Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/010403

Fractional Space Waves in Conductors

Jose F. Gómez-Aguilar[*]

Cátedras CONACYT, Centro Nacional de Investigación y Desarrollo Tecnológico, Tecnológico Nacional de México. Interior Internado Palmira S/N, Col. Palmira, C.P. 62490, Cuernavaca, Morelos, México

Received: 7 Apr. 2015, Revised: 26 Apr. 2015, Accepted: 28 Apr. 2015
Published online: 1 Oct. 2015

Abstract: This paper presents an alternative representation of the wave equation in fractional dimensional space, the order of the derivative is considered as $0 < \gamma \leq 2$, the fractional space derivative is described in the Caputo sense. We obtain the fractional phase velocity, the fractional dispersion relation, the fractional group velocity and the fractional skin depth. The response expressions are written in terms of the Mittag-Leffler function which describes physical systems with memory. The markovian nature of the system is recovered with $\gamma = 1$.

Keywords: Fractional calculus, Caputo derivative, wave equation, Maxwell's equations, Mittag-Leffler function.

1 Introduction

Fractional Calculus (FC) is a field of mathematical analysis where the concepts of integral and derivative operators of integer order to arbitrary order (real or complex) take place. Some fundamental definitions used in the literature are the Weyl, Hadamard, Caputo, Erdelyi-Kober, Riemann-Liouville and Grnwald-Letnikov fractional derivatives [1]-[3]. This mathematical representation involves non-local operators which can be applied to physical systems yielding new information about their behavior, for this reason the FC has became an interesting topic of research in science and engineering [4]-[14], the solutions of the electromagnetic wave equation have attracted much attention of the physicist since many years ago, for example, in the works [15]-[16] the Lagrangian and Hamiltonian formulation of dynamics and electromagnetic fields have been reported. Fractional curl operator and applications to the electromagnetic problems are discussed in [17]-[18]. Tarasov in [19]-[20] shows the fractional time electromagnetic waves in dielectric materials. Fractional dimensional space concept can be used in order to replace the complex anisotropic confining structures in a media with an effective space of non-integer dimension γ, where the non-integer dimension is the measure of anisotropy of the complex media [21]. In [22] a novel generalization of differential electromagnetic equations in fractional space is provided. These equations provide a basis for application of the concept of fractional in practical electromagnetic wave propagation and scattering problems in fractal media. In [23] a generalization of vector calculus for non-integer dimensional space by using a product measure method is presented. The integration over non-integer-dimensional spaces is considered and differential operators of first and second orders for fractional space and non-integer dimensional space are suggested. More applications of this concept are reported in [24]-[29]. Many times the authors replace the integer derivative by another of fractional order on a purely mathematical basis. However, from the physical point of view that is not totally correct, to be consistent with the dimensionality of the Fractional Differential Equations (FDE) in the work [30]; the authors have proposed a systematic way to construct FDE for the physical systems analyzing the dimensionality of the ordinary derivative operator and trying to bring it to a fractional derivative operator consistently.

Following [30], in this work these ideas are applied to the study of the fractional space wave equation in conductors, the fractional derivative in Caputo sense is used. The main goal of this work is to obtain the analytical solution of the wave equation in fractional dimensional space applying some basic properties of the FC, this representation preserves the physical units of the system for any value taken by the exponent of the fractional derivative.

This article is organized as follows. In Section 2 there is a description of the basic tools of fractional calculus. In Section 3 it is described the fractional space waves in conductors. Finally in Section 4 there are the conclusions.

[*] Corresponding author e-mail: jgomez@cenidet.edu.mx

2 Basic Tools

In the Caputo definition to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the Caputo Fractional Derivative (CFD), these additional conditions can be just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to ours [1]. The CFD For a function $f(t)$ is given by

$$ {}_0^C D_t^\gamma f(t) = \frac{1}{\Gamma(a-\gamma)} \int_0^t \frac{f^{(a)}(\eta)}{(t-\eta)^{\gamma-a+1}} d\eta, \qquad a-1 < \gamma \leq a \tag{1} $$

where $\frac{d^\gamma}{dt^\gamma} = {}_a^C D_t^\gamma$ is a CFD with respect to t, $\gamma \, \varepsilon \, R$ is the order of the fractional derivative, $a = 1, 2, \ldots \in A$ and $\Gamma(\cdot)$ represents the Euler's gamma function.

