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The McDonald Quasi Lindley Distribution and Its Statistical Properties with Applications

Rasool Roozegar and Fatemeh Esfandiyari*

Department of Statistics, Yazd University, P.O. Box 89195-741, Yazd, Iran

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Abstract: A new five-parameter distribution so-called the McDonald quasi Lindley distribution is proposed. The new distribution contains, as special submodels, several important distributions discussed in the literature, such as the beta quasi Lindley, Kumaraswamy quasi Lindley, beta Lindley, kumaraswamy Lindley and Lindley distributions, among others. The properties of this new distribution, including hazard function, reversed hazard function, shapes, moments, entropy and moment generating function are derived. We provide the density function of the order statistics and their moments. Method of maximum likelihood is used to estimate the parameters of the new and related distributions. The flexibility and usefulness of the new model are illustrated by means of an application to real data set.

Keywords: Quasi Lindley distribution, McDonald distribution, Maximum likelihood estimation, Moment generating function, Kumaraswamy distribution,Beta distribution,Lindley distribution

1 Introduction

Recently, several lifetime distributions have been used to model and analyze lifetime data. The Lindley (L) distribution was originally proposed by Lindley [15] in the context of Bayesian statistics as a counter example of fiducial statistics. This distribution is a mixture of exponential (E) and Length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. Ghitany et al. [12] have discussed the properties of this distribution. They have found that the Lindley distribution performs better than exponential model because of its time dependent/increasing hazard rate. Zakerzadeh and Dolati [25] obtained a generalized Lindley (GL) distribution and discussed its various properties and applications. Nadarajah et al. [20] studied the mathematical and statistical properties of the generalized Lindley distribution. Bakouch et al. [2] obtained an extended Lindley distribution and discussed its various properties. Merovci and Sharma [18] introduced a new generalization of Lindley distribution called beta-Lindley (BL) distribution. Shanker and Mishra [**?**] introduced and studied the mathematical and statistical properties of the quasi Lindley (QL) distribution where it has the L distribution as a particular case. The cumulative distribution function (cdf) of the QL distribution is given by

$$G(x;\alpha,\theta)=1-(1+\frac{\theta x}{\alpha+1})e^{-\theta x} \tag{1}$$

and the corresponding QL probability density function (pdf) is given by

$$g(x;\alpha,\theta)=\frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x}, \tag{2}$$

for $x>0$, $\theta>0$ and $\alpha>-1$. Elbatal and Elgarly [11] studied statistical properties of Kumaraswamy quasi Lindley (KumQL) distribution. The QL distribution reduces to L distribution when $\alpha=\theta$ and at $\alpha=0$, it reduces to the gamma distribution with parameters $(2,\theta)$.

The density function of QL model is a mixture of exponential and gamma distributions, that is

$$g(x;\alpha,\theta)=pf_1(x;\theta)+(1-p)f_2(x;\theta),$$

* Corresponding author e-mail: rroozegar@yazd.ac.ir

with $p = \dfrac{\alpha}{\alpha+1}$, where $f_1(x;\theta) = \theta e^{-\theta x}$ and $f_2(x;\theta) = \theta^2 x e^{-\theta x}$. It is also positively skewed. The hazard and mean residual life functions of the QL distribution are given by

$$h(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x)}{1+\alpha+\theta x}$$

and

$$m(x;\alpha,\theta) = \frac{2+\alpha+\theta x}{\theta(1+\alpha+\theta x)},$$

respectively. The hazard function $h(x;\alpha,\theta)$ is an increasing function whereas the mean residual life function $m(x;\alpha,\theta)$ is a decreasing function.

In recent years, many authors have proposed distributions which can arise as special submodels within the McDonald (Mc) generated or generalized beta (GB) generated class of distributions. Alexander et al. [1] introduce a class of generalized beta-generated distributions that have three shape parameters in the generator. They consider eleven different parents: normal, log-normal, skewed student-t, Laplace, exponential, Weibull, Gumbel, Brinbaum-Saunders, gamma, Pareto and logistic distributions. Other generalizations are McDonald inverted beta distribution by Corderio and Lemonte [5], McDonald gamma distribution by Marciano et al. [16], McDonald normal distribution by Corderio et al. [6] McDonald exponentiated exponential distribution by Corderio et al. [7], McDonald log-logistic distribution by Tahir et al. [24], McDonald arcsine distribution by Corderio and Lemonte [8], McDonald Weibull distribution by Corderio et al. [9]. and McDonald Extended Weibull Distribution by Hashimoto et al. [14].

One of the main reasons to consider the McDonald generated distribution is its ability of fitting skewed data, [19]. The McDonald generated family of densities allows for higher levels of flexibility of its tails and has a lot of applications in various fields such as economics, finance, reliability, engineering, biology and medicine. The main objective of this paper is to construct and explore the properties of the five-parameter model called the McDonald quasi Lindley (McQL) distribution. This distribution exhibits the desirable properties of increasing, decreasing, upside-down bathtub and bathtub shaped hazard function.

This paper is organized as follows. The pdf, cdf and hazard function of the McQL distribution are derived in Section 2. Some special models of the new distribution are described in this section. In Section 3, we present useful expansions of cdf and pdf of the McQL distribution. Some properties of the cdf, pdf, kth moment and moment generating function of the McQL distribution are discussed in Section 4. Moreover, the order statistics, their moments and entropy are investigated in this section. Maximum likelihood estimates (MLEs) of the model parameters are given in Section 5. An application of the McQL distribution by using a real data set is performed in Section 6.

2 The McQL model

The generalized beta distribution of the first kind or McDonald distribution (denoted with the prefix "Mc" for short) was introduced by [17]. McDonald (1984). The cdf of the McDonald distribution is given by

$$F(x) = I(x^c; a/c, b), \qquad 0 < x < 1,$$

for $a,\ b,\ c > 0$, where $I(y;a,b) = \frac{B_y(a,b)}{B(a,b)} = \dfrac{1}{B(a,b)} \int_0^y t^{a-1}(1-t)^{b-1}dt$ and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ are the incomplete beta function ratio and the beta function, respectively.

