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Upper Bound of the Second Order Hankel Determinant for a Class of Functions with Positive Real Part

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Abstract: In this paper, we find estimates on the upper bound of the Hankel determinant for a general class $\Sigma_n(\varphi)$ of functions defined by the known Sălăgean derivative operator as well as the $\varphi(z)$, where $\varphi(z)$ are positive real part functions and symmetric with respect to the real axis. Several useful consequences are obtained as special cases.

Keywords: Univalent functions, subordination, Hankel determinant, sălăgean derivative operator

1 Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1}$$

which are analytic in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of all univalent functions. Suppose that functions f and $\mathscr{F}$ be in the class $\mathscr{S}$, then function f is said to be subordinate to $\mathscr{F}$, or equivalently, the function $\mathscr{F}$ is said to be superordinate to f, if there exists a Schwarz function ω with $\omega(0)=0,|\omega(z)|<1(z\in\Delta)$ such that $f(z)=\mathscr{F}(\omega(z))$, written as $f(z)\prec\mathscr{F}(z)$.(see, for details, [8]).

For $f(z)\in\mathscr{A}$, Sălăgean [20] defined the following operator:

$$D^0f(z)=f(z),\ D^1f(z)=Df(z)=zf'(z),...,$$

$$D^nf(z)=D(D^{n-1}f(z)),$$

where $n\in N=\{1,2,...,\}$. We note that

$$D^nf(z)=z+\sum_{k=2}^{\infty}k^na_kz^k, n\in N_0=0\cup N. \tag{2}$$

In [13], Noonan and Thomas defined the qth Hankel determinant of $f(z)$ for $q\geq 1$ and $n\geq 1$ as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\\ . & & & \end{vmatrix}.$$

The Hankel determinant plays an important role in the study of the singularities and in the study of power series with integral coefficients. Also, Arif et al.[3] determined the rate of growth of the Hankel determinant $H_q(n)$ as $n\to\infty$ for different classes of functions. In fact, many authors(see[2,5,11,10,13,14,15]) have obtained sharp upper bounds of $H_2(2)$ for several classes of analytic functions.

We denote by $\mathscr{P}$ a class of analytic function in Δ with $\mathscr{P}(0)=0$ and $\mathscr{P}(z)>0$. Here we assume that $\varphi(z)\in\mathscr{P}$, satisfying $\varphi(0)=0$, $\varphi'(0)>0$ and $\varphi(\Delta)$ is symmetric with respect to the real axis. Also, $\varphi(z)$ has a series expansion of the form

$$\varphi(z)=1+\mathscr{B}_1z+\mathscr{B}_2z^2+\mathscr{B}_3z^3+...,(\mathscr{B}_1>0). \tag{3}$$

Using the Sălăgean operator and the $\varphi(z)$ defined as (3), we introduce the class $\Sigma_n(\varphi)$ as follows:

$$\Sigma_n(\varphi)=\left\{f\in\mathscr{A}:\frac{D^{n+1}f(z)}{D^nf(z)}\prec\varphi(z),z\in\Delta\right\}. \tag{4}$$

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By giving specific values to the parameters n and $\varphi(z)$ with $\Sigma_n(\varphi)$, we can obtain the following important subclasses studied by various authors in earlier works(see, for details, [1,4,9,12,16,17,18,19,21,22]), for instance,

$$\Sigma_0(\varphi) \equiv S^*(\varphi),\ \Sigma_1(\varphi) \equiv \mathscr{K}(\varphi);$$

$$\Sigma_0\left(\frac{1+Az}{1+Bz}\right) \equiv S^*(A,B),\ \Sigma_1\left(\frac{1+Az}{1+Bz}\right) \equiv \mathscr{K}(A,B);$$

$$\Sigma_0\left(\frac{1+(1-2\alpha)z}{1-z}\right) \equiv S^*(\alpha),\ \Sigma_1\left(\frac{1+(1-2\alpha)z}{1-z}\right) \equiv \mathscr{K}(\alpha);$$

$$\Sigma_0\left(\frac{1+z}{1-z}\right) \equiv S^*,\ \Sigma_1\left(\frac{1+z}{1-z}\right) \equiv \mathscr{K};$$

$$\Sigma_0\left((\frac{1+Az}{1+Bz})^\gamma\right) \equiv S^*_\gamma,\ \Sigma_1\left((\frac{1+Az}{1+Bz})^\gamma\right) \equiv \mathscr{K}_\gamma.$$

