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# Spin-Lattices as a Test Object for Quantum Foundations and Quantum Information

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**Abstract:** Here we show that spin-lattices can serve as a test object for quantum foundations and quantum information theory, revealing unsuspected properties. We propose a (thought) experiment on such lattices that mimics the Bell-experiment and calculate the outcomes and the Bell inequality. It appears that for a wide range of geometries the Bell-inequality is violated, even if these lattices are local according to the usual definitions. Some spin-1/2 lattices can even violate the Bell inequality more than the Tsirelson bound. We argue that there is no consensual explanation for this result. We make the link with problems in quantum foundations and quantum information theory.
**KeyWords:** Quantum Foundation, Quantum Information

## 1. Introduction

The standard or Copenhagen interpretation of quantum mechanics asserts that quantum theory is intrinsically indeterministic and cannot be causally completed. The question of completing quantum mechanics by a deeper-lying theory is posed in the sharpest manner by Bell's theorem [1-2]. In a slogan, Bell's theorem proves that any local hidden-variable model for a certain experiment will satisfy an inequality, while quantum mechanics predicts violation of the inequality. Since experiments have confirmed the quantum prediction, hidden-variable theories are nowadays considered to be a dead-end.

Now the Bell-inequality is only rarely tested on real-world physical systems that could, a priori, serve as a model for a sub-quantum hidden-variable reality. G. 't Hooft, a physics Nobel laureate, has recently argued that in the Cellular Automaton Theory for quantum mechanics, the Bell-inequality will be violated [17]. Here we will test the Bell-inequality in spin-lattices, by proposing a Bell-type (thought) experiment on such lattices. Note that a spin-lattice can be seen as a simple version of a cellular automaton. It can serve as a 'hidden-variable model' in the sense that it allows one to calculate the experimental Bell-probabilities $P(\sigma_1,\sigma_2|a,b)$, $P(\sigma_1|a)$ etc. (definitions given below) by summing over probabilities as $P(\sigma_1|a, \lambda)$ etc.; the hidden variables $\lambda$ are the spins on intermediate nodes (cf. Fig 1). We will calculate that for certain geometries the Bell-inequality can be violated in such an experiment (Sections 2-3). Perhaps surprisingly, some Ising lattices violate the Bell-inequality more than the Tsirelson bound [3]. Yet spin-lattices are local according to the usual definitions, as will be shown in Section 4.

Several explanations can be advanced for this result, but we will argue that any explanation contradicts a basic assumption in the general quantum foundations literature (Section 5). Therefore it seems that at least one of these basic wisdoms will have to be given up, or at least relaxed. We will also succinctly make a link with quantum information theory, where the Bell inequality plays e.g. an essential role as safeguard for secure communication. Note that terms as 'locality' (absence of superluminal interactions) will be used here following the physics jargon rather than the information-theoretic jargon, where it can have different be it meaningfully related meanings [4-9].

In the following I will focus on the classical spin-1/2 Ising Hamiltonian. In [10] some results on 2-D lattices were calculated numerically. Here I will present for the first time analytic results on 1-D and 2-D lattices and give a complete analysis of the results.

As an introduction, let us recall that, in a stochastic system, the Bell-inequality is derived starting from three assumptions – which are believed to describe any stochastic and local hidden-variable model for the Bell experiment. These three assumptions are usually termed 'outcome independence' (OI), 'parameter independence' (PI) and 'measurement independence' (MI) and are defined as follows [11, 8-9]:

$$P(\sigma_1|\sigma_2,a,b,\lambda) = P(\sigma_1|a,b,\lambda) \text{ for all } (\lambda,\sigma_1,\sigma_2) \text{ (OI)}, \tag{1a}$$

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$P(\sigma_1|a,b,\lambda) = P(\sigma_1|a,\lambda)$ for all $\lambda$ and similarly for $\sigma_2$ (PI), (1b)

$\rho(\lambda|a,b) = \rho(\lambda|a',b') \equiv \rho(\lambda)$ for all $(\lambda,a,b,a',b')$ (MI). (1c)

Here $\rho$ is the probability distribution of the hidden-variable set $\lambda$; a and a' are values of the left analyzer angle, and b and b' of the right analyzer angle; and $\sigma_1$ and $\sigma_2$ are the left and right measurement results, say spin values ('P' is a conditional probability). The set (or n-vector) of variables $\lambda$ may contain discrete or continuous variables; we might split them up in 'left' and 'right' variables; all such cases are subsumed in (1c). The conditions OI and PI are generally believed to express locality; in conjunction they form the factorability condition for local hidden-variable theories first proposed by Clauser and Horne [12], cf. Eq. (20) below.

The third condition, MI, is also a condition of stochastic independence. It is usually deemed 'obvious' because violating it would mean that the hidden variables $\lambda$ depend on (a,b), which means by standard rules of probability calculus that the analyzer angles (a,b) depend on the variables $\lambda$. But (a,b) can be freely or randomly chosen in experiments – so how could these angles depend on the $\lambda$ (variables which moreover determine the probabilities for the left and right outcomes) ? Ergo, MI must hold, unless one accepts a conspiratorial world. I will come back in detail to this subtle point, which will appear to be at the heart of our investigation.

## 2. Spin-lattices and the Bell inequality

Let us devise an experiment on a 2-D spin-lattice; such lattices are an approximate description of a variety of realistic systems, the best known case occurring in magnetic layers [13-14]. The experiment we consider is likely not yet feasible in the laboratory, so it is a thought experiment. Suppose, then, that Alice and Bob perform a Bell-type test on an ensemble of spin-lattices as schematized in Fig. 1.

