Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090428

Stability Analysis and Optimal Control of a Malaria Model with Larvivorous Fish as Biological Control Agent

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Received: 29 Oct. 2014, Revised: 29 Jan. 2015, Accepted: 30 Jan. 2015
Published online: 1 Jul. 2015

Abstract: This paper presents a non-linear mathematical model of malaria by considering the human reservoir and larvivorous fishes. The different equilibria of the model are computed and stability of these equilibria is investigated in-detail. Also, the basic reproduction number R_0 of the model is computed and we observe that the model exhibits backward bifurcation for some set of parameters implying the existence of multiple endemic equilibria for $R_0 < 1$. This existence of multiple endemic equilibria emphasizes the fact that $R_0 < 1$ is not sufficient to eradicate the disease from the population and the need is to lower R_0 much below one to make the disease-free equilibrium to be globally stable. The numerical simulation is performed to support analytical findings and the presented results show meaningful agreement. Additionally, the model is extended to incorporate optimal control by introducing the 'insecticide control' to control the mosquito population and Pontryagin's maximum principle[1] is used to analyze the optimal control model. Here too the numerical simulation is performed to demonstrate the effect of optimal control.

Keywords: Malaria; Larvivorous-fish; Backward-bifurcation; Biological-control; Predator-prey; Optimal Control.

1 Introduction

Malaria is a mosquito borne infectious disease and it is endemic in many countries around the world. It has become the economical and health related burden for the countries affected with endemic malaria. Malaria is caused by Plasmodium parasites which is transmitted to people through the bites of infected Anopheles mosquitoes, called "malaria vectors". These mosquitoes bite mainly between dusk and dawn. The CDC report reveals that malaria is the fifth leading killer among infectious diseases worldwide, and it is the second leading cause of death in Africa, following HIV/AIDS. Several malaria vector control methods are being implemented in order to reduce the density of malaria vector population to protect the human population against infectious mosquitoes.

Mathematical models are extensively used to predict the future of these kind of problems (see [2,3], *etc.*,). Many methods and models have been proposed to predict and control the dynamics of diseases like malaria, dengue, tb, HIV etc. The general SIR and SIRS epidemic model with some qualitative features are described in[4, 5]. The epidemiological impact of immunity to malaria has been investigated in [6,7]. The effect of vaccines for malaria has been described in [7,8]. Disease-modification and transmission-blocking concept have been discussed in detail through the model explored in [8]. In [9], visual representation of spatial aspects of malaria transmission in successive snap-shots in time, is presented with identification of mosquito vector breeding sites of defined shape and area. How the intensity of malaria transmission changes over the evolution of drug resistance is explained in [10]. A delay-differential equation model with partially immune population is discussed in [11]. In [12], the seasonal fluctuation of the mosquito density in Brazilian Amazon region is investigated. In [13], the authors considered two latent periods with non-constant host and vector populations in order to assess the potential impact of personal protection, treatment and possible vaccination strategies on the transmission dynamics of malaria. A mathematical model for malaria showing the impact of treatment and drug resistance is described in [14]. This model also considers delays in the latent periods in both mosquito and human populations. A host-vector interaction model with constant immigration in human population and a fraction of infective immigrants has been depicted in [15]. The possibility of Hopf bifurcation in a non-linear delay model for malaria is demonstrated in

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[16]. Threshold dynamics of a malaria transmission model in periodic environment is dealt in [17]. The phenomena of insecticide treated bed-nets usage to decrease the malaria vector population is described in [18]. This fact is also incorporated in [19]. Modelling malaria transmission by considering the variable human population is exhibited in [20], where it is assumed that the individuals recovered from malaria can act as infectives for susceptibles mosquitoes. This fact is true in endemic region. Infectious humans can recover through the treatment, but some of them can carry infection without affecting themselves and transmit infections to the mosquitoes biting them. As these people do not show any symptoms of malaria in their body, so they act as reservoir of malaria. The idea of reservoir class is also incorporated in [14,19,21,22]. Introduction of larvivorous fish is a promising biological control method to eradicate malaria. This method of control is inexpensive and is being used in many part of the World where this disease is endemic. The introduction of larvivorous fish to control malaria by decreasing the larvae population (*i.e.* the birth stage of mosquitoes population) is established in [23,24].

The optimal control strategies of an SIR epidemic model with time delay has been discussed in [25,26]. The optimal control approach is used to minimize the number of infectives. The paper [26] concentrated on the study of optimal control on vaccination program. There are several other research papers which have incorporated the optimal control problems for different types of diseases (see [24,27,28,29,30,31,32,33,34,35,36,37,38,39], *etc.*,). Out of these, few papers deal with the optimal control of malaria [24,27,28,29,30,31].

Keeping all the above aspects in consideration, we formulated a non-linear mathematical model by introducing Larvivorous fish population and the reservoir human population. We found the equilibria and analyzed the stability of these equilibria. Later we applied the optimal control by introducing insecticide control parameter to the model and analyzed it using the method of Pontragin's Maximum Principle[1].

This paper is organized as follows: Section 2 describes the basic model, section 3 elaborates the existence of equilibria and backward bifurcation using the center manifold theorem. Section 4 deals with the stability analysis and numerical simulation which affirms our analytical findings. Section 5 describes the application of optimal control to the proposed model and numerical simulation of optimal control model. At the last we have given our results as conclusion in section 6.

2 The Model

The whole human population $(N_h(t))$ under consideration is divided into three disjoint classes namely susceptible class $(S_h(t))$, infected class $(I_h(t))$ and the class of recovered individuals $(R_h(t))$ and the whole adult mosquito population $(N_v(t))$ of Anopheles species are divided into two disjoint classes namely susceptible $(S_v(t))$ and infected $(I_v(t))$. Additionally, we have two more classes corresponding to Larvae population $(L_v(t))$ and Larvivorous fish population $P(t)$. Larvae L_v and Larvivorous fish P populations are linked with prey-predator type interaction. Here it is assumed that the predatory fish population is fully dependent on mosquito larvae. Assuming the criss-cross interaction between susceptibles and infected humans and mosquitoes, the mathematical model is formulated as follows:

$$L_v' = gN_v - d_1L_v - \alpha L_v^2 - mL_v - \gamma L_vP, \tag{1a}$$

$$S_v' = mL_v - c\beta S_v\frac{(I_h+R_h)}{N_h} - d_2S_v, \tag{1b}$$

$$I_v' = c\beta S_v\frac{(I_h+R_h)}{N_h} - d_2I_v, \tag{1c}$$

$$P' = k\gamma L_vP - d_3P, \tag{1d}$$

$$S_h' = \Lambda - b\beta S_h\frac{I_v}{N_h} - d_4S_h + [(1-\rho)\tau]I_h + l_1R_h, \tag{1e}$$

$$I_h' = b\beta S_h\frac{I_v}{N_h} - (d_4+d_5+\tau)I_h, \tag{1f}$$

$$R_h' = \rho\tau I_h - (d_4+l_1)R_h, \tag{1g}$$

The parameters used in the model (1) are described in Table 1. For detailed descriptions of the parameters and transmission terms, one can refer [23,24].

The transfer diagram of the model is shown in Figure 1, where dotted line denotes interactions and the solid line denotes transfer from one class to another. The model (1) can be rewritten in the following form:

$$L_v' = gN_v - d_1L_v - \alpha L_v^2 - mL_v - \gamma L_vP, \tag{2a}$$

$$N_v' = mL_v - d_2N_v, \tag{2b}$$

$$I_v' = c\beta(N_v-I_v)\frac{(I_h+R_h)}{N_h} - d_2I_v, \tag{2c}$$

$$P' = k\gamma L_vP - d_3P, \tag{2d}$$

$$N_h' = \Lambda - d_4N_h - d_5I_h, \tag{2e}$$

$$I_h' = b\beta(N_h-I_h-R_h)\frac{I_v}{N_h} - k_1I_h, \tag{2f}$$

$$R_h' = \rho\tau I_h - (d_4+l_1)R_h, \tag{2g}$$

where $k_1 = (d_4+d_5+\tau)$.