The Mittag-Leffler function plays an important role in the solution of fractional differential equations [31]

$$ E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m + \beta)}, \qquad (\alpha > 0), \qquad (\beta > 0), \tag{2} $$

Some common Mittag-Leffler functions $E_\alpha(\chi)$, are [32]

$$ E_1(\chi) = e^\chi, \tag{3} $$

$$ E_2(-\chi^2) = \cos(\chi), \tag{4} $$

$$ E_3(\chi) = \frac{1}{2}\left[e^{\chi^{1/3}} + 2e^{-(1/2)\chi^{1/3}} \cos\left(\frac{\sqrt{3}}{2} \chi^{1/3} \right) \right], \tag{5} $$

$$ E_4(\chi) = \frac{1}{2}\left[\cos(\chi^{1/4}) + \cosh(\chi^{1/4}) \right], \tag{6} $$

where $E_\alpha(\chi) = E_{\alpha,1}(\chi)$.

3 Fractional Space Waves in Conductors

In the ohmic conductors we have

$$ \mathbf{J} = \eta \mathbf{E}, \tag{7} $$

where η is the conductivity, $\mathbf{E}$ is the electric field and $\mathbf{J}$ is the current density.

In a conducting material, Maxwell's equations take the form

$$ \nabla \mathbf{E} = 0, \tag{8} $$

$$ \nabla \mathbf{B} = 0, \tag{9} $$

$$ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{10} $$

$$ \nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mu \mathbf{J} \tag{11} $$

substituting (7) into (11) the wave equation for the electric field in conductors becomes

$$ \frac{\partial^2 \mathbf{E}(x,t)}{\partial x^2} - \mu\eta \frac{\partial \mathbf{E}(x,t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}(x,t)}{\partial t^2} = 0. \tag{12} $$

To be consistent with dimensionality and following [30] we introduce an auxiliary parameter α in the following way

$$ \frac{\partial^2}{\partial x^2} \rightarrow \frac{1}{\alpha^{2(1-\gamma)}} \cdot \frac{\partial^{2\gamma}}{\partial x^{2\gamma}}, \qquad n-1 < \gamma \leq n, \tag{13} $$

where n is integer and α has dimensions of length (meters) and characterizes the fractional space structures [30], when $\gamma = 1$ the expression (13) becomes a classical operator.

Considering (13) the fractional representation of (12) is

$$\frac{\partial^{2\gamma}\mathbf{E}(x,t)}{\partial x^{2\gamma}} - \mu\eta\alpha^{2(1-\gamma)}\frac{\partial \mathbf{E}(x,t)}{\partial t} - \mu\varepsilon\alpha^{2(1-\gamma)}\frac{\partial^2\mathbf{E}(x,t)}{\partial t^2} = 0. \tag{14}$$

the order of the derivative that is being considered is $0 < \gamma \leq 2$.

In this case we may consider electric fields of the form

$$\mathbf{E}(x,t) = \mathbf{E_0}\cdot e^{i\omega t}u(x), \tag{15}$$

where ω is the angular frequency, the wave propagation is considered in the x direction. Substituting (15) in (14) we obtain

$$\frac{d^{2\gamma}u(x)}{dx^{2\gamma}} + (\mu\varepsilon\omega^2 - i\mu\eta\omega)\alpha^{2(1-\gamma)}u(x) = 0, \tag{16}$$

where

$$v_x^2 = (\mu\varepsilon\omega^2 - i\mu\eta\omega), \tag{17}$$

is the dispersion relation and

$$\tilde{v_x}^2 = (\mu\varepsilon\omega^2 - i\mu\eta\omega)\alpha^{2(1-\gamma)} = v_x^2\alpha^{2(1-\gamma)}, \tag{18}$$

is the fractional dispersion relation. From the fractional dispersion relation (18), we can expect the fractional wave number $\tilde{v_x}$ in the x direction to obtain the real and imaginary parts, δ_x and φ_x. Let us write