The cdf of McQL model is defined by

$$F(x;a,b,c,\alpha,\theta) = I([1 - \frac{1+\alpha+\theta x}{\alpha+1} e^{-\theta x}]^c; a/c, b),\ x > 0, \tag{3}$$

where $\theta > 0$ and $\alpha > -1$. The pdf corresponding to (3) is given by

$$\begin{aligned} f(x;a,b,c,\alpha,\theta) \;=\; & \frac{c\theta(\alpha+\theta x)e^{-\theta x}}{(\alpha+1)B(a/c,b)} \left[1 - \left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^{a-1} \\ & \times \left[1 - \left[1 - \left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^c\right]^{b-1}. \end{aligned} \tag{4}$$

For random variable X with density function (4), we write $X \sim McQL(a,b,c,\alpha,\theta)$. In fact, the McQL distribution belongs to the new class of distributions called the McDonald-generated distributions with cdf and pdf as

$$F(x;a,b,c,\phi) = I(G^c(x;\phi); a/c, b) = \frac{1}{B(a/c,b)} \int_0^{G^c(x;\phi)} t^{a/c-1}(1-t)^{b-1}dt$$

and

$$f(x;a,b,c,\phi)=\frac{c}{B(a/c,b)}g(x;\phi)G^{a-1}(x;\phi)(1-G^{c}(x;\phi))^{b-1},$$

respectively. The cdf is given in (3) can also be represented by

$$F(x;a,b,c,\phi)=\frac{cG(x;\phi)^{a}}{aB(a/c,b)}{}_{2}F_{1}\left(\frac{a}{c},1-b,\frac{a}{c}+1;G(x;\phi)^{a}\right), \tag{5}$$

where

$${}_{2}F_{1}(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)}\frac{x^{j}}{j!},\quad |x|<1, \tag{6}$$

$\phi=(\alpha,\theta)$ and $G(x;\phi)=1-(1+\frac{\theta x}{\alpha+1})e^{-\theta x}$ is the cdf of QL model.

Theorem 2.1. Let $f(x;a,b,c,\alpha,\theta)$ be the pdf of McQL distribution given by (4). The limiting behavior of $f(x;a,b,c,\alpha,\theta)$ for different values of its parameters is given below:
i. If $a=1$, then $\lim_{x\to 0^{+}}f(x;a,b,c,\alpha,\theta)=\frac{c\theta\alpha}{(\alpha+1)B(1/c,b)}$.
ii. If $a>1$, then $\lim_{x\to 0^{+}}f(x;a,b,c,\alpha,\theta)=0$.
iii. If $a<1$, then $\lim_{x\to 0^{+}}f(x;a,b,c,\alpha,\theta)=\infty$.
iv. $\lim_{x\to+\infty}f(x;a,b,c,\alpha,\theta)=0$.

Proof. It is straightforwared to show the above from the McQL density in equation (4).

The hazard rate function (also known as the failure rate function) $h(t)$, which is an important quantity characterizing life phenomenon, is defined by $h(t)=\dfrac{f(t)}{1-F(t)}$. The hazard rate function (hrf) of the McQL distribution is given by

$$\begin{aligned}h(x;a,b,c,\alpha,\theta)=&\frac{c\theta(\alpha+\theta x)e^{-\theta x}}{(\alpha+1)\left[B(a/c,b)-B_{[1-(1+\frac{\theta x}{\alpha+1})e^{-\theta x}]^{c}}(a/c,b)\right]}\left[1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^{a-1}\\&\times\left[1-\left[1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^{c}\right]^{b-1}.\end{aligned} \tag{7}$$

The reversed hazard rate function $r(t)$ is defined by $r(t)=\dfrac{f(t)}{F(t)}$. The corresponding reversed hazard rate function of the McQL distribution is given as

$$\begin{aligned}r(x;a,b,c,\alpha,\theta)=&\frac{c\theta(\alpha+\theta x)e^{-\theta x}}{(\alpha+1)B_{[1-(1+\frac{\theta x}{\alpha+1})e^{-\theta x}]^{c}}(a/c,b)}\left[1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^{a-1}\\&\times\left[1-\left[1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}\right]^{c}\right]^{b-1}.\end{aligned} \tag{8}$$

Figure 1 illustrates some of the possible shapes of the density and hazard functions of the McQL distribution for selected values of the parameters. For instance, these plots show the hazard rate function of the new model is much more flexible than the beta Lindley (BL), quasi Lindley (QL) and Lindley distributions. The hazard rate function can be bathtub shaped, monotonically increasing or decreasing and upside-down bathtub shaped depending on the parameter values.

The McQL distribution contains as sub-models the beta quasi Lindley (BQL), the Kumaraswamy quasi Lindley (KumQL) [11], and McDonald Lindley (McL) distributions for $c=1$, $a=c$ and $\alpha=\theta$, respectively. For $c=1$ and $\alpha=\theta$, the McQL distribution reduces to the beta Lindley (BL) distribution, [18]. The subject distribution also includes as special cases the generalized quasi Lindley distribution (GQL), generalized Lindley (GL) distribution proposed by Nadarajah et al. [20] and McDonald gamma (McG) distribution. The classes of distributions that are included as special sub-models of the McQL distribution are displayed in Figure 2.

If the random variable X has the McQL distribution, then it has the following properties:
1. The random variable $V=[1-\left(1+\frac{\theta x}{\alpha+1}\right)e^{-\theta x}]^{c}$ satisfies the beta distribution with parameters a/c and b. Therefore, the random variable $T=\theta X-\ln(\dfrac{1+\alpha+\theta x}{\alpha+1})$ has the BGE (or McE) distribution, [3]. Furthermore, the random variable $X=G^{-1}(V)$ follows McQL distribution, where $G(.)$ is given in (3). This result helps us in simulating data from McQL distribution.
2. If $a=i$ and $b=n-i+1$, where i and n are positive integer values, then the $F(x;a,b,c,\alpha,\theta)$ is the cdf of the ith order statistic of GQL distribution.