2.1 The Basic Reproduction Number R_0

The basic reproduction number is defined as the number of secondary infections generated by a typical infected individual in an otherwise disease free population in his/her whole infectious period. The reproduction number (R_0) for our model is computed using the method described in [40] and using the same notation as in [40] the matrices $\mathscr{F}$ and $\mathscr{V}$ are given by

Table 1: Description of parameters

Parameter	Description
g	Vector's egg laying rate
k	Tropical convention efficiency
α	Density dependent mortality rate of larvae
m	Maturation rate of larvae
γ	Predation rate
β	Mosquitoes biting rate
b	Probability of disease transmission from infectious mosquitoes to humans
c	Probability of disease transmission from infectious humans to mosquitoes
Λ	Recruitment rate in the Susceptible humans class
ρ	Proportion recovered with temporary immunity
l_1	Loss of immunity rate for Recovered humans
τ	Rate of treatment
d_1	Natural mortality rate in mosquito larvae
d_2	Natural mortality rate in adult mosquitoes
d_3	Natural mortality rate in predators (Larvivorous fishes)
d_4	Natural mortality rate in humans
d_5	Disease induced mortality rate in humans

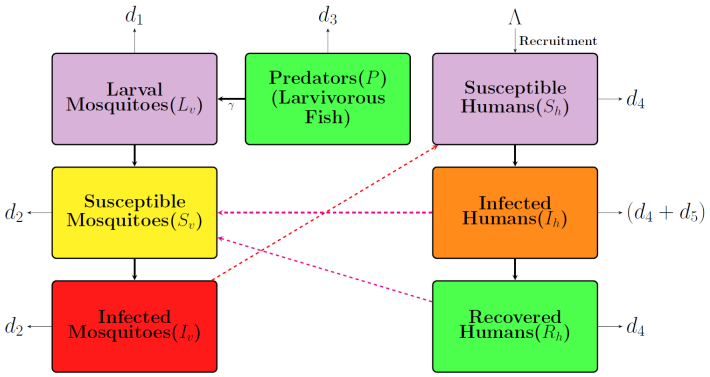

Fig. 1: Transfer diagram of the model.

$$\mathscr{F}=\begin{pmatrix} c\beta S_v\left(\frac{I_h+R_h}{N_h}\right) \\ b\beta S_h\frac{I_v}{N_h} \\ 0 \end{pmatrix}, \quad \mathscr{V}=\begin{pmatrix} d_2 I_v \\ k_1 I_h \\ -\rho\tau I_h+(d_4+l_1)R_h \end{pmatrix}.$$

Now, the matrix F and V evaluated at disease free equilibrium point are given by

$$F=\begin{pmatrix} 0 & c\beta\frac{A_1 d_4}{\Lambda} & c\beta\frac{A_1 d_4}{\Lambda} \\ b\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V=\begin{pmatrix} d_2 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & -\rho\tau & d_4+l_1 \end{pmatrix}.$$

And the matrix FV^{-1} is given by

$$FV^{-1}=\begin{pmatrix} 0 & \frac{c\beta d_4 A_1}{k_1\Lambda}+\frac{c\beta d_4 A_1\rho\tau}{k_1\Lambda(d_4+l_1)} & \frac{c\beta d_4 A_1}{\Lambda(d_4+l_1)} \\ \frac{b\beta}{d_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, the reproduction number R_0 which is the spectral radius of the matrix FV^{-1} is given by

$$R_0=\beta\sqrt{\frac{bcd_4A_1}{k_1\Lambda d_2}\left(\frac{d_4+l_1+\rho\tau}{d_4+l_1}\right)}.$$

$$i.e. \quad R_0=\beta\sqrt{\frac{bcd_4A_1(1+k_2)}{k_1\Lambda d_2}}=\sqrt{\frac{b\beta q_1 A_1}{k_1\Lambda d_2}}$$

with $q_1=c\beta d_4(1+k_2)$ and $k_2=\dfrac{\rho\tau}{(d_4+l_1)}$.

Here the number R_0^2 gives the average number of infected mosquitoes (humans) generated by one typical infected mosquito (human) in a fully susceptible population.

3 Equilibria

The equilibria for our model are determined by setting right hand sides of the model (2) to zero. The system (2) has following equilibria namely $E_1\left(0,0,0,0,\frac{\Lambda}{d_4},0,0\right)$, $E_2\left(L_v^*,m\frac{L_v^*}{d_2},0,0,\frac{\Lambda}{d_4},0,0\right)$, $E_3\left(L_v^*,m\frac{L_v^*}{d_2},0,P^*,\frac{\Lambda}{d_4},0,0\right)$, $E_4\left(L_v^*,N_v^*,I_v^*,P^*,N_h^*,I_h^*,R_h^*\right)$, where

$$
\begin{aligned}
L_v^* &= \frac{d_3}{k\gamma},\\
N_v^* &= \frac{mL_v^*}{d_2} = \frac{md_3}{d_2k\gamma} = A_1,\\
I_v^* &= \frac{c\beta A_1(1+k_2)d_4I_h^*}{c\beta(1+k_2)d_4I_h^* + d_2(\Lambda - d_5I_h^*)} = \frac{q_1A_1I_h^*}{q_1I_h^* + d_2(\Lambda - d_5I_h^*)},\\
P^* &= \frac{gm - d_2(d_1 + \alpha L_v^* + m)}{d_2\gamma},\\
N_h^* &= \frac{\Lambda - d_5I_h^*}{d_4},\\
R_h^* &= \left(\frac{\rho\tau}{d_4+l_1}\right)I_h^* = k_2I_h^*,
\end{aligned}
$$

and I_h^* is the positive root of the following quadratic equation

$$B_1I_h^2 + B_2I_h + B_3 = 0. \tag{3}$$

The expressions for B_1, B_2 and B_3 of (3) are as follows:

$$
\begin{aligned}
B_1 &= k_1d_5(d_2d_5 - q_1),\\
B_2 &= k_1d_2d_5\Lambda(R_0^2 - 2) + \Lambda k_1\left[q_1 + (1+k_2)d_2d_4R_0^2\right],\\
&= k_1d_2d_5\Lambda(R_0^2 - 1) + \Lambda k_1(q_1 - d_2d_5)\\
&\quad + (1+k_2)b\beta q_1A_1d_4,\\
&= H_1 + H_2 + H_3,\\
B_3 &= d_2k_1\Lambda^2\left(1 - \frac{bc\beta^2d_4A_1(1+k_2)}{d_2k_1\Lambda}\right)\\
&= d_2k_1\Lambda^2\left(1 - R_0^2\right),
\end{aligned}
$$

where

$$H_1 = k_1d_2d_5\Lambda(R_0^2 - 1), H_2 = \Lambda k_1(q_1 - d_2d_5),$$

$$H_3 = (1+k_2)b\beta q_1A_1d_4.$$

The two roots of this quadratic equation are given by

$$I_h = \frac{-B_2 \pm \sqrt{B_2^2 - 4B_1B_3}}{2B_1},$$

Now depending upon the signs of B_1, B_2 and B_3, we may have unique, two or no positive roots. These findings are summarized below:

Let us assume that $B_2^2 - 4B_1B_3 > 0$ and $disc = \sqrt{B_2^2 - 4B_1B_3}$.
It is easy to see that the discriminant can be obtained as follows:

$$disc^2 = (H_2 - H_1 + H_3)^2 + 4H_1H_3,$$

which implies that the discriminant $B_2^2 - 4B_1B_3$ is always positive for $R_0 > 1$.
Also, it is observed that we can have positive equilibrium only if $I_h < \dfrac{\Lambda}{d_5}$
i.e. We must have $\dfrac{-B_2}{2B_1} \pm \dfrac{disc}{2B_1} < \dfrac{\Lambda}{d_5} \implies \dfrac{disc^2}{(2B_1)^2} < \left(\dfrac{\Lambda}{d_5} + \dfrac{B_2}{2B_1}\right)^2$

$$i.e.\ \frac{-B_3}{B_1} < \left(\frac{\Lambda}{d_5}\right)^2 + \frac{\Lambda B_2}{d_5B_1}$$

Proceeding this way we get, $\dfrac{(1+k_2)b\beta q_1A_1d_4}{k_1d_5(d_2d_5 - q_1)} < 0$
$\implies q_1 - d_2d_5 < 0$
$\implies B_1 > 0$
Hence for the existence of positive equilibrium point we must have $B_1 > 0$ and $I_h < \dfrac{\Lambda}{d_5}$.