$$\tilde{v_x} = \delta_x - i\varphi_x, \tag{19}$$

substituting (19) into (18) we have

$$(\delta_x - i\varphi_x)^2 = {\delta_x}^2 - 2i\delta_x\varphi_x - {\varphi_x}^2, \tag{20}$$

where

$${\delta_x}^2 - 2i\delta_x\varphi_x - {\varphi_x}^2 = (\mu\varepsilon\omega^2 - i\mu\eta\omega)\alpha^{2(1-\gamma)}, \tag{21}$$

solving for φ_x we obtain

$$\varphi_x = \frac{\omega\mu\eta}{2\delta_x}\alpha^{2(1-\gamma)}, \tag{22}$$

and for δ_x

$$\delta_x = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}\alpha^{1-\gamma}, \tag{23}$$

substituting (23) into (22) we have

$$\varphi_x = \frac{\mu\eta}{2}\cdot\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}\alpha^{1-\gamma}. \tag{24}$$

Now the fractional wave number is, $\tilde{v_x} = \delta_x - i\varphi_x$, where δ_x and φ_x is given by (23) and (24) respectively

$$\tilde{v_x} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}\alpha^{1-\gamma} - i\frac{\mu\eta}{2}\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}\alpha^{1-\gamma}, \tag{25}$$

the equation (25) describes the real and imaginary part of the wave number in terms of the frequency ω, and material properties μ, ε and η, in presence of fractional space components α.

Consider (18) the equation (16) is

$$\frac{d^{2\gamma}u(x)}{dx^{2\gamma}} + \tilde{v_x}^2u(x) = 0, \tag{26}$$

the solution of the above equation is given by

$$u(x) = E_{2\gamma}(-\tilde{v_x}^2x^{2\gamma}), \tag{27}$$

where $E_{2\gamma,1}$ is the Mittag-Leffler function defined in (2).

Therefore the general solution of the equation (16) is given by

$$\mathbf{E}(x,t) = \mathbf{E_0} \cdot e^{i\omega t} \cdot E_{2\gamma}(-\tilde{v_x}^2 x^{2\gamma}). \tag{28}$$

The varying fractional electric field described in (28) must have a fractional magnetic field associated with it. This field has the same fractional dispersion relation (18) and frequency as the electric field, this is the only way that the Maxwell's equation is satisfied as function of all positions and times

$$\mathbf{B}(x,t) = \mathbf{B_0} \cdot e^{i\omega t} \cdot E_{2\gamma}(-\tilde{v_x}^2 x^{2\gamma}). \tag{29}$$

Using the Maxwell's equation (10) which gives

$$\mathbf{k} \times \mathbf{E_0} = \omega \mathbf{B_0}, \tag{30}$$

or

$$\mathbf{B_0} = \frac{\mathbf{k}}{\omega} \times \mathbf{E_0}, \tag{31}$$

the equation (31) is the relation between the fractional magnetic and electric fields in a wave in a conductor. In a non-conducting material, the fractional electric and magnetic fields are perpendicular to the direction of the motion and they are perpendicular to each other, these fields were in phase, in a conductor, the complex phase of **k** gives a phase difference between the fractional electric and magnetic fields.

In a bad conductor $(\eta \ll \varepsilon\omega)$ and from (25) we have

$$\delta_x \approx \omega\sqrt{\mu\varepsilon}\alpha^{1-\gamma}, \tag{32}$$

for φ_x from (25) we have

$$\varphi_x \approx \frac{\delta_x}{2} \cdot \frac{\eta}{\omega\varepsilon}\alpha^{1-\gamma}, \tag{33}$$

if $\varphi_x \ll \delta_x$, the fractional electric and magnetic components of the wave are approximately in phase $(\phi \approx 0)$, from (31) we have

$$\mathbf{B_0} \approx \frac{\mathbf{E_0}}{\tilde{\upsilon}}, \tag{34}$$

where the fractional phase velocity is $\tilde{\upsilon} = \frac{1}{\sqrt{\mu\varepsilon}}\alpha^{\gamma-1}$

In a good conductor $(\eta \gg \varepsilon\omega)$, in this case from (25) we have

$$\delta_x \approx \sqrt{\frac{\omega\mu\eta}{2}}\alpha^{1-\gamma}, \tag{35}$$

for φ_x from (25) we have

$$\varphi_x \approx \sqrt{\frac{\omega\mu\eta}{2}}\alpha^{1-\gamma} \approx \delta_x, \tag{36}$$

if $\varphi_x = \delta_x$, the real and imaginary part of the fractional wave number $\tilde{v_x}$ become equal. The decay of the wave is very fast, the term $\alpha^{1-\gamma}$ represents the fractional space structures of the wave. Now we analyze the case when β takes different values.