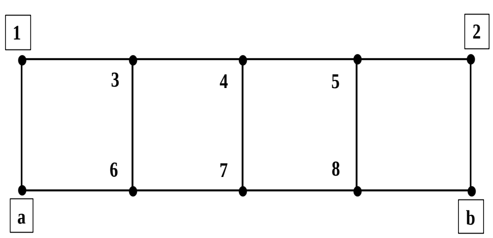


**Fig. 1.** 10 spins on a lattice

10 ions or electrons sit on a lattice, each having a spin $\sigma_i$ ($=\pm1$ for i =a, b, 1,…, 8). The Hamiltonian is the classical spin-1/2 Ising Hamiltonian (the $\sigma_i$ are numbers, not operators):

$$H(\theta) = -\Sigma_{i,j} J_{ij}.\sigma_i.\sigma_j - \Sigma_i h_i.\sigma_i. \quad (4)$$

Here $\theta$ is a 10-spin configuration $(\sigma_a,\sigma_b,\sigma_1,\ldots \sigma_8)$, the $h_i$ are local magnetic fields, and the $J_{ij}$ are the interaction constants between spin-i and spin-j, as usual assumed to be zero beyond nearest neighbours. Note that the Ising model explicitly relies only on local interaction between nearest neighbours[1], and that Hamiltonian (4) also describes a variety of purely classical phenomena: the $\sigma_i$ can represent atomic occupation in a 'lattice gas' or a crystal, deviation from equilibrium position in a network of springs, etc. [13]. The probability of a given spin configuration (at fixed temperature $1/\beta$) is the Boltzmann probability:

$$P(\theta) = e^{-\beta H(\theta)} / Z, \text{ with } Z = \Sigma_\theta e^{-\beta H(\theta)}, \text{ the partition function.} \quad (5)$$

We assume that Alice and Bob share an ensemble of such lattices, each in thermal equilibrium at the fixed temperature $1/\beta$. Suppose Alice has the means to measure the spin ($\pm1$) on nodes 1 and a, and Bob on 2 and b. They can then empirically determine joint probabilities as $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b) \equiv P(\sigma_1=\varepsilon_1, \sigma_2=\varepsilon_2 |\sigma_a=\varepsilon_a, \sigma_b=\varepsilon_b)$ (all $\varepsilon_i = \pm1$) simply by sitting together and counting relative frequencies over the ensemble. These 16 probabilities are the only ones needed to verify the Bell Inequality (BI), with $(\sigma_a,\sigma_b)$ taking the role of (a,b) and $\lambda \equiv \sigma_\lambda \equiv (\sigma_3,\sigma_4,\ldots\sigma_8)$ (or any subset of this set), as recalled below (cf. Eq. (6-7)).

Let us pause a moment and note that Alice and Bob could, in principle, do two equivalent experiments (Ex1 and Ex2) to determine the needed probabilities, just as in real Bell experiments. Either (Ex1) they 'postselect' 4 sub-ensembles out of one long run, each sub-ensemble corresponding to one of the 4 possible couples of $(\sigma_a, \sigma_b)$-values. They then determine $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b)$ within each sub-ensemble by counting relative frequencies. But if we push the thought experiment a little farther, and if Bob and Alice would have sufficiently sophisticated means to *control* $\sigma_a$ and $\sigma_b$, i.e. set $\sigma_a$ and $\sigma_b$ to either +1 or -1 at their free choice and keep these spins fixed, they could do 4 *consecutive* experiments each corresponding to a fixed value of $\sigma_a$ and $\sigma_b$ (Ex2). Then the probabilities $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b)$ determined in Ex2 should be identical (at $T \neq 0$) to those obtained in Ex1: the dynamics (Eq. (4-5)) is the same. (Actually Ex2 corresponds to how the first Bell experiments were done; most modern tests implement the scheme of Ex1.) But let us here first focus on Ex1, which does not require additional assumptions.

All probabilities just mentioned can be calculated. The Bell Inequality reads:

[1] The interaction $J_{ij}$ is of course not a direct spin-spin interaction. The interaction is mediated through a force, in the case of magnetic Ising lattices the Coulomb potential, as is well explained in [14].

$$X_{BI} = M(a,b) + M(a',b) + M(a,b') - M(a',b') \le 2 \quad \forall\, (a,a',b,b'). \tag{6}$$

For our experiment:

$$\begin{aligned} M(a,b) &= \langle \sigma_1.\sigma_2 \rangle_{a,b} \\ &= \sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1} \sigma_1.\sigma_2.P(\sigma_1,\sigma_2|\sigma_a,\sigma_b) \\ &= P(+,+|\sigma_a,\sigma_b) + P(-,-|\sigma_a,\sigma_b) - P(+,-|\sigma_a,\sigma_b) - P(-,+|\sigma_a,\sigma_b). \end{aligned} \tag{7}$$

Thus we can calculate $X_{BI}$ in (6) if we choose a ≡ b ≡ +1 and a' ≡ b' ≡ –1. In (7) we have:

$$P(\sigma_1,\sigma_2|\sigma_a,\sigma_b) = \frac{P(\sigma_1,\sigma_2,\sigma_a,\sigma_b)}{P(\sigma_a,\sigma_b)} \equiv \frac{P(\eta_1)}{P(\eta_2)}, \tag{8}$$

where $\eta_1$ is a 4-spin configuration and $\eta_2$ a 2-spin configuration. Any probability P(η) with η an m-spin configuration (m≤10) is given by:

$$P(\eta) = \sum_{\theta(\eta)}^{2^{10-m}} P(\theta), \tag{9}$$

where the sum runs over the $2^{10-m}$ 10-spin configurations θ(η) that contain η. P(θ) is the Boltzmann factor in (5). Thus we find:

$$P(\sigma_1,\sigma_2 \,|\, \sigma_a,\sigma_b) = \frac{\sum_{\theta(\eta_1)}^{2^6} e^{-\beta H(\theta)}}{\sum_{\theta(\eta_2)}^{2^8} e^{-\beta H(\theta)}} \tag{10}$$

Probabilities (10) are analytically tractable if one assumes that all $J_{ij}$ are equal (all $J_{ij}$ = J for nearest neighbours) and all $h_i$ = 0. Below the sums $\sum_{i,j}$ run over the 13 first-neighbour pairs (i,j) = (1,a), (1,3), (a,6),…, (2,b) as one reads on Fig. 1. In the sum $\sum_{\sigma_3,\sigma_4\ldots\sigma_8}$ all spin variables ($\sigma_3,\sigma_4,\ldots,\sigma_8$) run over the values +1 and -1. The numerator of Eq. (8) or (10) becomes:

$$\begin{aligned} Z.P(\eta_1) &= \sum_{\theta(\eta_1)}^{2^6} e^{-\beta H(\theta)} = \sum_{\sigma_3\sigma_4\ldots\sigma_8} e^{\beta J\sum_{i,j}\sigma_i\sigma_j} = \sum_{\sigma_3\sigma_4\ldots\sigma_8}\prod_{i,j} e^{\beta J\sigma_i\sigma_j} \\ &= \sum_{\sigma_3\sigma_4\ldots\sigma_8}\prod_{i,j}[\cosh(\beta J\sigma_i\sigma_j)+\sinh(\beta J\sigma_i\sigma_j)] \\ &= \sum_{\sigma_3\sigma_4\ldots\sigma_8}\prod_{i,j}[\cosh(\beta J)+\sigma_i\sigma_j\sinh(\beta J)] \\ &= \sum_{\sigma_3\sigma_4\ldots\sigma_8}(\cosh(\beta J))^{13}\prod_{i,j}[1+\sigma_i\sigma_j\tanh(\beta J)] \\ &= \sum_{\sigma_3\sigma_4\ldots\sigma_8}\alpha\prod_{i,j}[1+K.\sigma_i\sigma_j] = \end{aligned}$$

$$\begin{aligned} &\alpha(1+K.\sigma_1\sigma_a)(1+K.\sigma_2\sigma_b)\sum_{\sigma_3\sigma_4\ldots\sigma_8}(1+K.\sigma_1\sigma_3)(1+K.\sigma_a\sigma_6)\ldots(1+K.\sigma_5\sigma_2)(1+K.\sigma_8\sigma_b) \\ &= \alpha(1+K.\sigma_1\sigma_a)(1+K.\sigma_2\sigma_b) \times \\ &\Big\{\sum_{\sigma_3\sigma_4\ldots\sigma_8}[1+K(\sigma_1\sigma_3+\sigma_a\sigma_6+\ldots)+K^2(\sigma_1\sigma_3\sigma_a\sigma_6+\sigma_1\sigma_3^2\sigma_6+\ldots)+\ldots \\ &\ldots+K^{11}\sigma_1\sigma_a\sigma_3^3\sigma_4^3\sigma_5^3\sigma_6^3\sigma_7^3\sigma_8^3\sigma_2\sigma_b]\Big\} \end{aligned} \tag{11}$$

Here α and K are defined as follows: α = $(\cosh(\beta J))^{13}$ and K = tanh(βJ). Grouping the terms in powers of K, one sees that the only non-vanishing terms are those in which all $\sigma_i$ appearing as indices ($\sigma_3, \sigma_4,\ldots, \sigma_8$) are squared. The lowest-order terms in which this happens are $K^3\sigma_1\sigma_3^2\sigma_6^2\sigma_a$ and $K^3\sigma_2\sigma_5^2\sigma_8^2\sigma_b$. These terms correspond to a path linking the nodes 1-3-6-a and 2-5-8-b respectively (cf. Fig. 1); the power of K corresponds to the number of steps or segments in the path. To identify all non-vanishing terms, we thus have to count 1) all direct[2] paths linking nodes 1 and 2, 1 and a (and 2 and b), 1 and b (and 2 and a) and a and b; 2) all closed loops (such as 3-4-7-6-3); and all products of such paths that have no segments in common (such as 1-3-6-a and 4-5-8-7-4). This is a straightforward though somewhat tedious procedure leading to:

$$\begin{aligned} Z.P(\eta_1) &= \alpha(1+K.\sigma_1\sigma_a)(1+K.\sigma_2\sigma_b).2^6 \times \\ &\times\{1+(K^3+K^5+2K^7)(\sigma_1\sigma_a+\sigma_2\sigma_b)+(K^4+3K^6)(\sigma_1\sigma_2+\sigma_a\sigma_b)+ \\ &+(K^6+3K^8)\sigma_1\sigma_2\sigma_a\sigma_b+(3K^5+K^7)(\sigma_1\sigma_b+\sigma_2\sigma_a)+2K^4+K^6\}. \end{aligned} \tag{12}$$

Following the same procedure we find for the denominator:

$$Z.P(\eta_2) = \sum_{\sigma_1\sigma_2\ldots\sigma_8} e^{\beta J\sum_{i,j}\sigma_i\sigma_j} = \alpha\sum_{\sigma_1\sigma_2\ldots\sigma_8}\prod_{i,j}[1+K.\sigma_i\sigma_j]$$

$$= \alpha.2^8.[1+\sigma_a\sigma_b(K^4+10K^6+5K^8)+4K^4+3K^6+5K^8+3K^{10}] \tag{13}$$

Thus we obtain the desired expression for $P(\sigma_1,\sigma_2 \,|\, \sigma_a,\sigma_b)$ via Eq. (8), and $X_{BI}$ via (6-7). For instance,

$$P(+,+\,|\,+,+) = \frac{(1+K)^2[2K^3+4K^4+8K^5+8K^6+6K^7+3K^8]}{2^2[1+5K^4+13K^6+10K^8+3K^{10}]}, \tag{14}$$

implying that for a lattice with homogeneous interactions $J_{ij}$ = J = 1 and β = 1, P(+,+ | +,+) = 0.95 (J=1=β are common values used in simulations, cf. [13]). $X_{BI}$ is a combination of such terms (Eq. (6-7)); it is a complex expression that can numerically be determined for given parameters (β,J). E.g., for J=1=β, $X_{BI}$ = –0.667, a value that satisfies the BI. The expression for $X_{BI}$ can be simplified in the weak-interaction limit K << 1. In that case one finds: $X_{BI} \approx -2\,K^2$, satisfying the BI.

Formulas (12-14) were verified by an algorithm

[2] A direct path is one not intersecting itself.

that computes probabilities directly as sums of Boltzmann terms as in Eq. (10). Now, these numerical calculations reveal that if one introduces varying interaction constants $J_{ij}$ and local magnetic fields $h_i \neq 0$, *one can strongly violate the BI*, for a wide range of parameter values for β, $h_i$, $J_{ij}$. The parameter values leading to $X_{BI} > 2$ are standard values used in simulations; moreover one can maintain left-right symmetry in the lattice, such that e.g. $J_{a6} = J_{b8}$. For instance for $\beta = 1$, $h_i \in \{-1, 1, 3\}$, $J_{ij} \in \{1, 2, 3, 4\}$ one finds that $X_{BI} = 2.87$ at its local maximum[3]. This value is larger than $2\sqrt{2} \approx 2.83$, the Tsirelson bound and the value for the singlet state in a real Bell experiment. The value $X_{BI} = 2.87$ is likely close to the absolute maximum for the lattice of Fig. 1 as argued below; but other lattices may lead to even larger values (cf. below). Other examples are given in [10].