Also we concluded that when $B_1 < 0$, there is no equilibrium point.
So we consider only the cases under $B_1 > 0$
CASE (1): $R_0 > 1$ (Also $B_3 < 0$), we have the following sub-cases.

Sub-case (1a): If $B_2 > 0$, there is only one change of sign in the quadratic equation

$$B_1I_h^2 + B_2I_h + B_3 = 0.$$

So by Descartes's rule of signs, there exists at-most one positive root. It is easy to see that $\dfrac{-B_2 + disc}{2B_1}$ is the desired positive root (as $0 < |B_2| < disc$).

Sub-case (1b): If $B_2 < 0$, again there is only one change of sign and the expression to obtain the positive root is same as in the above sub-case.

CASE (2): $R_0 < 1$ (Also $B_3 > 0$), we have the following sub-cases.

Sub-case (2a): If $B_2 > 0$, then there is no change of sign and there is no equilibrium in this case.
Sub-case (2b): If $B_2 < 0$, then there are two changes of signs and of course there exists at most two positive roots of the quadratic equation. For the quadratic equation (3) to have real roots we must need to have $B_2^2 - 4B_1B_3 > 0$.
In this case, we have $disc < |B_2|$, $B_2 < 0$ and $B_1 > 0$. And

the roots are given by $I_h = \frac{-B_2}{2B_1} \pm \frac{disc}{2B_1}$, which must be less than $\frac{\Lambda}{d_5}$ to have two positive non-trivial equilibria. This result is summarized below:

The following conditions are needed for the existence of two positive roots
Condition 1: $B_2^2 - 4B_1B_3 > 0$
Condition 2: $B_2 < 0$
Condition 3: $\frac{-B_2}{2B_1} \pm \frac{disc}{2B_1} < \frac{\Lambda}{d_5}$
From condition 3 we proceed as follows: $\frac{-B_2}{2B_1} \pm \frac{disc}{2B_1} < \frac{\Lambda}{d_5}$

$$\implies \pm\frac{disc}{(2B_1)} < \left(\frac{\Lambda}{d_5} + \frac{B_2}{2B_1}\right) \tag{4}$$

The RHS of the inequality (4) can be simplified as

$\frac{1}{k_1 d_5(d_2 d_5 - q_1)}[-k_1 d_2 d_5 \Lambda(1-R_0^2) + (1+k_2) b\beta q_1 A_1 d_4]$
$= \frac{1}{B_1}[H_1 + H_3]$

Thus ineqality (4) can be rewritten as $\pm\frac{disc}{2} < H_1 + H_3$

From this we see that $H_1 + H_3 > 0$ is necessary for the existence of two positive roots.
$H_1 + H_3 > 0 \implies k_1 d_2 d_5 \Lambda(1-R_0^2) < (1+k_2) b\beta q_1 A_1 d_4$

$$\implies d_5 - d_5 R_0^2 < (1+k_2) b\beta q_1 A_1 R_0^2 d_4$$

$$\implies \frac{d_5}{(1+k_2)d_4 + d_5} < R_0^2 < 1$$

$$\implies \sqrt{\frac{d_5}{(1+k_2)d_4 + d_5}} < R_0 \ < 1$$

Hence the final condition for the existence of two positive roots and backward bifurcation is

$$\sqrt{\frac{d_5}{(1+k_2)d_4 + d_5}} < R_0 < 1$$

We denote the value $\sqrt{\frac{d_5}{(1+k_2)d_4 + d_5}}$ as R_0^c which we say the critical value of R_0.

3.1 Bifurcation Analysis

Let $L_v = x_1, N_v = x_2, I_v = x_3$, $P = x_4$, $N_h = x_5$, $I_h = x_6$, and $R_h = x_7$. Further, by using the vector notation $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$, the system (2) can be written in the form $\frac{dX}{dt} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$ as follows:

$$x_1' = gx_2 - d_1 x_1 - \alpha x_1^2 - mx_1 - \gamma x_1 x_4, \tag{5a}$$
$$x_2' = mx_1 - d_2 x_2, \tag{5b}$$
$$x_3' = c\beta(x_2 - x_3)\frac{(x_6 + x_7)}{x_5} - d_2 x_3, \tag{5c}$$
$$x_4' = k\gamma x_1 x_4 - d_3 x_4, \tag{5d}$$
$$x_5' = \Lambda - d_4 x_5 - d_5 x_6, \tag{5e}$$
$$x_6' = b\beta(x_5 - x_6 - x_7)\frac{x_3}{x_5} - k_1 x_6, \tag{5f}$$
$$x_7' = \rho\tau x_6 - (d_4 + l_1)x_7, \tag{5g}$$

The Jacobian of the system (5) at DFE is given by

$$J_{\beta^*} = \begin{bmatrix} a_{11} & g & 0 & -\gamma x_1^* & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & c\beta^* \frac{x_2^*}{x_5^*} & 0 \\ k\gamma x_4^* & 0 & 0 & k\gamma x_1^* - d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ 0 & 0 & b\beta & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4 + l_1) \end{bmatrix}$$

where $a_{11} = -(d_1 + 2\alpha x_1^* + m + \gamma x_4^*)$.

Consider the case when $R_0 = 1$. Suppose that $\beta = \beta^*$ is chosen as a bifurcation parameter. Solving for β from $R_0 = 1$ gives

$$\beta = \beta^* = \frac{\Lambda k_1 d_2}{b q_1 A_1}.$$

Using the following theorem which is reproduced from References [41] we will be able to determine whether or not the system (5) exhibits backward bifurcation at $R_0 = 1$.

Theorem 1. *Consider the following general system of ordinary differential equations with a parameter ϕ*

$$\frac{dx}{dt} = f(x, \phi)$$

,

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

and

$$f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R}),$$

where 0 is an equilibrium point of the system (i.e. $f(0, \phi) \equiv 0$ for all ϕ) and

1. *$A = D_x f(0,0) = (\frac{\partial f_i}{\partial x_j}(0,0))$ is the linearization matrix of the system around the equilibrium 0 with ϕ evaluated at 0;*
2. *Zero is a simple eigenvalue of A and other eigenvalues of A have negative real parts;*
3. *Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.*

Let f_k be the k^{th} component of f and

$$a_1 = \Sigma_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0),$$
$$b_1 = \Sigma_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0),$$

then the local dynamics of the system around the equilibrium point 0 is totally determined by the signs of a_1 and b_1. Particularly, if $a_1 > 0$ and $b_1 > 0$, then a backward bifurcation occurs at $\phi = 0$.