First case. When $\gamma = 2$, we have $\tilde{v_x}^2 = \frac{v_x^2}{\alpha}$

$$\tilde{v_x} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}\alpha^{-1} - i\frac{\mu\eta}{2}\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}\alpha^{-1}. \tag{37}$$

the equation (37) represents the fractional wave number. From equation (28) we have

$$\mathbf{E}(x,t) = \mathbf{E}_0 \cdot e^{i\omega t} \cdot E_4(-\tilde{v_x}^2 x^4), \tag{38}$$

where E_4 is given by (6)

$$\mathbf{E}(x,t) = \frac{\mathbf{E_0} \cdot e^{i\omega t}}{2} \cdot \left[\cos(-\tilde{v_x}^{1/2}x) + \cosh(-\tilde{v_x}^{1/2}x)\right]. \tag{39}$$

Second case. When $\gamma = 3/2$, we have $\tilde{v_x}^2 = \frac{v_x{}^2}{\alpha^{1/2}}$

$$\tilde{v_x} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}\alpha^{-1/2} - i\frac{\mu\eta}{2}\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}\alpha^{-1/2}. \tag{40}$$

the equation (40) represents the fractional wave number. From equation (28) we have

$$\mathbf{E}(x,t) = \mathbf{E}_0 \cdot e^{i\omega t} \cdot E_3(-\tilde{v_x}^2 x^3), \tag{41}$$

where E_3 is given by (5)

$$\mathbf{E}(x,t) = \frac{\mathbf{E_0} \cdot e^{i\omega t}}{2} \cdot \left[e^{-\tilde{v_x}^{2/3}x} + 2e^{\frac{\tilde{v_x}^{2/3}}{2}x} \cdot \cos\left(-\frac{\sqrt{3}}{2}\tilde{v_x}^{2/3}x\right)\right]. \tag{42}$$

Third case. When $\gamma = 1$, we have $\tilde{v_x} = v_x$

$$\tilde{v_x} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}} - i\frac{\mu\eta}{2}\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}. \tag{43}$$

the equation (43) represents the classical wave number. From equation (28) we have

$$\mathbf{E}(x,t) = \mathbf{E}_0 \cdot e^{i\omega t} \cdot E_2(-v_x x^2), \tag{44}$$

where E_2 is given by (4)

$$\mathbf{E}(x,t) = \Re[\mathbf{E_0} \cdot e^{i\omega t} \cdot e^{-iv_x x}], \tag{45}$$

where $\Re$ indicates the real part and $v_x = \delta_x - i\varphi_x$ is the wave number (43)

$$\mathbf{E}(x,t) = \Re[\mathbf{E_0} \cdot e^{i(\omega t - \delta_x x)} \cdot e^{-\varphi_x x}]. \tag{46}$$

The equation (46) represents the classical case for the wave equation in conductors. The first exponential $e^{i(\omega t-\delta_x x)}$ gives the usual plane-wave variation of the field with position x and time t. The second exponential $e^{-\varphi_x x}$ gives and exponential decay in the amplitude of the wave.

To find the physical field we have to take the real part to (43). From equation the (46) to obtain

$$\mathbf{E}(x,t) = \Re[\mathbf{E_0} \cdot e^{i\omega(t-\sqrt{\mu\varepsilon}x)}], \tag{47}$$

where for a bad conductor $(\eta \ll \omega\varepsilon)$ we take the real part of (43), $\Re(v_x) \approx \omega\sqrt{\mu\varepsilon}$ and $\upsilon = \frac{\omega}{v_x} = \frac{1}{\sqrt{\mu\varepsilon}}$ is the phase velocity (there is no dispersion due to its independent of the frequency). In this case the medium is considered transparent dispersive and the decay of the wave is very slow in terms of the number of wavelengths.