In the present paper, we study the Hankel determinant $H_2(2) = |a_2a_4 - a_3^2|$ for class $\Sigma_n(\varphi)$. Throughout this paper, for convenience of notation, we use the following notations:

$$\begin{aligned}
d_1 &= \mathscr{B}_1[2(\mathscr{B}_1+\mathscr{B}_3)+2\mathscr{B}_1\mathscr{B}_2+\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)\\
&\quad -2\mathscr{B}_1^2-4\mathscr{B}_2],\\
d_2 &= \mathscr{B}_1[8(\mathscr{B}_2-\mathscr{B}_1)+6\mathscr{B}_1^2],\\
d_3 &= (\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)^2,\\
d_4 &= 4\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1),\\
K_1 &= d_1-\frac{3}{2}(\frac{8}{9})^n d_3,\ K_2 = d_2-\frac{3}{2}(\frac{8}{9})^n d_4,\\
K_3 &= 8\mathscr{B}_1^2,\ K_4 = -6(\frac{8}{9})^n\mathscr{B}_1^2.
\end{aligned} \tag{5}$$

2 Preliminary Results

To prove our main results we need the following Lemmas:

Lemma 1.[6] *Let the function $\mathscr{P}_1 \in \mathscr{P}$, and be given by the power series $\mathscr{P}_1(z) = 1+\mathscr{E}_1z+\mathscr{E}_2z^2+\mathscr{E}_3z^3+\ldots(z\in\Delta)$, then*

$$2\mathscr{E}_2 = \mathscr{E}_1^2 + x(4-\mathscr{E}_1^2) \tag{6}$$

for some x, $|x|\le 1$, where the $\mathscr{P}$ is the class of functions with positive real part.

Lemma 2.[7] *Let the function $f(z)$ given by (1) be in the class $S^*[A,B]$. Then we have*

$$|a_k| \le \begin{cases} 1, & k=1,\\ \dfrac{1}{(k-1)!}\prod_{i=0}^{k-2}(A-B+i), & k\ge 2. \end{cases}$$

The bound is sharp.

Lemma 3.[8] *If $p\in\mathscr{P}$ and $p(z) = 1+\sum\limits_{n=1}^{\infty} p_nz^n, z\in\Delta$, then $|p_n| \leqslant 2$.*

3 Main Results

Theorem 1.*Let the function $f(z)\in\Sigma_n(\varphi)$ and $\mathscr{M} = |a_2a_4-a_3^2|$, we can obtain*

(1) If $0\le\sqrt{\frac{M_2}{M_1}}\le 2$, then

$$\mathscr{M} \le \max\left\{4T|K_4|, \mathscr{Q}^*, 8T|2K_1+K_2|+T|4K_3+K_4|\right\},$$

(2) If $\sqrt{\frac{M_2}{M_1}} > 2$ or $\frac{M_2}{M_1} < 0$ or $M_1 = 0$, then

$$\mathscr{M} \le \max\left\{4T|K_4|, 8T|2K_1+K_2|+T|4K_3+K_4|\right\},$$

where K_1,K_2,K_3,K_4 are defined by (5) and $T = \frac{1}{96}\frac{1}{8^n}$, $M_1 = 2T|2K_1+K_2|+T|K_4|-2T|K_2+K_4|$, $M_2 = 4T|K_4|-4T|K_2+K_4|-\frac{T}{2}|4K_3+K_4|$ and $\mathscr{Q}^* = \frac{T}{2}|2K_1+K_2|(\frac{M_2}{M_1})^2+\frac{T}{4}|4K_3+K_4|\frac{M_2}{M_1}+\frac{T}{2}|K_2+K_4|\frac{M_2}{M_1}(4-\frac{M_2}{M_1})+\frac{T}{4}|K_4|(4-\frac{M_2}{M_1})^2$.

*Proof.*If $f(z)\in\Sigma_n(\varphi)$, from (3) and (4), then there exists a Schwarz function $w(z)$, such that

$$\frac{D^{n+1}f(z)}{D^nf(z)} = \varphi(w(z))(z\in\Delta), \tag{7}$$

where $\varphi(z) = 1+\mathscr{B}_1z+\mathscr{B}_2z^2+\mathscr{B}_3z^3+\ldots(z\in\Delta)$.