## 3. Explanation: violation of Measurement Independence (MI)

If the BI is violated, at least one of the conditions MI, OI, PI does not hold. It appears that the resource for violation of the BI in our experiment on the lattice of Fig. 1 is violation of MI, as we will now prove. To verify MI in Eq. (1c) analytically, we again assume all $J_{ij} = J$ and all $h_i = 0$. One then finds, for $\lambda \equiv \sigma_\lambda \equiv (\sigma_3,\sigma_4,\ldots\sigma_8)$ and with the same notations as before:

$$P(\lambda \,|\, a,b) \equiv P(\sigma_3,\sigma_4,\ldots,\sigma_8 \,|\, \sigma_a,\sigma_b) = \frac{\sum_{\sigma_1\sigma_2} e^{\beta J \sum_{i,j}\sigma_i\sigma_j}}{\sum_{\sigma_1\sigma_2\ldots\sigma_8} e^{\beta J \sum_{i,j}\sigma_i\sigma_j}}$$

$$= \frac{\sum_{\sigma_1\sigma_2}\prod_{i,j}[1+K.\sigma_i\sigma_j]}{\sum_{\sigma_1\sigma_2\ldots\sigma_8}\prod_{i,j}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{\prod_{i,j\neq 1,2}[1+K.\sigma_i\sigma_j]\sum_{\sigma_1\sigma_2}(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K\sigma_5\sigma_2)(1+K\sigma_2\sigma_b)}{\sum_{\sigma_1\sigma_2\ldots\sigma_8}\prod_{i,j}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{2^2.[1+K^2(\sigma_a\sigma_3+\sigma_b\sigma_5)+K^4\sigma_a\sigma_b\sigma_3\sigma_5]\prod_{i,j\neq 1,2}[1+K.\sigma_i\sigma_j]}{\sum_{\sigma_1\sigma_2\ldots\sigma_8}\prod_{i,j}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{2^2.(1+K^2\sigma_a\sigma_3)(1+K^2\sigma_5\sigma_b)\prod_{i,j\neq 1,2}[1+K.\sigma_i\sigma_j]}{\sum_{\sigma_1\sigma_2\ldots\sigma_8}\prod_{i,j}[1+K.\sigma_i\sigma_j]} \quad . \qquad (15)$$

The term in $K^2$ in the numerator (second last line) corresponds to the two paths in which $\sigma_1$ or $\sigma_2$ are squared (namely a-1-3 and b-2-5); the term in $K^4$ to the product of these two paths; there are no other non-zero terms in the sum over $\sigma_1$ and $\sigma_2$. Using (13) for the denominator, we obtain:

$$P(\sigma_3,\sigma_4,\ldots,\sigma_8 \,|\, \sigma_a,\sigma_b)$$

$$= \frac{(1+K^2\sigma_a\sigma_3)(1+K^2\sigma_b\sigma_5)(1+K\sigma_a)(1+K\sigma_b)\prod_{i,j\neq 1,2,a,b}[1+K.\sigma_i\sigma_j]}{2^6[1+\sigma_a\sigma_b(K^4+10K^6+5K^8)+4K^4+3K^6+5K^8+3K^{10}]} . \qquad (16)$$

This clearly implies that MI is violated, except for the trivial case K=0 (i.e. J=0). For instance:

$$P(+++\ldots+ \,|\, \sigma_a,\sigma_b)$$

$$= \frac{(1+K^2\sigma_a)(1+K^2\sigma_b)(1+K\sigma_a)(1+K\sigma_b)(1+K)^7}{2^6[1+\sigma_a\sigma_b(K^4+10K^6+5K^8)+4K^4+3K^6+5K^8+3K^{10}]} . \qquad (17)$$

Thus one immediately sees that $P(+++\ldots+ \,|\, +,+) \neq P(+++\ldots+ \,|\, -,-)$, or numerically for β=J=1 (i.e. K = 0.762): $0.973 \neq 0.0012$. Again, numerical calculations can also be done by an algorithm that evaluates the sums over Boltzmann terms (first line in (15)).

In conclusion, in the lattice of Fig. 1 MI is always violated, for all non-trivial parameter values of β and J. Numerical simulations have shown that this conclusion remains valid when one introduces different interactions $J_{ij}$ and local fields $h_i$ over the nodes; and that it also holds for other 1-D and 2-D structures. We prove the latter claim for an arbitrary N-spin chain in Appendix 2. The latter calculation shows that MI is only asymptotically satisfied for $N = \infty$.

To quantify to which degree a hidden-variable model for given {h, J} violates MI we use a measure introduced by Hall in [8-9]. We term this parameter for self-explaining reasons 'measurement dependence' (MD):

$$MD = \sup_{(a,a',b,b')} \int d\lambda. \,|\, \rho(\lambda \,|\, a,b) - \rho(\lambda \,|\, a',b') \,| . \qquad (18a)$$

Here $\sup_{(X)}(Y)$ indicates the supremum of Y when varying the parameters X over all their values. Thus one sees that MD = 0 is equivalent to MI. We analogously define 'Outcome Dependence' (OD) and 'Parameter Dependence' (PD) [8-9]:

$$OD = \sup_{(a,b,\lambda)} \sum_{\sigma_1,\sigma_2} \,|\, P(\sigma_1,\sigma_2 \,|\, a,b,\lambda) - P(\sigma_1 \,|\, a,b,\lambda).P(\sigma_2 \,|\, a,b,\lambda) \,| \qquad (18b)$$

$$PD = \sup_{(a,a',b,\sigma_2,\lambda)} \,|\, P(\sigma_2 \,|\, a,b,\lambda) - P(\sigma_2 \,|\, a',b,\lambda) \,| \qquad (18c)$$

As an example, for the parameter set given in the former footnote, MD = 1.99 (its maximum possible value is 2), indicating that, with these parameters, we are likely close to the absolute maximum for $X_{BI}$ in the model [8-9].