For a good conductor $(\eta \gg \omega\varepsilon)$ and for (43) we have $\delta_x \approx \sqrt{\frac{\omega\eta\mu}{2}}$ and $\varphi_x \approx \sqrt{\frac{\omega\eta\mu}{2}} \approx \delta_x$, the real and imaginary parts of the wave number v_x become equal. The phase velocity is given by $\upsilon = \frac{\omega}{\delta_x} \approx \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\eta}}$ (there is dispersion that depends on the frequency). This means that the decay of the wave is very fast in terms of the number of wavelengths.

Fourth case. When $\gamma = \frac{1}{2}$, from equation (28) we have

$$\mathbf{E}(x,t) = \mathbf{E_0} \cdot e^{i\omega t} \cdot E_1(-\tilde{v_x}^2 x), \tag{48}$$

where E_1 is given by (3) and $\tilde{v_x}$ is

$$\tilde{v_x} = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}\alpha^{1/2} - i\frac{\mu\eta}{2}\frac{1}{\sqrt{\mu\varepsilon}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{\eta^2}{\varepsilon^2\omega^2}}\right]^{\frac{1}{2}}}\alpha^{1/2}. \tag{49}$$

The equation (49) describes the real (δ_x) and imaginary (φ_x) part of the fractional wave number with fractional components α in terms of the frequency ω, and material properties μ, ε and η. From the fractional dispersion relation (18), the solution for the equation (48) is

$$\mathbf{E}(x,t) = \Re[\mathbf{E_0} \cdot e^{i\omega(t+\eta\mu\alpha x)} \cdot e^{-\mu\varepsilon\omega^2\alpha x}], \tag{50}$$

where $\Re$ indicates the real part. To find the fractional physical field, we have to take the real part of the fractional dispersion (18). From equation (50) we have

$$\mathbf{E}(x,t) = \Re[\mathbf{E_0} \cdot e^{i\omega t} e^{-\omega^2 \mu \varepsilon \alpha x)}]. \tag{51}$$

For a bad conductor $(\eta \ll \omega\varepsilon)$ we take the real part of (49), $\tilde{\nu_x} \approx \omega\sqrt{\mu\varepsilon}\alpha^{1/2}$ and $\tilde{\upsilon} = \frac{\omega}{\tilde{\nu_x}} = \frac{1}{\sqrt{\mu\varepsilon}\alpha^{1/2}}$ is the fractional phase velocity (there is no dispersion due to its independent of the frequency). In this case the medium is considered fractional transparent dispersive and the decay of the wave is very slow in terms of the number of wavelengths.

For a good conductor $(\eta \gg \omega\varepsilon)$ and for the real part of (49) and the imaginary part of (49) results, $\delta_x \approx \sqrt{\frac{\omega\eta\mu}{2}}\alpha^{1/2}$ and $\varphi_x \approx \sqrt{\frac{\omega\eta\mu}{2}}\alpha^{1/2} \approx \delta$, the real and imaginary parts of the fractional wave number $\tilde{\nu_x}$ becomes equal. The fractional phase velocity is given by $\tilde{\upsilon} = \frac{\omega}{\delta_x} \approx \frac{1}{\sqrt{\mu\varepsilon}\alpha^{1/2}}\sqrt{\frac{2\omega\varepsilon}{\eta}}$ (there is dispersion due that depends on the frequency). This means that the decay of the wave is very fast in terms of the number of wavelengths.

The fractional dispersion relation χ for an electromagnetic wave in a good conductor is

$$\chi = \frac{2}{\mu\eta}{\delta_x}^2, \tag{52}$$

where δ_x is the real part of the fractional wave number (49).

The fractional group velocity ζ is then

$$\zeta = \frac{2}{\sqrt{\mu\varepsilon}\alpha^{1/2}} \cdot \sqrt{\frac{2\omega\varepsilon}{\eta}}, \tag{53}$$

the fractional group velocity ζ of a fractional electromagnetic wave in a good conductor is approximately twice the fractional phase velocity $\tilde{\upsilon}$.