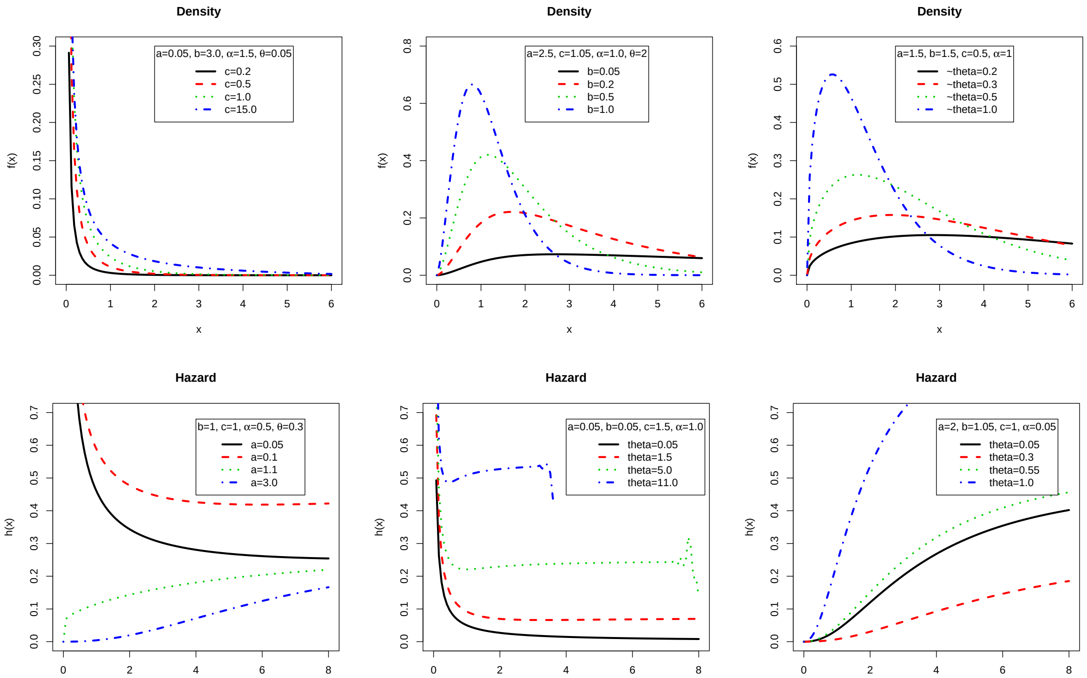

Fig. 1: The pdf and hrf of McQL model for some values of parameters.

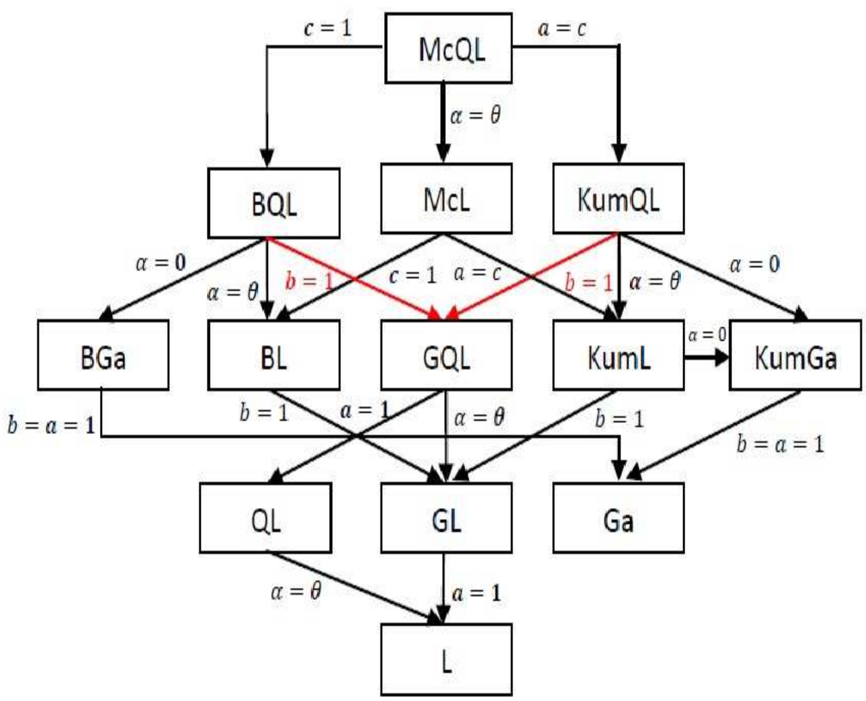

Fig. 2: Relationships of the McQL sub-models

3 Expansion of the model

In this section we derive some representations of cdf and pdf of McQL distibution. The binomial series expansion is defined by

$$(1-z)^m = \sum_{j=0}^{\infty}(-1)^j \binom{j}{m} z^j = \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(m+1)}{\Gamma(m-j+1)}\frac{z^j}{j!}, \tag{9}$$

where $|z|<1$ and m is a positive real non-integer.

The following proposition reveals that the McQL distribution can be expressed as a mixture of distribution function of GQL distribution, whereas Proposition 2 provides a useful expansion for the pdf in (4).

Proposition 1. *The cdf in (3) is a mixture of GQL distributions on the form*

$$F(x;a,b,c,\alpha,\theta) = \sum_{j=0}^{\infty} q_j G_j(x), \tag{10}$$

where $q_j = \frac{(-1)^j\Gamma(b)}{B(a/c,b)\Gamma(b-j)j!(a/c+j)}$, $\sum_{j=0}^{\infty} q_j = 1$ *and* $G_j(x) = (G(x;\alpha,\theta))^{a+jc}$ *is the distribution function of a random variable which has a GQL distribution with parameters* α, θ *and* $a+jc$.

If a is a real non-integer, we can expand $G_j(x)$ as follows:

$$\begin{aligned} G_j(x) &= (G(x;\alpha,\theta))^{a+jc} = [1-(1-G(x;\alpha,\theta))]^{a+jc} \\ &= \sum_{i=0}^{\infty}(-1)^i \binom{a+jc}{i}(1-G(x;\alpha,\theta))^i, \end{aligned} \tag{11}$$

with

$$(1-G(x;\alpha,\theta))^i = \sum_{r=0}^{i}(-1)^r \binom{i}{r} G^r(x;\alpha,\theta),$$

so that

$$G_j(x) = \sum_{i=0}^{\infty}\sum_{r=0}^{i}(-1)^{r+i}\binom{a+jc}{i}\binom{i}{r}G^r(x;\alpha,\theta). \tag{12}$$

Now, equation (3) becomes

$$F(x;a,b,c,\alpha,\theta) = \sum_{j=0}^{\infty} b_{j,r} G^r(x;\alpha,\theta),$$

where

$$b_{j,r} = \sum_{i=0}^{\infty}\sum_{r=0}^{i} q_j (-1)^{r+i}\binom{a+jc}{i}\binom{i}{r}.$$

If $b>0$ is an integer, then

$$F(x;a,b,c,\alpha,\theta) = \sum_{j=0}^{b-1} q_j G_j(x).$$

Proposition 2. *The pdf of McQL model can be expressed as an infinite mixture of GQL densities with parameters* α, θ *and* $(a+jc)$ *given by*

$$f(x;a,b,c,\alpha,\theta) = \sum_{j=0}^{\infty} q_j g_j(x), \tag{13}$$

where $g_j(x) = (a+jc)g(x;\alpha,\theta)[G(x;\alpha,\theta)]^{a+jc-1}$.