3.1.1 Eigenvalues of J_{β^*}

It can be easily seen that the Jacobian with $\beta = \beta^*$ of the linearized system has a simple zero eigenvalue and all the other eigenvalues have negative real parts. Hence, the center manifold theory can be used to analyze the dynamics of the system (5) near $\beta = \beta^*$. For the case when $R_0 = 1$, using the technique in Castillo-Chavez and Song [41], it can be shown that the matrix J_{β^*} has a right eigenvector (corresponding to the zero eigenvalue), given by $w = [w_1\ w_2\ w_3\ w_4\ w_5\ w_6\ w_7]^T$, where

$$w_1 = w_2 = 0,\ w_3 = \frac{k_1(d_4+l_1)}{b\beta\rho\tau} w_7,\ w_4 = 0,$$
$$w_5 = \frac{d_5(d_4+l_1)}{d_4\rho\tau} w_7,\ w_6 = \frac{(d_4+l_1)}{\rho\tau} w_7,\ w_7 = w_7 > 0.$$

Similarly, the matrix J_{β^*} has a left eigenvector (corresponding to the zero eigenvalue), denoted by $v = [v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_7]$, where

$$v_1 = v_2 = v_4 = v_5 = v_7 = 0,\ v_3 = v_3 > 0,\ v_6 = \frac{d_2}{b\beta} v_3.$$

Computation of a_1:

For the system (5), the associated non-zero partial derivatives are given by

$$\frac{\partial^2 f_3}{\partial x_3 \partial x_5} = \frac{\partial^2 f_3}{\partial x_5 \partial x_3} = \frac{c\beta(x_6+x_7)}{x_5^2}, \tag{6a}$$
$$\frac{\partial^2 f_3}{\partial x_3 \partial x_6} = \frac{\partial^2 f_3}{\partial x_6 \partial x_3} = -\frac{c\beta}{x_5}, \tag{6b}$$
$$\frac{\partial^2 f_3}{\partial x_3 \partial x_7} = \frac{\partial^2 f_3}{\partial x_7 \partial x_3} = -\frac{c\beta}{x_5}, \tag{6c}$$
$$\frac{\partial^2 f_3}{\partial x_5 \partial x_6} = \frac{\partial^2 f_3}{\partial x_6 \partial x_5} = -\frac{c\beta(x_2-x_3)}{x_5^2}, \tag{6d}$$
$$\frac{\partial^2 f_3}{\partial x_5 \partial x_7} = \frac{\partial^2 f_3}{\partial x_7 \partial x_5} = -\frac{c\beta(x_2-x_3)}{x_5^2}, \tag{6e}$$
$$\frac{\partial^2 f_3}{\partial x_5^2} = \frac{c\beta(x_2-x_3)(x_6+x_7)}{x_5^3}, \tag{6f}$$
$$\frac{\partial^2 f_6}{\partial x_3 \partial x_5} = \frac{\partial^2 f_6}{\partial x_5 \partial x_3} = \frac{b\beta(x_6+x_7)}{x_5^2}, \tag{6g}$$
$$\frac{\partial^2 f_6}{\partial x_3 \partial x_6} = \frac{\partial^2 f_6}{\partial x_6 \partial x_3} = -\frac{b\beta}{x_5^2}, \tag{6h}$$
$$\frac{\partial^2 f_6}{\partial x_3 \partial x_7} = \frac{\partial^2 f_6}{\partial x_7 \partial x_3} = -\frac{b\beta}{x_5^2}, \tag{6i}$$
$$\frac{\partial^2 f_6}{\partial x_5 \partial x_6} = \frac{\partial^2 f_6}{\partial x_6 \partial x_5} = \frac{b\beta x_3}{x_5^2}, \tag{6j}$$
$$\frac{\partial^2 f_6}{\partial x_3 \partial x_6} = \frac{\partial^2 f_6}{\partial x_6 \partial x_3} = \frac{b\beta x_3}{x_5^2}, \tag{6k}$$
$$\frac{\partial^2 f_6}{\partial x_5^2} = \frac{b\beta(x_5-x_6-x_7)x_3}{x_5^3}. \tag{6l}$$

It follows from the above expressions that

$$\begin{aligned} a_1 = {} & \frac{2c\beta v_3}{x_5}\left\{w_3\left[w_5\left(\frac{x_6+x_7}{x_5}\right)-(w_6+w_7)\right]\right\} \\ & + \frac{2c\beta v_3}{x_5}\left\{(x_2-x_3)w_5\left[2w_5\left(\frac{x_6+x_7}{(x_5)^2}\right)-(w_6+w_7)\right]\right\} \\ & + \frac{2b\beta v_6}{(x_5)^2}\left\{w_3\left[w_5(x_6+x_7)-(w_6+w_7)\right]\right\} \\ & + \frac{2b\beta v_6}{(x_5)^2}\left\{x_3 w_5\left[2w_5\left(\frac{x_5-(x_6+x_7)}{x_5}\right)+(w_6+w_7)\right]\right\} \\ = {} & 2\frac{k_1 d_2}{bq_1 A_1}\left\{(w_3-w_6-w_7)cv_3 d_4 - cv_3 d_4 A_1 w_5(w_6+w_7)\right\} \\ & - 2\frac{k_1 d_2}{bq_1 A_1}\left\{\frac{bv_6 d_4^2}{\Lambda} w_3(w_6+w_7)\right\} \end{aligned}$$

Computation of b_1:

For the computation of b_1, it is found that the associated

non-zero partial derivatives are

$$\frac{\partial^2 f_3}{\partial x_3 \partial \beta^*} = \frac{-c(x_6+x_7)}{x_5}, \tag{7a}$$

$$\frac{\partial^2 f_3}{\partial x_5 \partial \beta^*} = \frac{-c(x_2-x_3)(x_6+x_7)}{x_5^2}, \tag{7b}$$

$$\frac{\partial^2 f_3}{\partial x_6 \partial \beta^*} = \frac{c(x_2-x_3)}{x_5}, \tag{7c}$$

$$\frac{\partial^2 f_3}{\partial x_7 \partial \beta^*} = \frac{c(x_2-x_3)}{x_5}, \tag{7d}$$

$$\frac{\partial^2 f_3}{\partial x_3 \partial \beta^*} = \frac{b(x_5-x_6-x_7)}{x_5}, \tag{7e}$$

$$\frac{\partial^2 f_3}{\partial x_5 \partial \beta^*} = \frac{b(x_6+x_7)x_3}{x_5^2}, \tag{7f}$$

$$\frac{\partial^2 f_3}{\partial x_6 \partial \beta^*} = \frac{-bx_3}{x_5}, \tag{7g}$$

$$\frac{\partial^2 f_3}{\partial x_7 \partial \beta^*} = \frac{-bx_3}{x_5}. \tag{7h}$$

It follows from the above expressions that

$$\begin{aligned} b_1 &= v_3\left\{c\left(\frac{x_2-x_3}{x_5}\right)(w_6+w_7)\right\} \\ &\quad -v_3\left\{c\left(\frac{x_6^*+x_7}{x_5}\right)\times\left[w_3+w_5\left(\frac{x_2^*-x_3^*}{x_5}\right)\right]\right\} \\ &\quad +bv_6\left\{w_3-\frac{x_3}{x_5}(w_6+w_7)\right\} \\ &\quad -bv_6\left\{\left(\frac{x_6+x_7}{x_5}\right)\left[w_5\left(\frac{x_3}{x_5}\right)-w_3\right]\right\} \\ &= v_3c\frac{A_1d_4}{\Lambda}(w_6+w_7)+v_6bw_3>0 \end{aligned}$$

Thus, the following result is established.

Theorem 2.*The model exhibits backward bifurcation at $R_0=1$ whenever a_1 is positive.*

4 Stability Analysis and Numerical Simulation

4.1 Stability Analysis

The local stability results of the equilibria are established using variational matrix method and are stated in the following theorem.