From (49), the real part δ_x of the fractional wave number $\tilde{\nu_x}$ in a conductor gives the wavelength, ψ measures the distance that the wave travels before its amplitude fails to $\frac{1}{e}$ in a distance $\frac{1}{\varphi_x}$, we define the fractional skin depth $\tilde{\Omega}$ as

$$\tilde{\Omega} = \frac{1}{\varphi_x}. \tag{54}$$

For a good conductor $(\eta \gg \varepsilon\omega)$, from (49) the fractional skin depth is given by

$$\tilde{\Omega} = \sqrt{\frac{2}{\omega\mu\eta}} \cdot \frac{1}{\alpha^{1/2}}, \tag{55}$$

in a good conductor $\delta_x = \varphi_x$, the fractional phase difference between the electric and magnetic fields in a good conductor is given by

$$\tan\phi = \frac{\varphi_x}{\delta_x} = 1, \tag{56}$$

so the difference phase is approximately 45.

In this case exists a physical relation between the auxiliary parameter α and the wave number ν_x given by the order γ of the fractional differential equation

$$\beta = \nu_x\alpha = \frac{\alpha}{\lambda}, \qquad 0 < \alpha \leq \lambda, \tag{57}$$

where λ is the wavelength. We can use this relation in order to write to equation (28) and (20) for the fractional electric and magnetic fields respectively as

$$\mathbf{E}(\tilde{x},t) = \mathbf{E_0} \cdot e^{i\omega t} \cdot E_{2\gamma}\Big(-\gamma^{2(1-\gamma)}\tilde{x}^{2\gamma}\Big), \tag{58}$$

$$\mathbf{B}(\tilde{x},t) = \mathbf{B_0} \cdot e^{i\omega t} \cdot E_{2\gamma}\Big(-\gamma^{2(1-\gamma)}\tilde{x}^{2\gamma}\Big), \tag{59}$$

where, $\tilde{x} = \frac{x}{\lambda}$, is a dimensionless parameter and $t = t_0$. The Figure 1 shows the simulation of the equation (58) and (59) for $0 < \gamma \leq 2$ respectively, γ values were arbitrarily chosen.

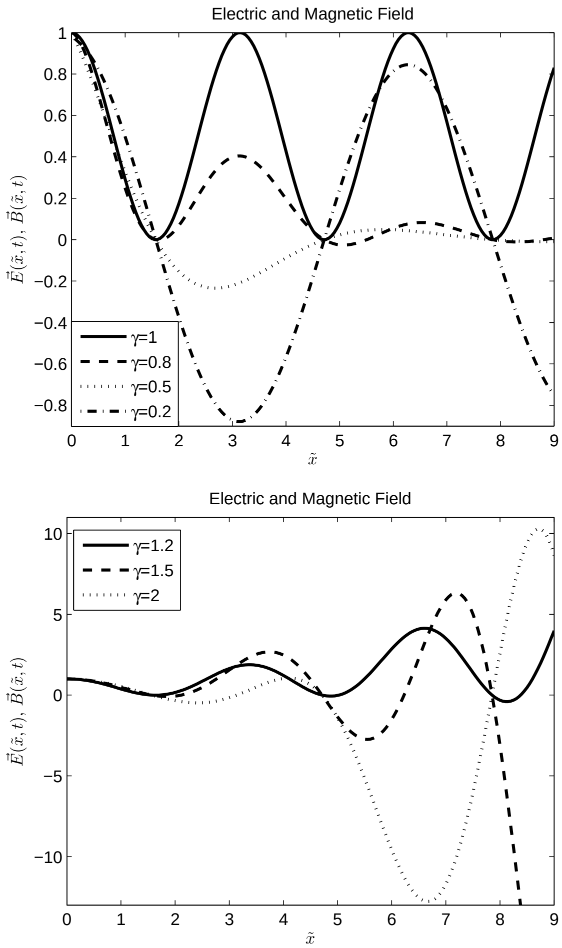

Fig. 1: Simulation of the equations (58) and (59) for $0 < \gamma \leq 2$.