Recall that while MI is always violated (MD ≠ 0), the BI only is for certain ranges of parameter values. Numerical simulations show that in 2-D structures the

[3] In detail, one obtains $X_{BI} = 2.87$ for following numerical values: $h_1 = 3$, $h_3 = h_4 = 1$, $h_6 = h_a = -1$ (and identical values at symmetric nodes on the right); $J_{1a} = J_{13} = 2$, $J_{36} = J_{34} = 1$, $J_{47} = J_{67} = 4$, $J_{6a} = 3$ (and identical values for symmetric interactions).

tendency is the same as in 1-D structures (cf. Appendix 2): for fixed parameters {h, J}, MD decreases with increasing size of the lattice; in parallel $X_{BI}$ decreases. This is not really a surprise [8-10]: the correlation in the system decreases – which can also be simulated by decreasing the interaction strengths $J_{ij}$.

### 4. Spin-lattices and locality

Are spin-lattices local ? To verify (the standard definition of) locality, we must calculate outcome independence (OI) and parameter independence (PI), or equivalently the Clauser-Horne factorability condition. Note that the conditions OI and PI in (1a-b) are indeed equivalent to the Clauser-Horne factorability condition [12], defined as:

$$P(\sigma_1,\sigma_2 \,|\, \lambda,a,b) = P(\sigma_1 \,|\, \lambda,a).P(\sigma_2 \,|\, \lambda,b) \quad \text{for all } (\lambda,\sigma_1,\sigma_2). \tag{20}$$

In the literature one always assumes that locality is equivalent to satisfying Eq. (20). Let us here verify this condition analytically for the case $J_{ij}$ = J and $h_i$ = 0, as applied to Fig. 1. One finds, using a result obtained in Eq. (15) in the denominator:

$$P(\sigma_1,\sigma_2 \,|\, \lambda,a,b) \equiv P(\sigma_1,\sigma_2 \,|\, \sigma_\lambda,\sigma_a,\sigma_b) = \frac{e^{\beta J\sum_{i,j}\sigma_i\sigma_j}}{\sum_{\sigma_1\sigma_2} e^{\beta J\sum_{i,j}\sigma_i\sigma_j}}$$

$$= \frac{\prod_{i,j}[1+K.\sigma_i\sigma_j]}{\sum_{\sigma_1\sigma_2}\prod_{i,j}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{\prod_{i,j}[1+K.\sigma_i\sigma_j]}{2^2.[1+K^2(\sigma_a\sigma_3+\sigma_b\sigma_5)+K^4\sigma_a\sigma_b\sigma_3\sigma_5].\prod_{i,j\neq 1,2}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K\sigma_5\sigma_2)(1+K\sigma_2\sigma_b)}{2^2.[1+K^2(\sigma_a\sigma_3+\sigma_b\sigma_5)+K^4\sigma_a\sigma_b\sigma_3\sigma_5]}$$

$$= \frac{(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K\sigma_5\sigma_2)(1+K\sigma_2\sigma_b)}{2^2.(1+K^2\sigma_a\sigma_3)(1+K^2\sigma_5\sigma_b)}. \tag{21}$$

On the other hand:

$$P(\sigma_1 \,|\, \sigma_\lambda,\sigma_a) = \frac{\sum_{\sigma_2\sigma_b} e^{\beta J\sum_{i,j}\sigma_i\sigma_j}}{\sum_{\sigma_1\sigma_2\sigma_b} e^{\beta J\sum_{i,j}\sigma_i\sigma_j}} = \frac{\sum_{\sigma_2\sigma_b}\prod_{i,j}[1+K.\sigma_i\sigma_j]}{\sum_{\sigma_1\sigma_2\sigma_b}\prod_{i,j}[1+K.\sigma_i\sigma_j]}$$

$$= \frac{\prod_{i,j\neq 2,b}[1+K.\sigma_i\sigma_j]\sum_{\sigma_2\sigma_b}(1+K\sigma_5\sigma_2)(1+K\sigma_8\sigma_b)(1+K\sigma_2\sigma_b)}{\prod_{i,j\neq 1,2,b}[1+K.\sigma_i\sigma_j]\sum_{\sigma_1\sigma_2\sigma_b}(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K\sigma_5\sigma_2)(1+K\sigma_8\sigma_b)(1+K\sigma_2\sigma_b)}$$

$$= \frac{(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K^3\sigma_5\sigma_8)}{2.(1+K^2\sigma_a\sigma_3+K^3\sigma_5\sigma_8+K^5\sigma_a\sigma_3\sigma_5\sigma_8)}$$

$$= \frac{(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)(1+K^3\sigma_5\sigma_8)}{2.(1+K^2\sigma_a\sigma_3)(1+K^3\sigma_5\sigma_8)}$$

$$= \frac{(1+K\sigma_1\sigma_a)(1+K\sigma_1\sigma_3)}{2.(1+K^2\sigma_a\sigma_3)}. \tag{22}$$

By symmetry we then immediately also have:

$$P(\sigma_2 \,|\, \sigma_\lambda,\sigma_b) = \frac{(1+K\sigma_2\sigma_b)(1+K\sigma_2\sigma_5)}{2.(1+K^2\sigma_b\sigma_5)}. \tag{23}$$

Thus indeed Eq. (20) and OI and PI are satisfied. Numerical simulation shows that OI and PI are always exactly satisfied also for other geometries and for any parameter set (including variable $J_{ij}$ and $h_i \neq 0$), implying that these systems are local in the sense of Clauser-Horne [11-12]. Note that the fact that PI is satisfied, implies that the system is 'non-signaling' in information-theoretic jargon [4-9].

Now, the result that the lattice satisfies Clauser-Horne factorability could be expected, or is at least in agreement with the fact that we forced the interactions to be local: we took the interaction constants $J_{ij}$ = J = 0 beyond nearest-neighbors. Interestingly, if one artificially introduces a delocalized 'left-right' interaction, e.g. $J_{12}$ or $J_{1b} \neq 0$, calculation shows that the Clauser-Horne factorability is *not* satisfied. This is for instance the case in the system of Fig. 2.

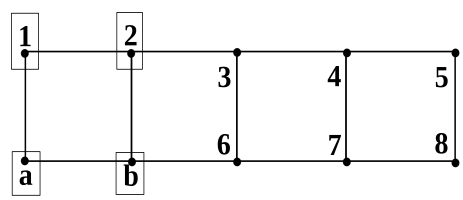


**Fig. 2**. A lattice with delocalized left-right interactions.

Let us take here interactions $J_{ij}$ that are ≠ 0 for first and second neighbours. If for instance all $h_i$=1, $J_{ij}$=1 (first neighbours), $J_{ij}$=0.5 (second neighbours), then one finds $X_{BI}$ = 2.32, MD = 0.03, PD = 0.78, and OD = 0.15 (cf. definitions (18)). In other words, none of the conditions OI, PI, MI holds.