Similarly, if $b>0$ is an integer, the pdf of McQL model is given by

$$f(x;a,b,c,\alpha,\theta) = \sum_{j=0}^{b-1} q_j g_j(x).$$

From equations (13) and (12), the McQL density can be written in the form

$$f(x;a,b,c,\alpha,\theta) = g(x;\alpha,\theta)\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{r=0}^{i} p_{i,j,r}G^{r}(x;\alpha,\theta).$$

where

$$p_{i,j,r} = \frac{c(-1)^{j+r+i}\Gamma(b)}{B(a/c,b)j!\Gamma(b-j)}\binom{a+jc}{i}\binom{i}{r}$$

and $\sum_{i,j=0}^{\infty}\sum_{r=0}^{i} p_{i,j,r} = 1$.

4 Statistical properties

In this section, we deal with the basic statistical properties of the McQL distribution, in particular, moments and moment generating function.

4.1 Moments and moment generating function

In this subsection we derive the kth non-central moment and moment generating function for the McQL distribution. Moments are necessary and important in any statistical analysis, especially in applications.

Proposition 3. *The kth moment, $E(X^k)$, of the McQL distributed random variable X, is given as*

$$\mu_k'(X) = E(X^k) = w_{i,j,r}\left[\frac{\alpha\Gamma(k+i+1)}{(\theta(r+1))^{k+i+1}} + \frac{\theta\Gamma(k+i+2)}{\theta(r+1))^{k+i+2}}\right],$$

where

$$w_{i,j,r} = \sum_{j,r=0}^{\infty}\sum_{i=0}^{r}\frac{c}{B(a/c,b)}\frac{\theta}{(\alpha+1)^{i+1}}(-1)^{i+1}\binom{b-1}{j}\binom{cj+a-1}{r}\binom{r}{i}.$$

Proposition 4. *If X has the McQL distribution then the moment generating function (mgf) of X is given as follows*

$$M_X(t) = w_{i,j,r}\left[\frac{\alpha\Gamma(i+1)}{(\theta(r+1)-t)^{i+1}} + \frac{\theta\Gamma(i+2)}{(\theta(r+1)-t)^{i+2}}\right]. \tag{14}$$

4.2 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let the random variable $X_{i:n}$ be the ith order statistic ($X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$) in a sample of size n from the McQL distribution. The pdf and cdf of $X_{i:n}$ for $i = 1,2,\ldots,n$ are given by

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k f(x)[F(x)]^{k+i-1}, \end{aligned} \tag{15}$$

and

$$F_{i:n}(x) = \int_0^x f_{i:n}(t)dt = \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\frac{(-1)^k}{k+i}\binom{n-i}{k}[F(x)]^{k+i}, \tag{16}$$

respectively, where $F(x) = F(x;\alpha,\theta) = \sum_{j=0}^{\infty} b_{j,r}G^r(x;\alpha,\theta)$. We use throughout an equation by Gradshteyn and Ryzhik (2007, page 17) for a power series raised to a positive integer m given by

$$\left(\sum_{r=0}^{\infty} b_r u^r\right)^m = \sum_{r=0}^{\infty} c_{m,r}u^r, \tag{17}$$

where the coefficients $c_{m,r}$ (for $r = 1, 2, \ldots$) are easily determined from the recurrence equation

$$c_{m,r} = (r\,b_0)^{-1} \sum_{k=1}^{r} [k\,(m+1) - r + k]\, b_k\, c_{m,r-k},$$

where $c_{m,0} = b_0^m$. Hence, the coefficients $c_{m,r}$ can be calculated from $c_{m,0}, \ldots, c_{m,r-1}$ and therefore, from the quantities $b_0, \ldots, b_r$. Using (17), the equations (15) and (16) can be written as

$$f_{i:n}(x;\alpha,\theta) = \frac{1}{B(i,n-i+1)} \sum_{k=0}^{n-i} \sum_{r=1}^{\infty} \frac{r}{k+i} (-1)^k \binom{n-i}{k} c_{i+k,r}\, g(x;\alpha,\theta)[G(x;\alpha,\theta)]^{r-1},$$

$$F_{i:n}(x;\alpha,\theta) = \frac{1}{B(i,n-i+1)} \sum_{k=0}^{n-i} \sum_{r=0}^{\infty} \frac{1}{k+i} (-1)^k \binom{n-i}{k} c_{i+k,r}[G(x;\alpha,\theta)]^{r}.$$

An explicit expression for the sth moments of $X_{i:n}$ can be obtained as

$$\begin{aligned} E[X_{i:n}^s] &= \frac{1}{B(i,n-i+1)} \sum_{k=0}^{n-i} \sum_{r=1}^{\infty} \frac{r}{k+i} (-1)^k \binom{n-i}{k} c_{i+k,r} \int_0^{+\infty} t^s g(t;\alpha,\theta)[G(t;\alpha,\theta)]^{r-1} dt \\ &= \frac{\theta\Gamma(s+1)}{B(i,n-i+1)} \sum_{k=0}^{n-i} \sum_{r=1}^{\infty} \frac{r}{k+i} (-1)^k \binom{n-i}{k} c_{i+k,r} \\ &\quad \times \frac{1}{\theta^s} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{i_1} \sum_{i_3=0}^{i_2+1} \alpha^{i_2-i_3+1} (-1)^{i_1} \binom{i_1}{i_2} \binom{r-1}{i_1} \frac{\Gamma(s+i_3+1)}{(\alpha+1)^{i_1+1}(i_1+1)^{s+i_3+1}}. \end{aligned} \tag{18}$$

4.3 Entropy

The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. The Rényi entropy is an extension of Shannon entropy. The Rényi entropy is defined to be

$$I_R(\gamma) = \frac{1}{1-\gamma} \log\Big(\int_R [f(x)]^\gamma dx \Big), \tag{19}$$

where $\gamma > 0$ and $\gamma \neq 1$. The Rényi entropy tends to the Shannon entropy as $\gamma \to 1$. By using the pdf of McQL model, we have

$$\begin{aligned} \int_0^{+\infty} f^\gamma(x;a,b,c,\alpha,\theta) dx &= \left(\frac{c\theta\alpha}{(\alpha+1)B(a/c,b)} \right)^\gamma \\ &\quad \times \int_0^{+\infty} (1+\frac{\theta}{\alpha}x)^\gamma e^{-\theta\gamma x} [G(x;\alpha,\theta)]^{a\gamma-\gamma} [1-G^c(x;\alpha,\theta)]^{b\gamma-\gamma} dx. \end{aligned}$$