4 Conclusions

In this manuscript was presented the analysis of the fractional space waves in conductors from the point of view of the FC. The fractional Caputo derivative is used and the order of the derivative is $0 < \gamma \leq 2$. In particular, a model one-dimensional fractional wave equation was considered in detail. In this alternative representation an auxiliary parameter α is introduced, this parameter is related to the equation's result in a fractal space geometry representation and characterizes the existence of the fractional space components presented an entire new family of solutions for the electric and magnetic field, see equations (58) and (59). These equations just depend on the order γ of the fractional differential equation due to the physical relation (57) preserving the physical units of the system.

In the case when γ is in the range $0 < \gamma \leq 1/2$ the electric or magnetic field exhibits an exponential behavior and in the range $1/2 < \gamma \leq 1$ the displacement represented a fractional oscillatory motion, see Figure 1. In the case when γ is in the range $1 < \gamma \leq 2$ the oscillations of the electric field or magnetic field increases with increasing order of derivative and the vibrational frequency increases, this frequency is depending on the properties of the system itself, see Figure 1. When integer dimension $\gamma = 1$ is considered, we recovered the classical cases.

Since the solutions are given in terms of the Mittag-Leffler functions depending only on a small number of parameters, the universality concept (when the class of behavior does not depend on the details of the physical system) can be considered through this methodology since the analytic solutions presented only need a few parameters to describe their behavior. The methodology proposed in this work can be applied in the critical phenomena theory, self-similarity, scale-invariance, transient effects and insulation in the electrical systems, electromagnetic hysteresis, wave propagation in transmission lines, scattering in random media, renormalization group and the description of anomalous complex processes.

Among problems for further research, we mention two- and three-dimensional fractional wave equations consider fractional variational calculus (see [33] and the references therein) with different initial or/and boundary conditions, of course, it would be interesting to consider the fractional wave equations with fractional derivatives defined in different ways.

Acknowledgments

The author appreciates the constructive remarks and suggestions of the anonymous referees that helped to improve the paper. We would like to thank to Mayra Martnez for the interesting discussions. Jos Francisco Gmez Aguilar acknowledges the support provided by CONACYT: ctedras CONACYT para jovenes investigadores 2014.

References

[1] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[2] A. Atangana and A. Secer, *A note on fractional order derivatives and table of fractional derivatives of some special functions*, *Abstr. Appl. Anal.* **2013**, (2013). Doi:10.1155/2013/279681.

[3] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus models and numerical methods*, Series on Complexity, Nonlinearity and Chaos. World Scientific, 2012.

[4] J. F. Gmez Aguilar and D. Baleanu, *Solutions of the telegraph equations using a fractional calculus approach*, *Proc. Rom. Acad. Ser A* **1** (15), 27-34 (2014).

[5] D. Baleanu, H.M. Srivastava and X.J. Yang, *Local Fractional variational iteration algorithms for the parabolic Fokker-Planck equation defined on Cantor sets*, *Progr. Fract. Differ. Appl.* **1**(1), 1-10 (2015).

[6] O. Shulika and I. Sukhoivanov (Eds.), *Description of the dynamics of charged particles in electric fields, an approach using fractional calculus*. Advanced Lasers. Laser Physics and Technology for Applied and Fundamental Science. Springer Series in Optical Sciences. Springer Netherlands. **193** (2015). DOI: 10.1007/978-94-017-9481-7.

[7] J. F. Gmez-Aguilar, R. Razo-Hernndez and D. Granados-Lieberman, *A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response*, *Rev. Mex. Fís.* **60**, 32-38 (2014).

[8] H. Jafari and H. Tajadodi, *Numerical solutions of the fractional advection- dispersion equation*, *Progr. Fract. Differ. Appl.***1** (1), 37-45 (2015).

[9] F. Gmez, J. Bernal, J. Rosales and T. Crdova, *Modeling and simulation of equivalent circuits in description of biological systems - a fractional calculus approach*, *J. Electr. Bioimped.* **3**, 2-11 (2012).

[10] D. A. Bensona, M. M. Meerschaert and J. Reviellea, *Fractional calculus in hydrologic modeling: a numerical perspective*, *Adv. Water Res.* **51**, 479-497 (2013).