Define the function $\mathscr{P}_1(z)$ by

$$\mathscr{P}_1(z) = \frac{1+w(z)}{1-w(z)} = 1+\mathscr{E}_1z+\mathscr{E}_2z^2+\ldots \prec \frac{1+z}{1-z}, \tag{8}$$

we can see that the $\mathscr{P}_1(z)$ is a function with positive real part. In view of the (8), we have

$$\begin{aligned}
\varphi(w(z)) &= \varphi\left(\frac{\mathscr{P}_1(z)-1}{\mathscr{P}_1(z)+1}\right) = \varphi\left(\frac{\mathscr{E}_1z+\mathscr{E}_2z^2+\mathscr{E}_3z^3+\ldots}{2+\mathscr{E}_1z+\mathscr{E}_2z^2+\mathscr{E}_3z^3+\ldots}\right)\\
&= \varphi\left(\frac{1}{2}\mathscr{E}_1z+\frac{1}{2}(\mathscr{E}_2-\frac{\mathscr{E}_1^2}{2})z^2+\frac{1}{2}(\mathscr{E}_3-\mathscr{E}_1\mathscr{E}_2+\frac{\mathscr{E}_1^3}{4})z^3+\ldots\right)\\
&= 1+\frac{\mathscr{B}_1\mathscr{E}_1}{2}z+\left[\frac{\mathscr{B}_1}{2}\left(\mathscr{E}_2-\frac{\mathscr{E}_1^2}{2}\right)+\frac{\mathscr{B}_2\mathscr{E}_1^2}{4}\right]z^2+\\
&\left[\frac{\mathscr{B}_1}{2}\left(\mathscr{E}_3-\mathscr{E}_1\mathscr{E}_2+\frac{\mathscr{E}_1^3}{4}\right)+\frac{\mathscr{B}_2\mathscr{E}_1}{2}\left(\mathscr{E}_2-\frac{\mathscr{E}_1^2}{2}\right)+\frac{\mathscr{B}_3\mathscr{E}_1^3}{8}\right]z^3+\ldots
\end{aligned} \tag{9}$$

From the (7), we obtain

$$D^{n+1}f(z) = D^nf(z)\,.\,\varphi(w(z)). \tag{10}$$

After a simple calculation and using (2) and (9), the (10) yields the relationship:

$$\begin{aligned}
z+\sum_{k=2}^{\infty}k^{n+1}a_kz^k &= (z+\sum_{k=2}^{\infty}k^na_kz^k)\,.\left\{1+\frac{\mathscr{B}_1\mathscr{E}_1}{2}z\right.\\
&+\left[\frac{\mathscr{B}_1}{2}(\mathscr{E}_2-\frac{\mathscr{E}_1^2}{2})+\frac{\mathscr{B}_2\mathscr{E}_1^2}{4}\right]z^2+\left[\frac{\mathscr{B}_1}{2}(\mathscr{E}_3-\mathscr{E}_1\mathscr{E}_2+\frac{\mathscr{E}_1^3}{4})\right.\\
&\left.\left.+\frac{\mathscr{B}_2\mathscr{E}_1}{2}(\mathscr{E}_2-\frac{\mathscr{E}_1^2}{2})+\frac{\mathscr{B}_3\mathscr{E}_1^3}{8}\right]z^3+\ldots\right\}.
\end{aligned} \tag{11}$$

Following the (11) and equating coefficients, we obtain

$$
\begin{aligned}
&a_2=\frac{1}{2^{n+1}}\mathscr{B}_1\mathscr{E}_1,\ a_3=\frac{1}{8}\frac{1}{3^n}[2\mathscr{B}_1\mathscr{E}_2+\mathscr{E}_1^2(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)],\\
&a_4=\frac{1}{48}\frac{1}{4^n}\{[2(\mathscr{B}_1+\mathscr{B}_3)+2\mathscr{B}_1\mathscr{B}_2+\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)\\
&-2\mathscr{B}_1^2-4\mathscr{B}_2]\mathscr{E}_1^3+[8(\mathscr{B}_2-\mathscr{B}_1)+6\mathscr{B}_1^2]\mathscr{E}_1\mathscr{E}_2+8\mathscr{B}_1\mathscr{E}_1\}.
\end{aligned}\tag{12}
$$