We reached identical conclusions for a variety of 2-D lattices that are small enough to be numerically tractable. To further confirm our conclusions, we analytically investigated an arbitrary 1-D lattice in Appendix 2. This exercise is also useful to treat the case N → ∞.

### 5. Contradictions.

Let us now show that any reasonable explanation for above results leads to a conflict with generally accepted claims; it seems there is no explanation that is free of surprises and problems. Thus below we list and discuss the possible explanations (E1-E3) one could a priori invoke for the violation of the Bell-inequality in our test system.

(E1). "*The BI is violated because MI (measurement independence) is violated in the system*". Correct, the BI can only be derived if MI, OI and PI hold for the considered hidden-variable system; if one of these conditions is violated then the BI does not necessarily hold (it may or it may not, depending on the detailed system parameters); and in Section 3 it was proven that MI is always violated. So this interpretation is satisfactory; and yet it points towards a contradiction with a generally accepted wisdom. Indeed, in Section 1 we emphasized that MI is usually considered to be an 'obvious' assumption for Bell experiments, and that MI-violation amounts to superdeterminism and violation of free will. However, it seems that spin-lattices are an example of a physical system in which MI can be violated *without any superdeterministic or conspiratorial mechanism*, and thus in a manner that is fully compatible with free will. Even if this conclusion follows from a thought experiment, it seems that the important point is that the latter does not violate a physical law. Indeed, all above calculations and in particular BI-violation and MI-violation remain valid for an experiment on spin-lattices in which Alice and Bob are manifestly free agents – namely experiment Ex2 discussed in Section 2 (above Eq. (6)). In Section 2 we argued that all relevant probabilities for a lattice evolving 'on its own' (Ex1) are identical to those obtaining when free experimenters intervene on a and b (Ex2) (if $T \neq 0$). If the dynamics of the system (Eqs. (4) and (5)) remains unchanged in this second experiment, then it is intuitive that all relevant probabilities are identical in Ex1 and Ex2; an explicit proof is given in Appendix 1. Then, since MI (and the BI) are violated in Ex1, they also are in Ex2, an experiment in which Alice and Bob manifestly freely set the analyzer spins ($\sigma_a$ and $\sigma_b$)[4].

If this reasoning, based on a thought experiment, can be generalized, this conclusion is essential. If there indeed exist local systems in nature in which MI and the BI can be violated without superdeterminism, their physics may be at the basis of theories completing quantum mechanics, much in the sense of 't Hooft's Cellular Automaton Theory [17]. This claim will be further investigated in a forthcoming publication [15]. The key point is to use MI-violation (MD) as a resource for BI-violation and reproduction of the quantum statistics. In this context it was recently shown that MD (compared to OD and PD) is the strongest resource for reproduction of the quantum statistics in a Bell-experiment [8-9].

(E2). "*The BI is violated because spin-lattices are non-local systems*". Granted, spin-lattices are 'extended' and the Coulomb-interaction between the nodes ensures a 'communication' between them – so these systems have something vaguely 'delocalized' about them. But in these lattices there surely are no non-local interactions in the sense of Bell, the sense that matters: local means 'only involving subluminal, Lorentz-invariant interactions'. Moreover these systems are well modeled by assuming that the spin-spin (Coulomb) interaction is localized between nearest neighbours only. Most importantly, all spin-lattices satisfy the Clauser-Horne factorability condition (20) (Section 4). Thus, if one still maintains that spin-lattices are nonlocal then our result would be the first to prove that the generally accepted Clauser-Horne factorability is not a good definition of Bell-locality.

(E3). "*The BI is violated because spin-lattices are quantum systems*". True, when Bell originally devised his theorem in [1] he was probably thinking in the first place of hidden variables that are classical, not quantum. Recall that the test system I consider is a hidden-variable model in the sense that it allows one to calculate the probabilities $P(\sigma_1,\sigma_2|a,b)$, $P(\sigma_1|a)$ etc. by summing over probabilities as $P(\sigma_1|a, \lambda)$ etc.; the hidden variables are the spins on the nodes in between ($\sigma_1$, $\sigma_a$) and ($\sigma_2$, $\sigma_b$), cf. Fig. 1. Now, nowhere in Bell's derivation it is required that the hidden variables are classical and not quantum, as Bell explicitly mentions in a more recent article [2]. On p. 56 he says: "It is notable that in this argument nothing is said about the locality, or even localizability, of the variable $\lambda$. These variables could well include, for example, quantum mechanical state vectors, which have no particular localization in ordinary space time. It is assumed only that the outputs [$\sigma_1$] and [$\sigma_2$], and the particular inputs a and b, are well localized" [2]. Therefore the claim that the Ising-lattice is a quantum system should not be a valid explanation of the above results. Moreover, the Hamiltonian (4) is usually termed *classical*: the spins are numbers and not operators as in the quantum Ising Hamiltonian [16]. Hamiltonian (4) also describes purely classical systems as lattice gases.

Thus we see that none of the above explanations (E1-3) is free of controversy.

## 6. Conclusion.

We presented here well-known model systems from statistical mechanics, namely Ising spin-lattices, in which the Bell inequality (BI) can be strongly violated even if they are Bell-local. Some of these lattices violate the Bell inequality more than the Tsirelson bound. We showed that any explanation of the above results invites fundamental

[4] This implies that we also have in these lattice models that $P(a,b|\lambda) \neq P(a,b|\lambda')$ in general, again even if Alice & Bob may choose to set (a,b) in whatever sequence, with whatever frequency, they fancy. The point is that one should not understand this as a manifestation of a causal determination of the freely chosen (a,b) by $\lambda$. An infinity of such systems exist. Think e.g. of $P(x|T)$ with x = half-life of a nucleus, T = experimental temperature (suppose that a few discrete values of x and T are sampled). If one performs 1000 experiments measuring x at different T's, $P(T|x) \neq P(T)$ in general even if one chooses T freely.

questions. The best explanation hinges on the fact that these lattices violate measurement independence (MI). But we argued that MI (and the BI) can be violated in these systems in a non-conspiratorial way: it appears that MI can be violated in an experiment in which Alice and Bob are manifestly free-willed, in contradiction with the claim that MI-violation amounts to superdeterminism and violation of free will.