Setting

$$\begin{aligned} [G(x;\alpha,\theta)]^{a\gamma-\gamma} &= \left[1 - \frac{\alpha+1+\theta x}{\alpha+1} e^{-\theta x} \right]^{a\gamma-\gamma} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^k \binom{a\gamma-\gamma}{k} \binom{k}{j} \frac{(-1)^k \alpha^j (1+\frac{\theta}{\alpha}x)^j}{(1+\alpha)^k} e^{-\theta k x} \end{aligned} \tag{20}$$

and

$$\begin{aligned} [1-G^c(x;\alpha,\theta)]^{b\gamma-\gamma} &= \sum_{l=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{m} \binom{b\gamma-\gamma}{l} \binom{cl}{m} \binom{m}{n} (-1)^{l+m} \\ &\quad \times \frac{\alpha^n (1+\frac{\theta}{\alpha}x)^n}{(1+\alpha)^m} e^{-\theta m x}. \end{aligned} \tag{21}$$

By using (20) and (21), we obtain

$$\begin{aligned}
\int_0^{+\infty} f^{\gamma}(x;a,b,c,\alpha,\theta)dx &= \left(\frac{c\theta\alpha}{(\alpha+1)B(a/c,b)}\right)^{\gamma} \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m} \binom{a\gamma-\gamma}{k}\binom{k}{j} \\
&\quad \times \binom{b\gamma-\gamma}{l}\binom{cl}{m}\binom{m}{n}\frac{(-1)^{k+l+m}\alpha^{n+j}}{(1+\alpha)^{k+m}} \int_0^{+\infty}(1+\frac{\theta}{\alpha}x)^{\gamma+j+n}e^{-\theta(\gamma+k+m)x}dx \\
&= \left(\frac{c\theta\alpha}{(\alpha+1)B(a/c,b)}\right)^{\gamma} \sum_{k,m,l,u=0}^{\infty}\sum_{j=0}^{k}\sum_{n=0}^{m}\binom{k}{j}\binom{b\gamma-\gamma}{l} \\
&\quad \times l\binom{cl}{m}\binom{m}{n}\binom{\gamma+j+n}{u}\frac{(-1)^{k+l+m}\alpha^{n+j-u}\Gamma(u+1)}{\theta(1+\alpha)^{k+m}(\gamma+k+m)^{u+1}}. \qquad (22)
\end{aligned}$$

Therefore, the Rényi entropy for McQL distribution is obtained by above relation and (19).

The Shannon entropy for the McQL distribution is defined as follows:

$$H_{SH}(f) = -E_f[\log(f(X))] = -\int_0^{\infty} f(x)\log f(x)dx.$$

Hence, the Shannon entropy for the McQL distribution can be expressed as

$$\begin{aligned}
H_{Sh}(f) &= \log(\frac{B(a/c,b)(\alpha+1)}{c\theta\alpha}) - E[\log(1+\frac{\theta}{\alpha}X)] \\
&\quad + \theta E(X) + (1-a)E[\log G(x;\alpha,\theta)] + (1-b)E[\log(1-G^c(x;\alpha,\theta))].
\end{aligned}$$

We note that

$$E[\log(1-G^c(X;\alpha,\theta))] = -\sum_{k=1}^{\infty}\frac{1}{k}E[G^{ck}(x;\alpha,\theta)]$$

and

$$E[\log G(X;\alpha,\theta)] = -\sum_{k=1}^{\infty}\sum_{l=0}^{k}\frac{1}{k}\binom{k}{l}\frac{\theta^l}{k(1+\alpha)^l}E[X^l e^{-k\theta x}].$$

Therefore by using the results in Lemma 1 in Nadarajah et al. (2011), we have

$$\begin{aligned}
H_{Sh}(f) &= \log(\frac{B(a/c,b)(\alpha+1)}{c\theta\alpha}) + \sum_{k=1}^{\infty}\frac{(-1)^k\theta^k}{k\alpha^k}E(X^k) \\
&\quad + \theta E(X) + (1-a)E[\log G(x;\alpha,\theta)] + (1-b)E[\log(1-G^c(x;\alpha,\theta))],
\end{aligned}$$

where

$$\begin{aligned}
E[\log G(x;\alpha,\theta)] &= -\frac{c}{B(a/c,b)}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}\binom{k}{l}\binom{a/c+i-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\quad \times \frac{(-1)^{i+j+r}\theta^{l+2}}{k(1+\alpha)^{l+1}}A((r+1)c,\alpha,l,\alpha(1+k))
\end{aligned}$$

and

$$\begin{aligned}
E[\log(1-G^c(x;\alpha,\theta))] &= -\frac{c}{B(a/c,b)}\sum_{k=1}^{\infty}\sum_{i,j=0}^{\infty}\sum_{r=0}^{j}\binom{k}{l}\binom{a/c+i-1}{j}\binom{b-1}{i}\binom{j}{r} \\
&\quad \times \frac{(-1)^{i+j+r}\theta^2}{k(1+\alpha)}A((r+k+1)c,\alpha,0,\alpha),
\end{aligned}$$

and

$$A(r,s,t,u) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n+1}\binom{r-1}{m}\binom{m}{n}\binom{n+1}{p}\frac{(-1)^m s^n\Gamma(t+p+1)}{(1+s)^m(sm+u)^{t+p+1}}.$$

5 Estimation

Let $X_1,\ldots,X_n$ be a random sample of size n from the $McQL(a,b,c,\alpha,\theta)$ distribution and $\boldsymbol{\Theta}=(a,b,c,\alpha,\theta)$ be the unknown parameter vector. The log-likelihood function is given by

$$
\begin{aligned}
l(\boldsymbol{\Theta}) &= n\log(c\theta)-n\log(B(a/c,b))-n\log(\alpha+1)+\sum_{i=1}^{n}\log(\alpha+\theta x_i)-\theta\sum_{i=1}^{n}x_i \\
&\quad +(a-1)\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}\right] \\
&\quad +(b-1)\sum_{i=1}^{n}\log\left[1-[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}]^{c}\right]. \qquad (23)
\end{aligned}
$$