From (12), therefore we have

$$
\begin{aligned}
a_2a_4-a_3^2=&\frac{1}{96}\frac{1}{8^n}\{\mathscr{B}_1[2(\mathscr{B}_1+\mathscr{B}_3)+2\mathscr{B}_1\mathscr{B}_2\\
&+\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)-2\mathscr{B}_1^2-4\mathscr{B}_2]\mathscr{E}_1^4\\
&+\mathscr{B}_1[8(\mathscr{B}_2-\mathscr{B}_1)+6\mathscr{B}_1^2]\mathscr{E}_1^2\mathscr{E}_2+8\mathscr{B}_1^2\mathscr{E}_1^2\}\\
&-\frac{1}{64}\frac{1}{9^n}\{4\mathscr{B}_1^2\mathscr{E}_2^2+(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)^2\mathscr{E}_1^4\\
&+4\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)\mathscr{E}_2\mathscr{E}_1^2\}.
\end{aligned}\tag{13}
$$

Using (13), we get

$$
\begin{aligned}
|a_2a_4-a_3^2|=&\frac{1}{96}\frac{1}{8^n}|\mathscr{B}_1[2(\mathscr{B}_1+\mathscr{B}_3)+2\mathscr{B}_1\mathscr{B}_2\\
&+\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)-2\mathscr{B}_1^2-4\mathscr{B}_2]\mathscr{E}_1^4\\
&+\mathscr{B}_1[8(\mathscr{B}_2-\mathscr{B}_1)+6\mathscr{B}_1^2]\mathscr{E}_1^2\mathscr{E}_2+8\mathscr{B}_1^2\mathscr{E}_1^2\\
&-\frac{3}{2}(\frac{8}{9})^n\{4\mathscr{B}_1^2\mathscr{E}_2^2+(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)^2\mathscr{E}_1^4\\
&+4\mathscr{B}_1(\mathscr{B}_1^2+\mathscr{B}_2-\mathscr{B}_1)\mathscr{E}_2\mathscr{E}_1^2\}|\\
=&T|d_1\mathscr{E}_1^4+d_2\mathscr{E}_1^2\mathscr{E}_2+8\mathscr{B}_1^2\mathscr{E}_1^2-P\{4\mathscr{B}_1^2\mathscr{E}_2^2\\
&+d_3\mathscr{E}_1^4+d_4\mathscr{E}_2\mathscr{E}_1^2\}|\\
=&T|(d_1-Pd_3)\mathscr{E}_1^4+(d_2-Pd_4)\mathscr{E}_1^2\mathscr{E}_2\\
&+8\mathscr{B}_1^2\mathscr{E}_1^2-4P\mathscr{B}_1^2\mathscr{E}_2^2|\\
=&T|K_1\mathscr{E}_1^4+K_2\mathscr{E}_1^2\mathscr{E}_2+K_3\mathscr{E}_1^2+K_4\mathscr{E}_2^2|,
\end{aligned}\tag{14}
$$

where $T=\frac{1}{96}\frac{1}{8^n}, P=\frac{3}{2}(\frac{8}{9})^n$ and $d_1,d_2,d_3,d_4,K_1,K_2,K_3,K_4$ are defined as (5).

Since the functions $\mathscr{P}_1(z)$ are members of the class $\mathscr{P}$ of functions with positive real part, we assume without loss of generality that $\mathscr{E}_1>0$. For convenience of notation, we take $\mathscr{E}_1=c(c\in[0,2],$ *Lemma* 3). Also, substituting the values of $\mathscr{E}_2$ from (6) in (14), we have

$$
\begin{aligned}
|a_2a_4-a_3^2|=&T\left|K_1c^4+K_2c^2\frac{c^2+x(4-c^2)}{2}+K_3c^2\right.\\
&\left.+K_4\frac{c^4+x^2(4-c^2)^2+2xc^2(4-c^2)}{4}\right|\\
=&\frac{T}{4}\Big|(4K_1+2K_2)c^4+(4K_3+K_4)c^2\\
&+2(K_2+K_4)\mu c^2(4-c^2)+K_4\mu^2(4-c^2)^2\Big|\\
\leqslant&\frac{T}{4}|4K_1+2K_2|c^4+\frac{T}{4}|4K_3+K_4|c^2+\frac{T}{2}|K_2\\
&+K_4|\mu c^2(4-c^2)+\frac{T}{4}|K_4|\mu^2(4-c^2)^2\\
=&F(c,\mu)
\end{aligned}\tag{15}
$$

with $\mu=|x|\leqslant 1$.