Theorem 3.*The equilibrium point $E_1(0,0,0,0,\frac{\Lambda}{d_4},0,0)$ is locally asymptotically stable if $d_2(d_1+m)>gm$. The equilibrium point $E_2\left(L_v^*,N_v^*=m\frac{L_v^*}{d_2},0,0,\frac{\Lambda}{d_4},0,0\right)$ is locally asymptotically stable if $(d_1+2\alpha L_v^*+m)d_2>gm$. The disease free equilibrium point $E_3\left(L_v^*,N_v^*,0,P^*,\frac{\Lambda}{d_4},0,0\right)$ is locally asymptotically stable when $R_0<1$ and $h_1h_2-h_3>0$, where h_1, h_2 and h_3 are given in the proof of this theorem. The endemic equilibrium point $E_4\left(L_v^*,N_v^*,I_v^*,P^*,N_h^*,I_h^*,R_h^*\right)$ is locally asymptotically stable provided $g_1>0$, $g_1g_2-g_3>0$, $u_i>0$, $i=1,3,4$ & $u_1u_2u_3>u_3^2+u_1^2u_4$, where g_i and u_i are given in the proof of the theorem.*

Proof. The variational matrix corresponding to the system (2) is given by

$$M=\begin{pmatrix} a_{11} & g & 0 & -\gamma L_v & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & -a_{32}-d_2 & 0 & a_{35} & a_{36} & 0 \\ k\gamma P & 0 & 0 & k\gamma L_v-d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ 0 & 0 & a_{63} & 0 & a_{65} & a_{67}-k_1 & a_{67} \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4+l_1) \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= -(d_1+2\alpha L_v+m+\gamma P), \\ a_{32} &= c\beta\left(\frac{I_h+R_h}{N_h}\right), \\ a_{35} &= -c\beta\left(\frac{(N_v-I_v)(I_h+R_h)}{N_h^2}\right), \\ a_{36} &= c\beta\left(\frac{N_v-I_v}{N_h}\right), \\ a_{63} &= b\beta\left(\frac{N_h-I_h-R_h}{N_h}\right), \\ a_{65} &= b\beta\left(\frac{(I_h+R_h)I_v}{N_h^2}\right), \\ a_{67} &= -b\beta\left(\frac{I_v}{N_h}\right) \end{aligned}$$

The variational matrix corresponding to the system ((2)) at the equilibrium point $E_1\left(0,0,0,0,\frac{\Lambda}{d_4},0,0\right)$ is given by

$$M_1=\begin{pmatrix} -(d_1+m) & g & 0 & 0 & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ 0 & 0 & b\beta & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4+l_1) \end{pmatrix}.$$

Here five roots of the characteristic polynomial corresponding to the matrix M_1 are $-(d_4+l_1), -d_4, -d_3, -d_2$ and $-k_1=-(d_4+d_5+\tau)$ and other two roots are given by the roots of the following quadratic equation,

$$\lambda^2+(d_1+d_2+m)\lambda+\{d_2(d_1+m)-gm\}=0.$$

From the above quadratic equation, it is clear this equilibrium point E_1 is stable if $d_2(d_1+m)>gm$.

The variational matrix at the equilibrium point $E_2\left(L_v^*,m\frac{L_v^*}{d_2},0,0,\frac{\Lambda}{d_4},0,0\right)$ is given by

$$M_2 = \begin{pmatrix} -(d_1+2\alpha L_v^*+m) & g & 0 & -\gamma L_v^* & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ 0 & 0 & b\beta & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4+l_1) \end{pmatrix}$$

Here also five roots of the characteristic polynomial of this variational matrix are $-(d_4+l_1), -d_4, -d_3, -d_2$ and $-k_1 = -(d_4+d_5+\tau)$ and other two roots are given by the roots of the following quadratic equation,

$$\lambda^2 + (d_1+d_2+2\alpha L_v^*+m)\lambda + \{(d_1+2\alpha L_v^*+m)d_2 - gm\} = 0.$$

From above quadratic equation, it is clear that the equilibrium point E_2 is locally asymptotically stable if $(d_1+2\alpha L_v^*+m)d_2 > gm$.

The variational matrix at the equilibrium point $E_3\left(L_v^*, N_v^*, 0, P^*, \frac{\Lambda}{d_4}, 0, 0\right)$ is given by

$$M_3 = \begin{pmatrix} -f_{11} & g & 0 & -\gamma L_v & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 & 0 & c\beta\left(\frac{N_v}{N_h}\right) & 0 \\ k\gamma P & 0 & 0 & k\gamma L_v - d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ 0 & 0 & b\beta & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4+l_1) \end{pmatrix}$$

where $f_{11} = (d_1 + 2\alpha L_v^* + m + \gamma P^*)$. Clearly two eigenvalues of this matrix are $-(d_4+l_1)$, and $-d_4$. Two of the remaining five eigenvalues are the eigenvalues of the following matrix

$$\begin{pmatrix} -d_2 & c\beta\frac{N_v}{N_h} \\ b\beta & -k_1 \end{pmatrix}.$$

The two eigenvalues of this matrix are given by following quadratic equation

$$\lambda^2 + (d_2+k_1)\lambda + d_2k_1 - \frac{bc\beta^2 N_v^*}{N_h^*} = 0.$$

Substituting the values of N_h^* and N_v^*, the above equation becomes

$$\lambda^2 + (d_2+k_1)\lambda + d_2k_1\left(1 - \frac{bc\beta^2 A_1 d_4}{\Lambda k_1 d_2}\right) = 0.$$

So roots of this quadratic have negative real parts provided the basic reproduction number $R_0 < 1$.

The remaining three eigenvalues of the matrix M_3 are the eigenvalues of the following matrix

$$\begin{pmatrix} -f_{11} & g & -\gamma L_v^* \\ m & -d_2 & 0 \\ k\gamma P^* & 0 & k\gamma L_v^* - d_3 \end{pmatrix}.$$

The three eigenvalues of this matrix are given by the following cubic equation in λ,

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0,$$

where

$$h_1 = -[f_{11} + d_2 + d_3 - k\gamma L_v^*],$$
$$h_2 = -d_2(k\gamma L_v^* - d_3) + [-f_{11}(k\gamma L_v^* - d_3) + k\gamma^2 L_v^* P^*] + f_{11}d_2 - mg,$$
$$h_3 = -k\gamma^2 d_2 L_v^* P^* + (k\gamma L_v^* - d_3)(f_{11}d_2 - mg).$$

By Routh Hurwitz criteria, roots of this cubic equation will have negative real parts if $h_1h_2 - h_3$ is positive. Hence the equilibrium point E_3 is locally asymptotically stable provided $R_0 < 1$ & $h_1h_2 > h_3$.

The variational matrix at the equilibrium point $E_4\left(L_v^*, N_v^*, I_v^*, P^*, N_h^*, I_h^*, R_h^*\right)$ is given by

$$M_4 = \begin{pmatrix} b_{11} & g & 0 & -\gamma L_v & 0 & 0 & 0 \\ m & -d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{32} & -b_{32}-d_2 & 0 & b_{35} & b_{36} & 0 \\ k\gamma P & 0 & 0 & k\gamma L_v - d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_4 & -d_5 & 0 \\ .0 & 0 & b_{63} & 0 & b_{65} & b_{67}-k_1 & b_{67} \\ 0 & 0 & 0 & 0 & 0 & \rho\tau & -(d_4+l_1) \end{pmatrix}$$

where

$$b_{11} = -(d_1 + 2\alpha L_v + m + \gamma P),$$
$$b_{32} = c\beta\left(\frac{I_h + R_h}{N_h}\right),$$
$$b_{35} = -c\beta\left(\frac{(N_v - I_v)(I_h + R_h)}{N_h^2}\right),$$
$$b_{36} = c\beta\left(\frac{N_v - I_v}{N_h}\right),$$
$$b_{63} = b\beta\left(\frac{N_h - I_h - R_h}{N_h}\right),$$
$$b_{65} = b\beta\left(\frac{(I_h + R_h)I_v}{N_h^2}\right),$$
$$b_{67} = -b\beta\left(\frac{I_v}{N_h}\right).$$