The maximum likelihood estimation (MLE) of $\boldsymbol{\Theta}$ is obtained by solving the nonlinear equations, $U(\boldsymbol{\Theta})=(U_a(\boldsymbol{\Theta}),U_b(\boldsymbol{\Theta}),U_c(\boldsymbol{\Theta}),U_\alpha(\boldsymbol{\Theta}),U_\theta(\boldsymbol{\Theta}))^T=\mathbf{0}$, where

$$
U_a(\boldsymbol{\Theta})=\frac{\partial l(\boldsymbol{\Theta})}{\partial a}=n/c\left[\psi(b+a/c)+\psi(a/c)\right]+\sum_{i=1}^{n}\log[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}], \qquad (24)
$$

$$
U_b(\boldsymbol{\Theta})=\frac{\partial l(\boldsymbol{\Theta})}{\partial b}=n\psi(b+a/c)-n\psi(b)+\sum_{i=1}^{n}\log[\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}]^{c}, \qquad (25)
$$

$$
U_c(\boldsymbol{\Theta})=\frac{\partial l(\boldsymbol{\Theta})}{\partial c}=n/c^2\left[c-a\psi(b+a/c)+a\psi(a/c)\right]+(b-1)\sum_{i=1}^{n}\log[\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}], \qquad (26)
$$

$$
\begin{aligned}
U_\alpha(\boldsymbol{\Theta}) &= \frac{\partial l(\boldsymbol{\Theta})}{\partial \alpha}=\frac{n}{\alpha+1}+\sum_{i=1}^{n}\frac{1}{\alpha+\theta x_i}+(a-1)\sum_{i=1}^{n}\frac{e^{-\theta x_i}\frac{\theta x_i}{(\alpha+1)^2}}{1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}} \\
&\quad -(b-1)\sum_{i=1}^{n}\frac{ce^{-\theta x_i}\frac{\theta x_i}{(\alpha+1)^2}\left[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}\right]^{c-1}}{1-[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}]^{c}}, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
U_\theta(\boldsymbol{\Theta}) &= \frac{\partial l(\boldsymbol{\Theta})}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{x_i}{\alpha+\theta x_i} \\
&\quad +(b-1)\sum_{i=1}^{n}\frac{c\frac{x_i e^{-\theta x_i}}{\alpha+1}-x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)\left[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}\right]^{c-1}}{1-[1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}]^{c}} \\
&\quad +(a-1)\sum_{i=1}^{n}\frac{\frac{x_i e^{-\theta x_i}}{\alpha+1}-x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)}{1-\left(1+\frac{\theta x_i}{\alpha+1}\right)e^{-\theta x_i}}, \qquad (28)
\end{aligned}
$$

where $\psi(.)=\frac{\Gamma'(.)}{\Gamma(.)}$ denotes the digamma function.

We need the observed information matrix for interval estimation and hypotheses tests on the model parameters. The 5×5 Fisher information matrix, $J=J_n(\boldsymbol{\Theta})$, is given by

$$
J=-\begin{bmatrix}
J_{aa} & J_{ab} & J_{ac} & J_{a\alpha} & J_{a\theta} \\
J_{ba} & J_{bb} & J_{bc} & J_{b\alpha} & J_{b\theta} \\
J_{ca} & J_{cb} & J_{cc} & J_{c\alpha} & J_{c\theta} \\
J_{\alpha a} & J_{\alpha b} & J_{\alpha c} & J_{\alpha\alpha} & J_{\alpha\theta} \\
J_{\theta a} & J_{\theta b} & J_{\theta c} & J_{\theta\alpha} & J_{\theta\theta}
\end{bmatrix},
$$

where the expressions for the elements of J are in the appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, asymptotically

$$
\sqrt{n}(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})\sim N_5(0,I(\boldsymbol{\Theta})^{-1}),
$$

where $I(\boldsymbol{\Theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\Theta})$ replaced by $J_n(\hat{\boldsymbol{\Theta}})$, i.e., the observed information matrix evaluated at $\hat{\boldsymbol{\Theta}}$ (Cox and Hinkley, 1979).

6 Application of McQL to a real data set

In this section, we fit the McQL distribution to a real data set and compare it with some models and submodels such as: the McDonald Dagum (McD) by Rajasooriya [21], the McDonald Weibull (McW) and the McDonald log-logistic (McLL) distributions and the KumQL, KumG, BQL, QL and L distributions to show the superiority of the McQL distribution. The data set is given by Suprawhardana and Prayoto [23], that refers to the time between failures (thousands of hours) of secondary reactor pumps. The data set consists of 23 observations.

The MLEs of the parameters, -2log-likelihood, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), AICC (Consistent Akaike Information Criterion), the KS statistic with its p-value and LRT statistic for this data set are displayed in Table 1.

From the values of these statistics, we conclude that the McQL distribution provides a better fit to this data than the McD, McW, McLL, KumQL, BQL, QL and L distributions. Moreover, the plots of empirical cdf of the data set and estimated cdf of seven models are displayed in Figure 3. These plots suggest that the McQL distribution is superior to the other distributions in terms of model fitting.

Table 1: MLEs of the model parameters for the time between failures data, the corresponding AIC, AICC, BIC, KS and LRT statistics.

Dist.	MLE	-2 Log L	AIC	AICC	BIC	KS (p-value)	LRT	p-value
McQL(a,b,c,α,θ)	$\hat{a}=1.5594, \hat{b}=0.1193, \hat{c}=22.8140,$ $\hat{\alpha}=3.8165, \hat{\theta}=3.5611$	61.5037	71.5037	75.0332	67.9918	0.1136 (0.8955)	–	–
McD($a,b,c,\alpha,\theta,\delta$)	$\hat{a}=7.8434, \hat{b}=41.7895, \hat{c}=6.8431,$ $\hat{\alpha}=0.0816, \hat{\theta}=3.3260, \hat{\delta}=0.1362$	63.6685	75.6685	80.9185	71.4542	0.121 (0.8497)	5.6044	0.0052
McW(a,b,c,α,θ)	$\hat{a}=46.0331, \hat{b}=31.9924, \hat{c}=0.0314,$ $\hat{\alpha}=0.1288, \hat{\theta}=0.0915$	63.5928	73.5928	77.1222	70.0809	0.1204 (0.8539)	3.2457	0.0212
McLL(a,b,c,α,θ)	$\hat{a}=12.3203, \hat{b}=7.0308, \hat{c}=3.5592,$ $\hat{\alpha}=0.3011, \hat{\theta}=0.0100$	74.1614	84.1614	87.6909	80.6495	0.2061 (0.2466)	12.0816	0.0021
KumQL(a,b,a,α,θ)	$\hat{a}=0.7982, \hat{b}=18.2196, \hat{c}=0.7982,$ $\hat{\alpha}=16.1428, \hat{\theta}=0.0195$	65.1652	73.1652	73.7260	67.9918	0.1417 (0.6934)	3.0141	0.0454
BQL($a,b,1,\alpha,\theta$)	$\hat{a}=0.7456, \hat{b}=8.0368, \hat{c}=1,$ $\hat{\alpha}=13.2208, \hat{\theta}=0.0642$	65.5198	73.5198	73.7260	70.7102	0.1994 (0.2808)	3.2306	0.0319