Next, we assume that the upper bound for $F(c,\mu)$ occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we have

$$
\frac{\partial F}{\partial\mu}=\frac{T}{2}|K_2+K_4|c^2(4-c^2)+\frac{T}{2}|K_4|\mu(4-c^2)^2. \tag{16}
$$

We observe that $\frac{\partial F}{\partial\mu}>0$. Therefore $F(c,\mu)$ is an increasing function of μ, this contradicts our assumption that the maximum value of $F(c,\mu)$ occurs at an interior point of the rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$
\begin{aligned}
\max F(c,\mu)=F(c,1)=&\frac{T}{4}|4K_1+2K_2|c^4+\frac{T}{4}|4K_3+K_4|c^2\\
&+\frac{T}{2}|K_2+K_4|c^2(4-c^2)+\frac{T}{4}|K_4|(4-c^2)^2=G(c).
\end{aligned}\tag{17}
$$

Next, differentiating $G(c)$ with respect to c, it gives

$$
\begin{aligned}
G'(c)=&2T|2K_1+K_2|c^3+\frac{T}{2}|4K_3+K_4|c\\
&+2T|K_2+K_4|(2-c^2)c-T|K_4|(4-c^2)c\\
=&c[2T|2K_1+K_2|c^2+\frac{T}{2}|4K_3+K_4|\\
&+2T|K_2+K_4|(2-c^2)-T|K_4|(4-c^2)]\\
=&c[(2T|2K_1+K_2|+T|K_4|-2T|K_2+K_4|)c^2\\
&+4T|K_2+K_4|+\frac{T}{2}|4K_3+K_4|-4T|K_4|].
\end{aligned}\tag{18}
$$

Now, taking $M_1=2T|2K_1+K_2|+T|K_4|-2T|K_2+K_4|$, $M_2=4T|K_4|-4T|K_2+K_4|-\frac{T}{2}|4K_3+K_4|$, we need to discuss with different cases:

Case 1: If $M_1=0, M_2\neq 0$, then $G'(c)=0$ implies that the real critical point $c^*=0$. Also, if $M_1=0, M_2=0$, we can note that $G'(c)\equiv 0$, thus, the $G(c)$ is a constant. In words, the maximum value of $G(c)$ occurs at $c=0$ or $c=2$.

Case 2: If $\frac{M_2}{M_1} > 0$, then $G'(c) = 0$ implies that the real critical point $c^* = 0$ or $c^* = \sqrt{\frac{M_2}{M_1}}$. Suppose that $0 \le c^* \le 2$, Since $G(c)$ is a continuous function in $[0,2]$, so we can know that the maximum value of $G(c)$ occurs at $c = 0$, or $c = c^*$, or $c = 2$. Furthermore, we have $G(0) = 4T|K_4|$, $G(2) = 8T|2K_1 + K_2| + T|4K_3 + K_4|$ and $\mathscr{Q}^* = G(c^*)$. Hence $|a_2a_4 - a_3^2| \le \max\{4T|K_4|, \mathscr{Q}^*, 8T|2K_1 + K_2| + T|4K_3 + K_4|\}$.

Case 3: Suppose that $c^* > 2$ or $\frac{M_2}{M_1} < 0$, this implies there isn't real critical in $c \in (0,2)$, in other words, $G(c)$ is a monotone function on c, so we have $|a_2a_4 - a_3^2| \le \max\{4T|K_4|, 8T|2K_1 + K_2| + T|4K_3 + K_4|\}$. This completes the proof of the Theorem.

4 Some consequences of the main results

In this section, we give some applications of the results produced in the third section.