The eigenvalues of this variational matrix are given by the roots of the following two equations in λ: $\lambda^3 + g_1\lambda^2 + g_2\lambda + g_3 = 0$ and $\lambda^4 + u_1\lambda^3 + u_2\lambda^2 + u_3\lambda + u_4 = 0$ which are the characteristic equations of the following matrices respectively:

$$\begin{pmatrix} -f_{11} & g & -\gamma L_v^* \\ m & -d_2 & 0 \\ k\gamma P^* & 0 & k\gamma L_v^* - d_3 \end{pmatrix},$$

$$\begin{pmatrix} b_{33} & b_{35} & b_{36} & 0 \\ 0 & -d_4 & -d_5 & 0 \\ b_{63} & b_{65} & b_{66} & b_{67} \\ 0 & 0 & \rho\tau & b_{77} \end{pmatrix}.$$

where

$$g_1 = -[f_{11} + d_2 + d_3 - k\gamma L_v^*],$$
$$g_2 = -d_2(k\gamma L_v^* - d_3) + [-f_{11}(k\gamma L_v^* - d_3) + k\gamma^2 L_v^* P^*] + f_{11}d_2 - mg,$$
$$g_3 = -k\gamma^2 d_2 L_v^* P^* + (k\gamma L_v^* - d_3)(f_{11}d_2 - mg),$$

Table 2: Parameter Values and References

Parameter	Baseline value	Reference
g	30	[23]
k	0.0005	[23]
α	0.001	Assumed
m	1/16	[23]
γ	0.0005	Assumed
β	0.25	Assumed
b	0.5	[23]
c	0.5	[23]
Λ	0.5	[24]
ρ	0.01	[23]
l_1	0.014	Assumed
τ	0.1	Assumed
d_1	0.1	[23]
d_2	0.1	[23]
d_3	1/365	[23]
d_4	$1/(70\times 365)$	[23,24]
d_5	$10/(70\times 365)$	Assumed

$$u_1 = b_{33} - d_4 + b_{66} + b_{77},$$

$$u_2 = \begin{vmatrix} b_{33} & b_{35} \\ 0 & -d_4 \end{vmatrix} + \begin{vmatrix} -d_4 & -d_5 \\ b_{65} & b_{66} \end{vmatrix} + \begin{vmatrix} b_{66} & b_{67} \\ \rho & b_{77} \end{vmatrix} + \begin{vmatrix} b_{33} & b_{36} \\ b_{63} & b_{66} \end{vmatrix} + \begin{vmatrix} -d_4 & 0 \\ 0 & b_{77} \end{vmatrix} + \begin{vmatrix} b_{33} & 0 \\ 0 & b_{77} \end{vmatrix},$$

$$u_3 = \begin{vmatrix} b_{33} & b_{35} & b_{36} \\ 0 & -d_4 & -d_5 \\ b_{63} & b_{65} & b_{66} \end{vmatrix} + \begin{vmatrix} b_{33} & b_{36} & 0 \\ b_{63} & b_{66} & b_{67} \\ 0 & \rho\tau & b_{77} \end{vmatrix} + \begin{vmatrix} b_{33} & b_{35} & 0 \\ 0 & -d_4 & 0 \\ 0 & \tau & b_{77} \end{vmatrix} + \begin{vmatrix} -d_4 & -d_5 & 0 \\ b_{65} & b_{66} & b_{67} \\ 0 & \rho\tau & b_{77} \end{vmatrix},$$

$$u_4 = \begin{vmatrix} b_{33} & b_{35} & b_{36} & 0 \\ 0 & -d_4 & -d_5 & 0 \\ b_{63} & b_{65} & b_{66} & b_{67} \\ 0 & 0 & \rho\tau & b_{77} \end{vmatrix}.$$

By Routh Hurwitz criteria, roots of this cubic equation will have negative real parts if $u_i > 0,\ i = 1,3,4$ & $u_1u_2u_3 > u_3^2 + u_1^2u_4$. Hence the equilibrium point E_4 is locally asymptotically stable provided $g_1 > 0,\ g_1g_2 - g_3 > 0,\ u_i > 0,\ i = 1,3,4$ & $u_1u_2u_3 > u_3^2 + u_1^2u_4$.

4.2 Numerical simulation

At first we demonstrate the backward bifurcation for the model (2) by considering β as the bifurcation parameter. All other parameter values are as in Table 2. The bifurcation diagram is obtained by varying β and corresponding values of R_0 is placed along the x-axis for better visualization of this phenomenon. Here Figures 2 & 3 are showing the backward bifurcation. Figure 3 is showing the effect of rate of treatment τ on the dynamics of this disease. From the Figure 3, it can be observed that there is a shift in the backward bifurcation curves with the increase in the value of τ, i.e., the increase in the rate of treatment is causing the backward bifurcation curve to shift to right, which leads to increase in the R_0^c (the critical value of the R_0). From this figure it is easy to visualize that the further increase in τ can force R_0^c to shift towards 1. As in the case of backward bifurcation, one need to lower the R_0 value below R_0^c to get the disease-free equilibrium to be stable, so increase in it showing the positive impact of the treatment. The biological interpretation of this is that the increase in the rate of treatment can lead to disappearance of the backward bifurcation curve and in this case lowering R_0 below one will be sufficient to eliminate the disease from the population. So if the rate of treatment is high enough, we will have only forward bifurcation and lowering R_0 below one would be sufficient to make the disease-free equilibrium to be globally stable. This fact is demonstrated in Figure 4, where the rate of treatment τ is taken as 1 and we have only the forward bifurcation. The system (2) is simulated for various set of parameters satisfying the conditions of local asymptotic stability of different equilibria E_1 and E_2 by fourth order Runge–Kutta method. To exhibit the stability of the DFE E_1 we considered the parameters $g = 10$, $\beta = 0.14$ and all the other parameters are taken from Table 2. In a similar way to show the stability of EE E_2, we considered the parameters $g = 10$ and all the other parameters are taken from Table 2. The stability of these equilibria are demonstrated in Figures 5, 6, 7 & 8.

5 The Optimal Control Model

Here in this section we have extended the basic model (1) to optimal control model by introducing the time dependent variable $u(t)$ which is representing the insecticide control. We shall use Pontryagin's Maximum Principle (see [25,42,43,44], *etc.*) to analyze this model. Our aim is to find the minimal effort required to decrease the mosquitoes population considering the cost of insecticide application, while minimizing the cost of implementation of such measures. The optimal control

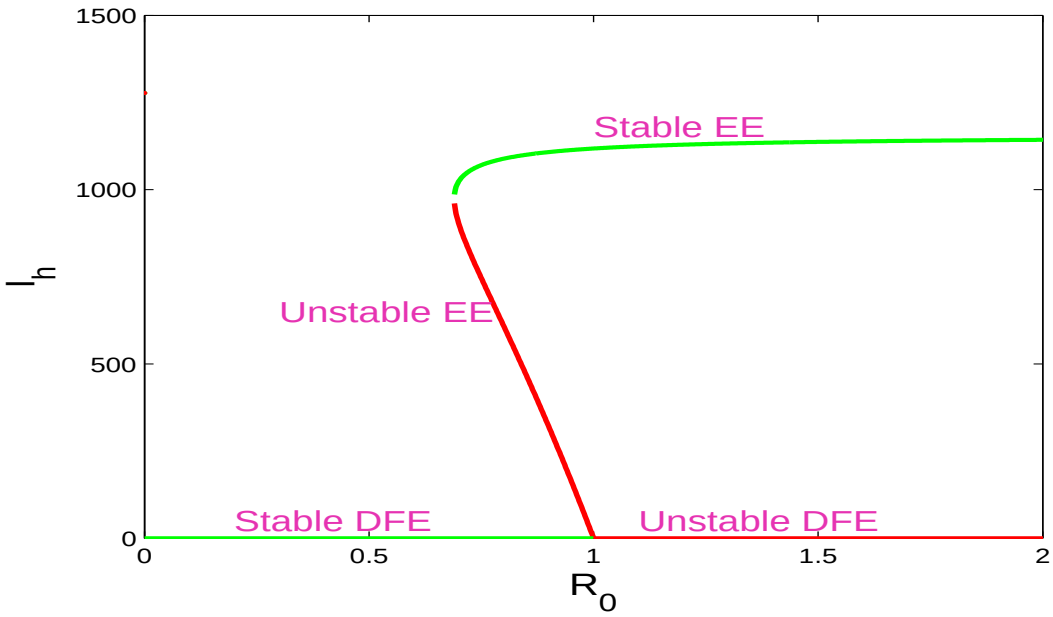

Fig. 2: Variation of the equilibrium level of I_h with β showing the backward bifurcation of the model (2) where all the other parameters are given in Table 2.