[11] T. Srivastava, A. P. Singh and H. Agarwal, *Modeling the under-actuated mechanical system with fractional order derivative*, *Progr. Fract. Differ. Appl.* **1 (1)**, 57-64 (2015).

[12] J. F. Gmez Aguilar and D. Baleanu, *Fractional transmission line with losses*, Z. Naturforsch. **69a**, 539-546 (2014). Doi:10.5560/ZNA.2014-0049.

[13] M. Sharma, M.F. Ali and R. Jain, *Advanced generalized fractional kinetic equation in astrophysics*, *Progr. Fract. Differ. Appl.***1** (1), 65-71, (2015).

[14] D. Baleanu, R. Garra and I. Petras, *A fractional variational approach to the fractional Basset-type equation*, *Rep. Math. Phys.* **72** (1), 57-64 (2013).

[15] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, *Phys. Rev. E* **53**, 1890-1899 (1996).

[16] D. Baleanu, I. Muslih Sami and M. Rabei Eqab, *On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative*, *Nonlinear Dyn.* **53** (1-2), 67-74 (2008).

[17] Q. A. Naqvi and M. Abbas, *Complex and higher order fractional curl operator in electromagnetics*, *Opt. Commun.* **241**, 349-355 (2004).

[18] A. Hussain, M. Faryad and Q. A. Naqvi, *Fractional curl operator and fractional chiro-waveguide*, *J. Electr. Waves Appl.* **21 (8)**, 1119-1129 (2007).

[19] V. E. Tarasov, *Fractional equations of Curie-von Schweidler and Gauss laws*, *J. Phys. Condens. Matter.* **20**, 145-212 (2008).

[20] V. E. Tarasov, *Universal electromagnetic waves in dielectric*, *J. Phys. Condens. Matter.* **20**, 175-223 (2008).

[21] X. He, *Anisotropy and isotropy: a model of fraction-dimensional space*, *PSolid State Commun.* **75**, 111-114 (1990).

[22] M. Zubair, M. J. Mughal and Q. A. Naqvi, *The wave equation and general plane wave solutions in fractional space*, *Progr. Electr. Res. Lett.* **19**, 137-146, (2010).

[23] V. E. Tarasov, *Anisotropic fractal media by vector calculus in non-integer dimensional space*, *J. Math. Phys.* **55**, 083510 (2014).

[24] M. Zubair, M. J. Mughal and Q. A. Naqvi, *Electromagnetic fields and waves in fractional dimensional space*,Springer.Briefs in Applied Sciences and Technology, 7-16 (2012).

[25] M. Zubair, M. J. Mughal and Q. A. Naqvi, *An exact solution of the cylindrical wave equation for electromagnetic field in fractional dimensional space, Prog. Electr. Res.* **114**, 443-455 (2011).

[26] N. Engheta, *On the role of fractional calculus in electromagnetic theory*, *Antenn. Propag. Magaz.* **39**, 35-46 (1997).
[27] A. S. Balankin, B. Mena, J. Patio and D. Morales, *Electromagnetic fields in fractal continua*, *Phys. Lett. A* **377**, 738-788 (2013).
[28] V. E. Tarasov and J. J. Trujillo, *Fractional power-law spatial dispersion in electrodynamics*, *Ann. Phys.New-York* **334**, 1-23 (2013).
[29] V. E. Tarasov, *Fractional dynamics*, Springer, 2011.
[30] J. F. Gómez-Aguilar, J. J. Rosales-García, J. J. Bernal-Alvarado, T. Córdova-Fraga and R. Guzmán-Cabrera, *Fractional mechanical oscillators*, *Rev. Mex. Fís.* **58**, 348-352 (2012).
[31] A. Ansari, M. A. Darani and M. Moradi, *On fractional Mittag-Leffler operators*, *Rep. Math. Phys.* **70** (1), 119-131 (2012).
[32] H. J. Haubold, A. M. Mathai and R. K. Saxena, *Mittag-Leffler functions and their applications*, *J. Appl. Math.*, 298628 (2011).
[33] V. E. Tarasov, *Fractional vector calculus and fractional Maxwell's equations*, *Ann. Phys.New-York* **323**(11), 2756-2778 (2008).