If this conclusion can be generalized, it may have weighty consequences. Indeed, if MI can be violated in real-world physical systems by mechanisms that are compatible with free will (and of course Bell-local), then searching for MI-violating theories may be a promising road for finding theories completing quantum mechanics. Such theories are evidently not affected by Bell's no-go result, which presupposes MI. We explore this avenue further in [15]. Although we did not investigate here 't Hooft's Cellular Automaton Theory for quantum mechanics [17] (which is still an unfinished research project), there yet might be a link: spin-lattices are cellular automata.

The above results invite also other questions. What is the exact link with Tsirelson's theorem ? What is the exact link with non-locality in quantum information theory ? The Bell inequality is the essential safeguard for secure quantum communication. Some of the above systems allow to violate the Bell inequality more than the singlet state while being non-signaling (Section 4). They may thus be compared to results published in an information-theoretic context [4-9], in particular those involving correlation between the hidden variables and the settings (a,b) [5, 8-9]. For instance, our results confirm that it is essential for quantum communication that the particle source and the random number generators are sufficiently decorrelated [5]. What is the exact link with 'non-local Popescu-Rohrlich boxes' [4, 6-7], violating the Bell-inequality more than the singlet state while being non-signaling – just as certain types of spin-lattices ? We hope that also these information-theoretic questions will be addressed in the context of spin-lattices.

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## Appendix 1. 'Free will' and violation of MI in spin-lattices.

Here we calculate the 16 probabilities $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b) \equiv P(\sigma_1=\varepsilon_1,\ \sigma_2=\varepsilon_2\ |\sigma_a=\varepsilon_a,\ \sigma_b=\varepsilon_b)$ (all $\varepsilon_i = \pm1$) needed to verify the BI for the experiment of Section 2. More precisely, we compare two experiments that Alice and Bob can do to determine these probabilities (Ex1 and Ex2 of Section 2).

Ex1) In this experiment the ensemble evolves fully 'on its own' (there is no intervention of Alice and Bob on any of the spins). Alice just measures $\sigma_1$ and $\sigma_a$, Bob $\sigma_2$ and $\sigma_b$ for the whole ensemble of lattices. From this set of results they 'postselect' 4 sub-sets, each sub-set corresponding to one for the 4 possible couples of ($\sigma_a$, $\sigma_b$)-values. They then compute the probabilities $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b)$ by counting relative frequencies within the sub-sets. They will find following values (these are Eqs. (8-10) in the main text):

$$P(\sigma_1,\sigma_2|\sigma_a,\sigma_b) \;=\; \frac{P(\sigma_1,\sigma_2,\sigma_a,\sigma_b)}{P(\sigma_a,\sigma_b)} \;=$$

$$\frac{P(\eta_1)}{P(\eta_2)} = \frac{\sum_{\theta(\eta_1)}^{2^6} e^{-\beta H(\theta)}}{\sum_{\theta(\eta_2)}^{2^8} e^{-\beta H(\theta)}} \;. \qquad \text{(A1)}$$

Ex2) A second way to determine the $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b)$ is available to Bob and Alice if they can intervene on $\sigma_a$ and $\sigma_b$. If they have sufficient technological means to control $\sigma_a$ and $\sigma_b$ (and keep their values fixed) they can do 4 consecutive experiments each corresponding to a given value of $\sigma_a$ and $\sigma_b$. In that case and under the assumptions stipulated in Section 2, they will find:

$$\begin{aligned} P^*(\sigma_1,\sigma_2|\sigma_a,\sigma_b) \;&=\; \frac{P^*(\sigma_1,\sigma_2,\sigma_a,\sigma_b)}{P^*(\sigma_a,\sigma_b)} \\ &=\; \frac{P^*(\eta_1)}{1}, \end{aligned} \qquad \text{(A2)}$$

where the asterisk reminds us that the probability is determined in an experiment in which $\sigma_a$ and $\sigma_b$ have a given value. With Eq. (5) of the main text we have:

$$P^*(\eta_1) \;=\; \sum_{\theta(\eta_1)}^{2^6} P^*(\theta) \;=\; \sum_{\theta(\eta_1)}^{2^6} \frac{e^{-\beta H(\theta)}}{Z^*}. \qquad \text{(A3)}$$

Here the partition function $Z^*$ is the sum over all Boltzmann terms given that $\sigma_a$ and $\sigma_b$ are fixed, i.e.:

$$Z^* \;=\; \sum_{\theta(\eta_2)}^{2^8} \; e^{-\beta H(\theta)}. \qquad \text{(A4)}$$

Thus comparing (A2-A3) and (A1) proves that the probabilities $P(\sigma_1,\sigma_2|\sigma_a,\sigma_b)$ determined in Ex1 and Ex2 are identical. This was to be expected: the dynamics is the same in both experiments. Also, this nicely reflects what happens in real Bell experiments. Finally, it is clear that any other relevant probability, such as $P(\lambda|\sigma_a,\sigma_b)$ $\equiv P(\sigma_\lambda|\sigma_a,\sigma_b)$ is also identical in both experiments: the partition function Z* in Ex2 always corresponds to $P(\sigma_a,\sigma_b)$, the denominator in Ex1. Ergo, the BI and MI can be violated in an experiment compatible with free will.

## Appendix 2. 1-D spin-lattices: test of MI, OI and PI.

Here we investigate a 1-D spin-lattice with N+2 spins, as in Fig. 4. We assume all $J_{ij}$ = J and all $h_i$ = 0, which makes analytical calculations tractable.