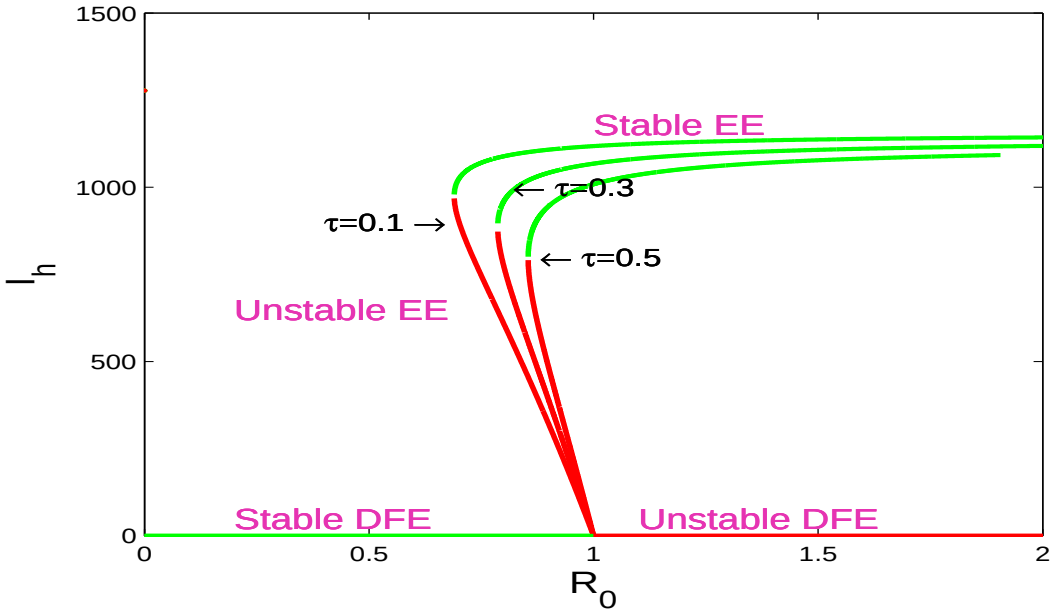

Fig. 3: Effect of τ on the backward bifurcation curve where all the other parameters are given in Table 2.

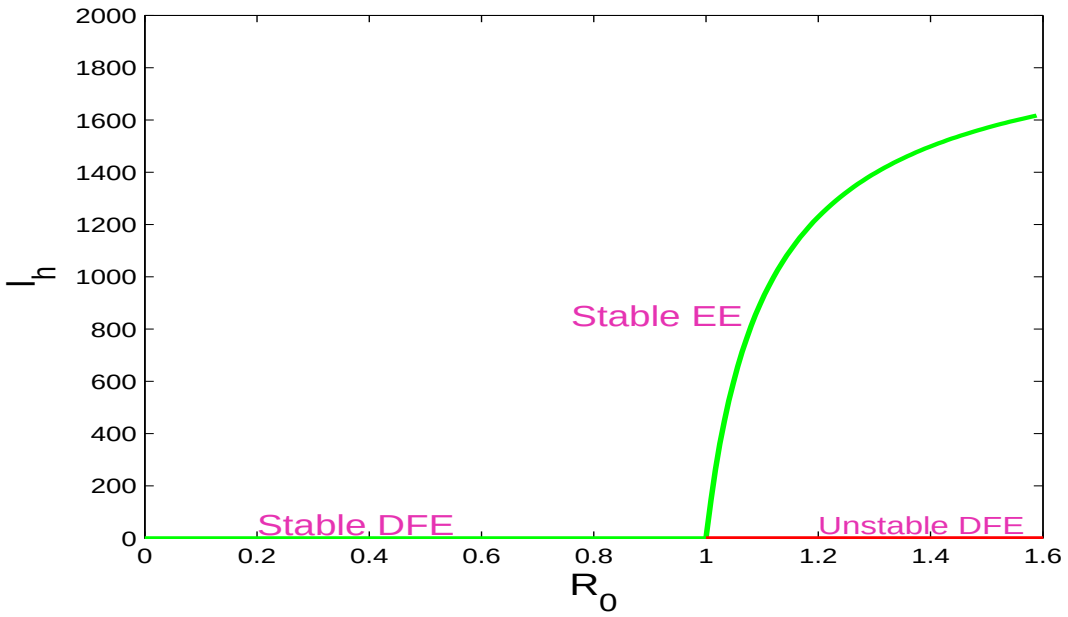

Fig. 4: Variation of the equilibrium level of I_h with β showing the forward bifurcation of the model (2) for the parameter values $\Lambda = 0.9$, $k = 0.00101, \tau = 1$ and all the other parameters are given in Table 2.

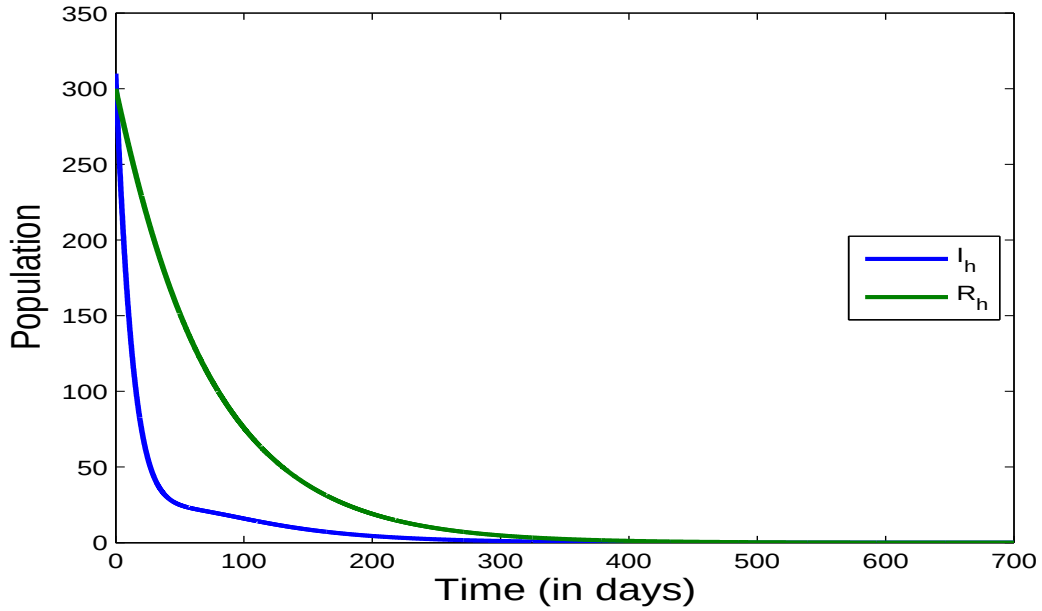

Fig. 5: Variation of I_h and R_h with time showing the stability of the disease-free Equilibrium when $R_0 < 1$ for the parameter values $g = 10$, $\beta = 0.14$ and all the other parameters are given in Table 2.