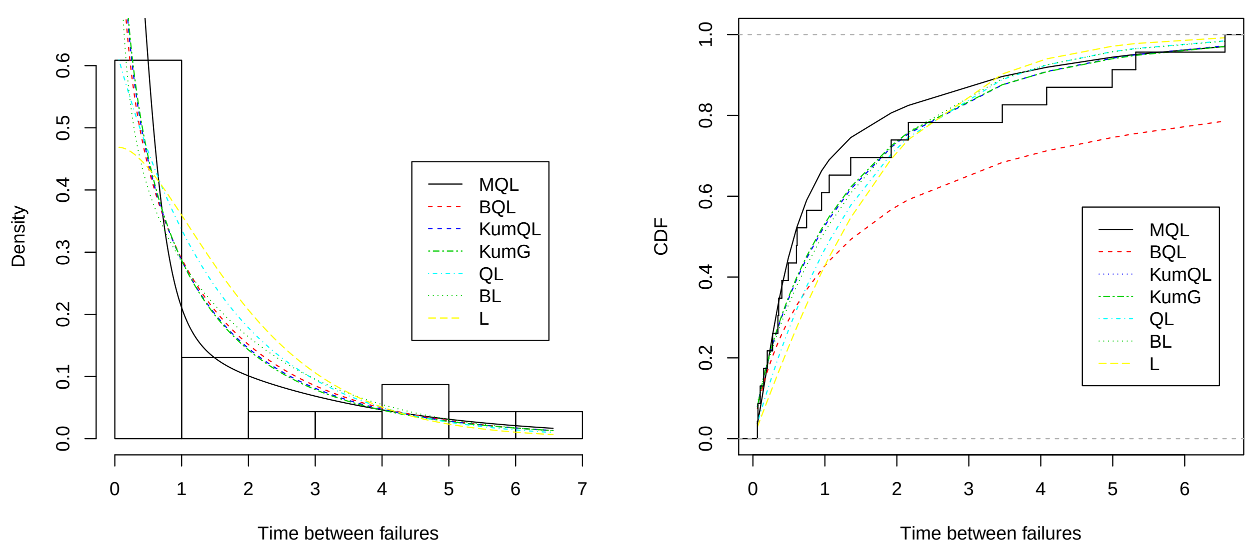

Fig. 3: Plots of the estimated pdfs and cdfs of KumQL, KumG, BQL, QL,BL and L models using the time between failures data.

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Appendix

We can compute the elements of the observed information matrix J for the five parameters (a,b,c,α,θ). We obtain the following:

$$J_{aa} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial a^2} = \frac{n\left(\psi^{(1)}(b+a/c)-\psi^{(1)}(a/c)\right)}{c^2},$$

$$J_{ab} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial a \partial b} = \frac{n\left(\psi^{(1)}(b+a/c)\right)}{c},$$

$$J_{ac} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial a \partial c} = \frac{n(-c\psi(b+a/c)+c\psi(a/c)+a(-\psi^{(1)}(b+a/c)+\psi^{(1)}(a/c))}{c^3},$$

$$J_{a\alpha} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial a \partial \alpha} = \sum_{i=1}^{n} \frac{\theta x_i e^{-\theta x_i}}{(\alpha+1)^2(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))},$$

$$J_{a\theta} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial a \partial \theta} = \sum_{i=1}^{n} \frac{-\frac{x_i e^{-\theta x_i}}{\alpha+1}+x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)},$$

$$J_{bb} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial b^2} = n(-\psi^1(b)+\psi^1(b+a/c)),$$

$$J_{bc} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial b \partial c} = \frac{-an(\psi^{(1)}(b+a/c))}{c^2} + \sum_{i=1}^{n} \log[\left(1+\frac{\theta x_i}{\alpha+1}\right) e^{-\theta x_i}],$$

$$J_{b\alpha} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial b \partial \alpha} = \sum_{i=1}^{n} \frac{-c\theta x_i}{(\alpha+1)^2\left(1+\frac{\theta x_i}{\alpha+1}\right)},$$

$$J_{b\theta} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial b \partial \theta} = \sum_{i=1}^{n} \frac{ce^{\theta x_i}(\frac{x_i e^{-\theta x_i}}{\alpha+1}-x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{\left(1+\frac{\theta x_i}{\alpha+1}\right)},$$

$$J_{cc} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial c^2} = -\frac{n(c^2+a\left[-2c\psi(b+a/c)+2c\psi(a/c)+a(-\psi^{(1)}(b+a/c)+\psi^{(1)}(a/c))\right]}{c^4},$$

$$J_{c\alpha} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial c \partial \alpha} = (b-1)\sum_{i=1}^{n} \frac{-\theta x_i}{(\alpha+1)^2\left(1+\frac{\theta x_i}{\alpha+1}\right)},$$

$$J_{c\theta} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial c \partial \theta} = (b-1)\sum_{i=1}^{n} \frac{e^{\theta x_i}[\frac{x_i e^{-\theta x_i}}{\alpha+1}-x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)]}{\left(1+\frac{\theta x_i}{\alpha+1}\right)},$$

$$\begin{aligned}
J_{\alpha\alpha} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial \alpha^2} &= \frac{n}{(1+\alpha)^2} + \sum_{i=1}^{n} \frac{-1}{\alpha+\theta x_i} \\
&+(b-1)\sum_{i=1}^{n}\left[-\frac{c\theta^2 x_i{}^2}{(1+\alpha)^4\left(1+\frac{\theta x_i}{\alpha+1}\right)} + \frac{2c\theta x_i}{(\alpha+1)^3\left(1+\frac{\theta x_i}{\alpha+1}\right)}\right] \\
&+(a-1)\sum_{i=1}^{n}\left[-\frac{e^{-2\theta x_i^2}\theta^2 x_i{}^2}{(1+\alpha)^4(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)^2)}\right. \\
&\left.-\frac{2e^{-\theta x_i}\theta x_i}{(\alpha+1)^3(1-e^{-\theta x_i}(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)}\right],
\end{aligned}$$

