta-function-type pulses (kicks) [53,54,55], the situation is somewhat different. Equations of these maps contain special finite differences with power-law weights. At the same time the derivation of these maps from fractional differential equations of kicked motions is not used approximations. This fact allows us to study the time-fractional dynamics by computer simulations without approximations. The special situation is related to the fact that the fractional differential equations contain the terms of periodic delta-function-type kicks. The fractional partial differential equations of nonlocal continuum do not contain terms with delta-functions. Therefore, the discrete (lattice) models, which are connected with these equations without approximations, should contain the infinite differences (for example, the Grünvald-Letnikov type [24,32,49]) and the long-range lattice interactions.

There is an interesting question about a connection between the finite fractional differences and the Grünvald-Letnikov fractional differences, which are infinite differences, and the corresponding derivatives. It should be noted that lattice models with long-range interactions, which are based on fractional-order differences of Grünwald-Letnikov type, have been suggested in [24,49]. These differences, which is represented by infinite series, allow us to describe long-range interactions in chains and lattices [32]. In paper [48], an equivalence between the discrete maps with power-law memory and the Grünvald-Letnikov fractional difference equations has been proved. In the continuous limit, this connection leads to the equivalence of some fractional differential equations and the Volterra integral equations of the second kind.

We can mention some possible extensions of the proposed lattice models to formulate generalizations of fractional nonlocal elasticity theories. We suppose that the lattice fractional derivatives [32,33] can be used for nonlocal elasticity theory to generalize for different types of Bravais lattices such as monoclinic, triclinic, hexagonal and rhombohedral. We suppose that the lattice fractional calculus [32,33] can be used to get lattice models for dislocations in the gradient elasticity continuum and in the fractional generalization of nonlocal dislocations. The suggested lattice approach that based on the three-dimensional lattice with long-range interactions can play an important role in the description of nonlocal materials and continua [50,51,52] at micro and nano scales since the long-range intermolecular interactions are prevalent in determining the elastic properties at these scales.

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