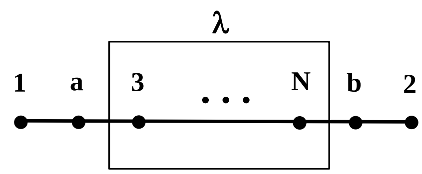


**Fig. 3.** 1-D chain of N+2 spins.

To verify whether MI holds, we calculate P(λ | a,b) for $\lambda \equiv \sigma_\lambda \equiv (\sigma_3,\sigma_4,\ldots\sigma_N)$. First:

$$P(\lambda|a,b) \;\equiv\; P(\sigma_\lambda|\sigma_a,\sigma_b) = \frac{P(\lambda,a,b)}{P(a,b)} = \frac{P_1}{P_2}. \qquad \text{(A5)}$$

Introducing analogous definitions of α and K as in the 2-D case (cf. main text), we now have:

$$\begin{aligned} P_1 \;&=\; \frac{1}{Z}\sum_{\sigma_1\sigma_2} e^{\beta J\sum_i \sigma_i\sigma_{i+1}} \\ &= \frac{1}{Z}\sum_{\sigma_1\sigma_2} (\cosh(\beta J))^{N+1}\prod_i [1+\sigma_i\sigma_{i+1}\tanh(\beta J)] \\ &= \frac{1}{Z}\sum_{\sigma_1\sigma_2} \alpha\prod_i [1+K.\sigma_i\sigma_{i+1}] \\ &= \frac{1}{Z}\,\alpha(1+K\sigma_a\sigma_3)(1+K\sigma_3\sigma_4)\ldots \\ &\quad \ldots(1+K\sigma_{N-1}\sigma_N)(1+K\sigma_N\sigma_b).2^2 \end{aligned} \qquad \text{(A6)}$$

The last step follows from the fact that in the sum $\sum_{\sigma_1\sigma_2}$ both $\sigma_1$ and $\sigma_2$ run over +1 and -1. For $P_2$ we find likewise:

$$\begin{aligned} \frac{Z}{\alpha}P_2 \;&=\; \sum_{\sigma_1\sigma_2\sigma_3\ldots\sigma_N}\prod_i [1+K.\sigma_i\sigma_{i+1}] = \\ &\sum_{\sigma_1\sigma_2\sigma_3\ldots\sigma_N} \{\; 1+K(\sigma_1\sigma_a+\sigma_a\sigma_3+\ldots)+K^2(\sigma_1\sigma_a^2\sigma_3+\sigma_1\sigma_a\sigma_3\sigma_4+\ldots)+\ldots \\ &+K^{N-1}(\sigma_a\sigma_3^2\sigma_4^2\ldots\sigma_{N-1}^2\sigma_N^2\sigma_b+\ldots)+\ldots \;\} \end{aligned} \qquad \text{(A7)}$$

Grouping the terms in powers of K, one sees that in only one term all $\sigma_i$ appearing as indices are squared, namely in $K^{N-1}.\sigma_a\sigma_3^2\sigma_4^2\ldots\sigma_{N-1}^2\sigma_N^2\sigma_b$. All other terms vanish, so that we obtain:

$$P_2 \;=\; \frac{1}{Z}\,\alpha\sum_{\sigma_1\sigma_2\sigma_3\ldots\sigma_N}(1+K^{N-1}\sigma_a\sigma_b) = \frac{1}{Z}\,\alpha.2^N(1+K^{N-1}\sigma_a\sigma_b). \qquad \text{(A8)}$$

Finally

$$\begin{aligned} P(\lambda|a,b) &= P(\sigma_3,\ldots\sigma_N|\sigma_a,\sigma_b) \\ &= \frac{(1+K\sigma_a\sigma_3)(1+K\sigma_3\sigma_4)\ldots(1+K\sigma_{N-1}\sigma_N)(1+K\sigma_N\sigma_b)}{2^{N-2}(1+K^{N-1}\sigma_a\sigma_b)}. \end{aligned} \qquad \text{(A9)}$$

Thus in general $P(\sigma_\lambda|\sigma_a,\sigma_b) \neq P(\sigma_\lambda|\sigma_{a'},\sigma_{b'})$. In other words, according to Eq. (1c) MI is violated (MD ≠ 0), except for the trivial case K=0, i.e. J=0, corresponding to a non-interacting lattice. For instance:

$$P(+,+,...,+\,|\,+,+) = \frac{(1+K)^{N-3}(1+K)^2}{2^{N-2}(1+K^{N-1})}$$

$$\neq \frac{(1+K)^{N-3}(1-K)^2}{2^{N-2}(1+K^{N-1})} = P(+,+,...,+\,|\,-,-). \qquad \text{(A10)}$$

Notice that MD → 0 for N → ∞. Formulas as (A9-A10) can be checked by a short computer program. Such numerical calculations also show that nothing substantially alters by introducing different interactions $J_{ij}$ between sites and local fields $h_i \neq 0$.

For verifying the Clauser-Horne factorability condition (OI and PI), we need to calculate:

$$P(\sigma_1,\sigma_2\,|\,\sigma_\lambda,\sigma_a,\sigma_b) = \frac{e^{\beta J\sum_i \sigma_i\sigma_{i+1}}}{\sum_{\sigma_1\sigma_2} e^{\beta J\sum_i \sigma_i\sigma_{i+1}}} = \frac{\prod_i (1+K.\sigma_i\sigma_{i+1})}{\sum_{\sigma_1\sigma_2}\prod_i [1+K.\sigma_i\sigma_{i+1}]}$$

$$= \frac{\prod_i (1+K.\sigma_i\sigma_{i+1})}{(1+K.\sigma_a\sigma_3)...(1+K.\sigma_N\sigma_b)\sum_{\sigma_1\sigma_2}(1+K.\sigma_1\sigma_a)(1+K.\sigma_b\sigma_2)}$$

$$= \frac{(1+K.\sigma_1\sigma_a)(1+K.\sigma_2\sigma_b)}{4}. \qquad \text{(A11)}$$

On the other hand,

$$P(\sigma_1\,|\,\sigma_\lambda,\sigma_a) = \frac{\sum_{\sigma_2\sigma_b} e^{\beta J\sum_i \sigma_i\sigma_{i+1}}}{\sum_{\sigma_1\sigma_2\sigma_b} e^{\beta J\sum_i \sigma_i\sigma_{i+1}}} =$$

$$\frac{(1+K.\sigma_1\sigma_a)\sum_{\sigma_2\sigma_b}(1+K.\sigma_N\sigma_b)(1+K.\sigma_b\sigma_2)}{\sum_{\sigma_1\sigma_2\sigma_b}(1+K.\sigma_1\sigma_a)(1+K.\sigma_N\sigma_b)(1+K.\sigma_b\sigma_2)}$$

$$= \frac{(1+K.\sigma_1\sigma_a)}{2}, \qquad \text{(A12)}$$

and similarly

$P(\sigma_2\,|\,\sigma_\lambda,\sigma_b) = \frac{(1+K.\sigma_2\sigma_b)}{2}$, so that we do satisfy OI and PI, i.e. locality à la Clauser-Horne for an arbitrary N-chain.