$$
\begin{aligned}
J_{\alpha\theta} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\alpha\partial\theta} &= \sum_{i=1}^{n} \frac{-x_i}{(\alpha+\theta x_i)^2} + (b-1)\sum_{i=1}^{n} \frac{ce^{\theta x_i}(\frac{-x_i e^{-\theta x_i}}{(\alpha+1)^2} + \frac{x_i e^{-\theta x_i}}{(\alpha+1)^2})}{\left(1+\frac{\theta x_i}{\alpha+1}\right)} \\
&+ \frac{c\theta x_i e^{\theta x_i}(\frac{x_i e^{-\theta x_i}}{\alpha+1} - x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{(1+\alpha)^2(1+\frac{\theta x_i}{\alpha+1})^2} \\
&+ (a-1)\sum_{i=1}^{n} \frac{\frac{x_i e^{-\theta x_i}}{(\alpha+1)^2} - \frac{\theta x_i e^{-\theta x_i}}{(\alpha+1)^2}}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)} - \frac{\theta x_i e^{\theta x_i}(-\frac{x_i e^{-\theta x_i}}{\alpha+1} + x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{(\alpha+1)^2(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))^2},
\end{aligned}
$$

$$
\begin{aligned}
J_{\theta\theta} = \frac{\partial^2 l(\boldsymbol{\Theta})}{\partial\theta^2} &= -\frac{n}{\theta^2} + \sum_{i=1}^{n} \frac{-x_i{}^2}{(\alpha+\theta x_i)^2} \\
&+ (a-1)\sum_{i=1}^{n}\left[-\frac{(-\frac{e^{-\theta x_i} x_i}{(1+\alpha)} + e^{-\theta x_i} x_i\left(1+\frac{\theta x_i}{\alpha+1}\right))^2}{(1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))^2} + \frac{\frac{2e^{-\theta x_i} x_i}{\alpha+1} - e^{-\theta x_i} x_i{}^2(1+\frac{\theta x_i}{\alpha+1})}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)}\right] \\
&+ (b-1)\sum_{i=1}^{n}\left[-\frac{ce^{\theta x_i} x_i(\frac{x_i e^{-\theta x_i}}{\alpha+1} - x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{(\alpha+1)\left(1+\frac{\theta x_i}{\alpha+1}\right)^2}\right. \\
&\left.+ \frac{ce^{\theta x_i} x_i(\frac{x_i e^{-\theta x_i}}{\alpha+1} - x_i e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{\left(1+\frac{\theta x_i}{\alpha+1}\right)} + \frac{ce^{\theta x_i}(-\frac{2x_i e^{-\theta x_i}}{\alpha+1} + x_i{}^2 e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right))}{\left(1+\frac{\theta x_i}{\alpha+1}\right)}\right].
\end{aligned}
$$

References

[1] C. Alexander, G.M. Cordeiro, E.M.M. Ortega and J.M. Sarabia, Computational Statistics and Data Analysis **56**, 1880-1897 (2012).
[2] H.S. Bakouch, B.M. Al-Zahrani, A.A. Al-Shomrani, V.A. Marchi and F. Louzada, Journal of the Korean Statistical Society **41**, 75-85 (2012).
[3] W. Barreto-Souza, A. H.S. Santos and G.M. Cordeiro, Journal of Statistical Computation and Simulation **80**, 159-172 (2010).
[4] G.M. Cordeiro and M.D. Castro, Journal of Statistical Computation and Simulation **81**, 883-898 (2011).
[5] G.M. Cordeiro and A.J. Lemonte, Journal of the Franklin Institute **349**, 1174-1197 (2012).
[6] G.M. Cordeiro, R.J. Cintra, L.C. Rêgo and E.M. Ortega, Pakistan Journal of Statistics and Operation Research **8**, 301-329 (2012a).
[7] G.M. Cordeiro, E.M. Hashimoto, E.M. Ortega and M.A. Pascoa, AStA Advances in Statistical Analysis **96**, 409–433 (2012b).
[8] G.M. Cordeiro and A.J. Lemonte, Statistics **48**, 182-199 (2014).
[9] G.M. Cordeiro, E.M. Hashimoto and E.M.M. Ortega, Statistics **48**, 256-278 (2014).
[10] D. R. Cox and D. V. Hinkley, Chapman and Hall, London, 1979.
[11] I. Elbatal and M. Elgarhy, International Journal of Mathematics Trends and Technology, **4**, 237-246 (2013).
[12] M.E. Ghitany, B. Atieh and S. Nadarajah, Mathematics and computers in simulation **78**, 493-506 (2008).
[13] I.S. Gradshteyn and I.M. Ryzhik, Academic Press, New York, 7th edition, 2007.
[14] E.M. Hashimoto, E.M. Ortega, G.M. Cordeiro and M.A. Pascoa, Journal of Statistical Theory and Practice **9**, 608-632 (2015).
[15] D.V. Lindley, Journal of the Royal Statistical Society, Series B **20**, 102-107 (1958).
[16] F.W. Marciano, A.D.C. Nascimento, M. Santos-Neto and G.M. Cordeiro, International Journal of Statistics and Probability **1**, 53-71 (2012).
[17] J.B. McDonald, Econometrica **52**, 647-663 (1984).
[18] F. Merovci and V. K. Sharma, Journal of Applied Mathematics (2014).
[19] G.S. Mudholkar and H. Wang, Journal of Statistical Planning and Inference **137**, 3655-3671 (2007).
[20] S. Nadarajah, H.S. Bakouch and R. Tahmasbi, Sankhya B **73**, 331-359 (2011).
[21] S. Rajasooriya, Electronic Theses and Dissertations Paper 45, (2013).
[22] R. Shanker and A. Mishra, African Journal of Mathematics and Computer Science Research **6**, 64-71 (2013).
[23] M.S. Suprawhardana and S. Prayoto, Atom Indones **25**, (1999).
[24] M. Tahir, M. Mansoor, M. Zubair and G. Hamedani, Journal of Statistical Theory and Applications **13**, 65-82 (2014).
[25] H. Zakerzadeh, A. Dolati, Journal of Mathematical Extension **3**, 13-25 (2009).