Corollary 1. *Let the function $f(z)$ given by (1) be in the class S^*, then we have $|a_2a_4 - a_3^2| \le \frac{29}{12} \approx 2.417$.*

Proof. Letting $n = 0$, $\varphi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \ldots$ in Theorem 1, then $K_1 = -8, K_2 = 0, K_3 = 32, K_4 = -24, T = \frac{1}{96}, M_1 = \frac{1}{12}, M_2 = -\frac{13}{24}$, thus, it is easy to obtain the estimate on S^*.

Corollary 2. *Let the function $f(z)$ given by (1) be in the class $\mathscr{K}$, then we have $|a_2a_4 - a_3^2| \le \frac{7}{36} \approx 0.194$.*

Proof. Letting $n = 1$, $\varphi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \ldots$ in Theorem 1, then $K_1 = -\frac{16}{3}, K_2 = \frac{16}{3}, K_3 = 32, K_4 = -\frac{64}{3}, T = \frac{1}{768}, M_1 = 0, M_2 = -\frac{1}{24}$, thus, it is easy to obtain the estimate on $\mathscr{K}$.

Corollary 3. *Let the function $f(z)$ given by (1) be in the class $S^*_{\frac{1}{2}}$, then we have $|a_2a_4 - a_3^2| \le \frac{1}{3} \approx 0.333$.*

Proof. Letting $n = 0$, $\varphi(z) = (\frac{1+z}{1-z})^{\frac{1}{2}} = 1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \ldots$ in Theorem 1, then $K_1 = \frac{1}{8}, K_2 = -1, K_3 = 8, K_4 = -6, T = \frac{1}{96}, M_1 = -\frac{13}{192}, M_2 = -\frac{5}{12}, \frac{M_2}{M_1} \approx 6.15 > 4$, thus, it is easy to obtain the estimate on $S^*_{\frac{1}{2}}$.

Corollary 4. *Let the function $f(z)$ given by (1) be in the class $\mathscr{K}_{\frac{1}{2}}$, then we have $|a_2a_4 - a_3^2| \le \frac{1}{24} \approx 0.0417$.*

Proof. Letting $n = 1, \varphi(z) = (\frac{1+z}{1-z})^{\frac{1}{2}} = 1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \ldots$ in Theorem 1, then $K_1 = \frac{1}{6}, K_2 = -\frac{2}{3}, K_3 = 8, K_4 = -\frac{16}{3}, T = \frac{1}{768}, M_1 = -\frac{1}{128}, M_2 = -\frac{1}{48}, \frac{M_2}{M_1} \approx 2.67 < 4$, thus, it is easy to obtain the estimate on $\mathscr{K}_{\frac{1}{2}}$.

Remark (1) In fact, we can know the corresponding bounds for kinds of classes of functions $f(z)$ by Theorem 1, where $f(z) \in \{S^*(\varphi), \mathscr{K}(\varphi), S^*[A,B], \mathscr{K}[A,B], S^*(\alpha), \mathscr{K}(\alpha), S^*_\gamma, \mathscr{K}_\gamma, -1 \leqslant B < A \leqslant 1, 0 \leqslant \alpha < 1, 0 < \gamma \leqslant 1\}$. Furthermore, obviously, various other interesting consequences of our main results can be derived by appropriately specializing those parameters as A, B, α, γ, n and function $\varphi(z)$.

(2) It seems that we can estimate the Hankel determinant $|a_2a_4 - a_4^2|$ for class $\Sigma^{n,m}_{\lambda_1,\lambda_2}(A,B)$ by Lemma 2, but the the results are too rough relative to the corresponding inequalities with Theorem 1. For this point, we gives a special example: let $f(z) \in \mathscr{S}^*[1,-1] \equiv S^*$, by Lemma 2, then we have $|a_2| \le 2, |a_3| \le 3, |a_4| \le 4$, thus, we observe that $|a_2a_4 - a_3^2| \le |a_2a_4| + |a_3|^2 \le 8 + 9 = 17$. Furthermore, in Corollary 1, we have $|a_2a_4 - a_3^2| \le \frac{29}{12} \approx 2.417$. It is obvious that $2.417 < 17$.

5 Conclusions

This paper gives an estimation of the coefficients for a general class of analytic functions, which is related to the Hankel determinant of order two. In fact, with giving some applications of the main results, various other interesting consequences (old and new) can be derived by appropriately specializing parameters. In corresponding places the obtained results have been supplemented by Corollarys and Remarks.

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