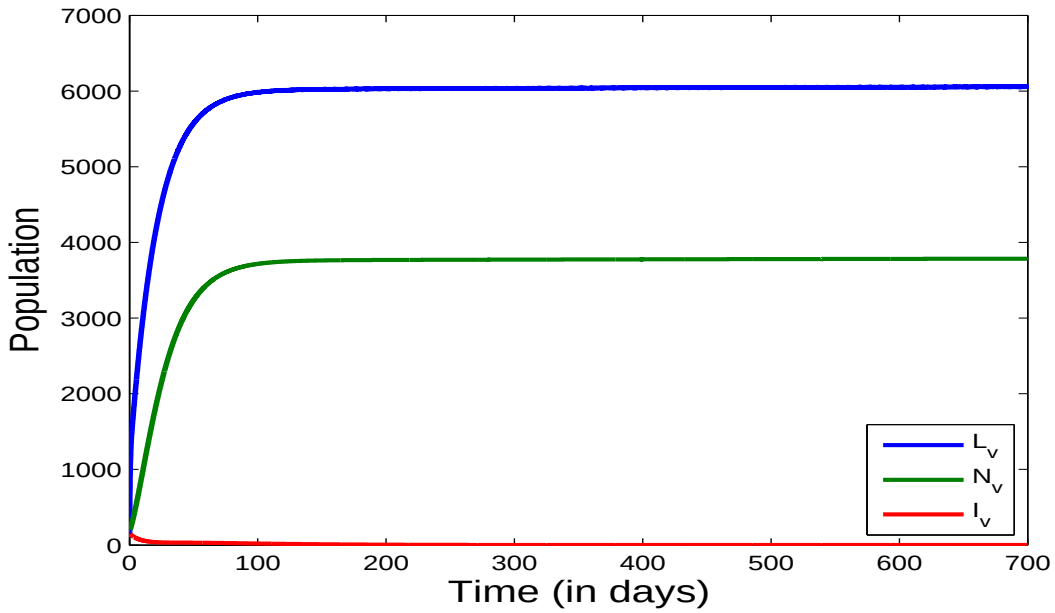

Fig. 6: Variation of L_v, N_v and I_v with time showing the stability of the disease-free Equilibrium when $R_0 < 1$ for the parameter values $g = 10$, $\beta = 0.14$ and all the other parameters are given in Table 2.

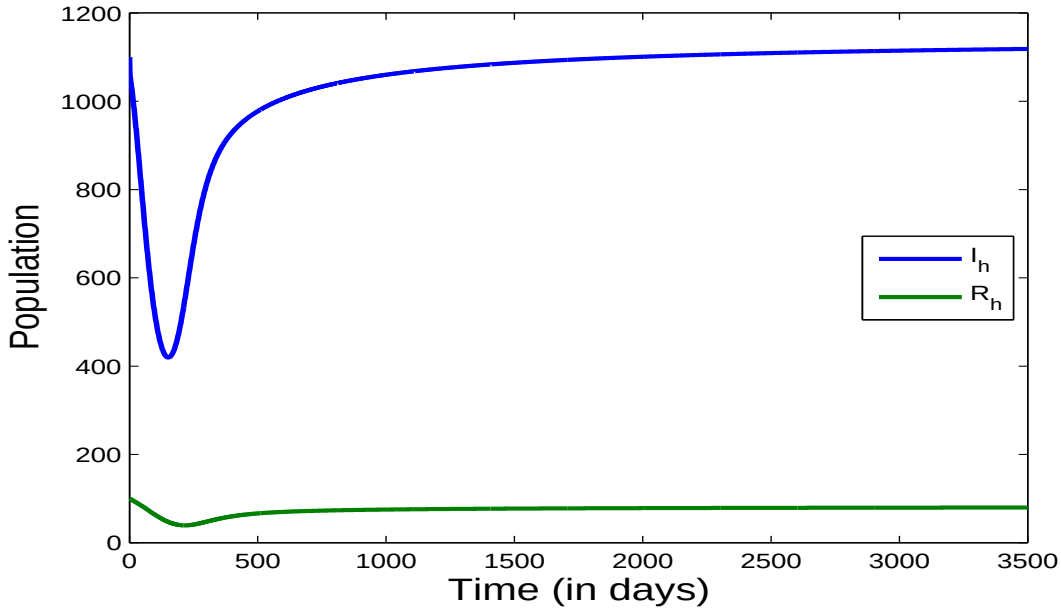

Fig. 7: Variation of I_h and R_h with time showing the stability of endemic equilibrium when $R_0 > 1$ for the parameter values $g = 10$ and all the other parameters are given in Table 2.

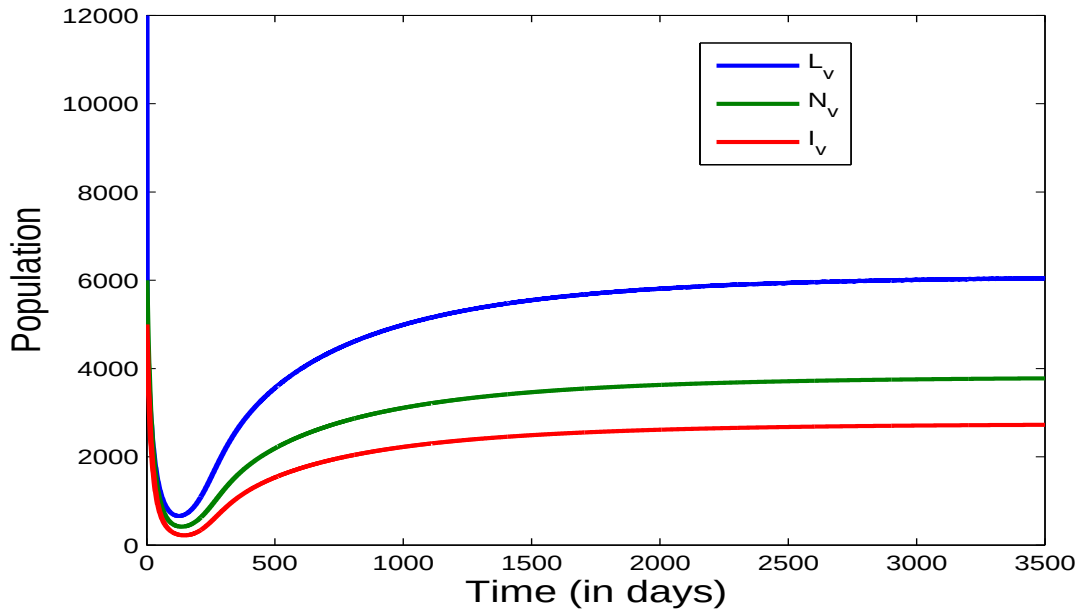

Fig. 8: Variation of L_v, N_v and I_v with time showing the stability of endemic equilibrium when $R_0 > 1$ for the parameter values g = 10 and all the other parameters are given in Table 2.

model to be optimized is given below:

$$L_v' = gN_v - d_1L_v - \alpha L_v^2 - mL_v - \gamma L_v P, \tag{8a}$$

$$S_v' = mL_v - c\beta S_v\frac{(I_h+R_h)}{N_h} - (d_2+u(t))S_v, \tag{8b}$$

$$I_v' = c\beta S_v\frac{(I_h+R_h)}{N_h} - (d_2+u(t))I_v, \tag{8c}$$

$$P' = k\gamma L_v P - d_3 P, \tag{8d}$$

$$S_h' = \Lambda - b\beta S_h\frac{I_v}{N_h} - d_4S_h + [(1-\rho)\tau]I_h + l_1R_h, \tag{8e}$$

$$I_h' = b\beta S_h\frac{I_v}{N_h} - (d_4+d_5+\tau)I_h, \tag{8f}$$

$$R_h' = \rho\tau I_h - (d_4+l_1)R_h, \tag{8g}$$

Here insecticide control $u(t)$ is applied only to adult form of mosquitoes assuming that this control is effective only in the adult stage and not in the aquatic phase. The objective (cost) functional corresponding to this optimal control model (8) is given by

$$J(u) = \int_0^T \left(C_1S_v + C_2I_v + \frac{1}{2}C_3u^2\right)dt, \tag{9}$$

subject to the state system given by (8).

Our objective is to find a control u^* such that $J(u^*) = \min_{u\in\Omega} J(u)$ where $\Omega = \{u$: is measurable and $0 \le u(t) \le 1\}$ for $t \in [0,T]$ is the set for the control.

Here, the value $u(t) = 1$ represents the maximal control due to insecticide effect. The quantities C_1 and C_2 represent, respectively, the weight constants of the susceptible and infected mosquito populations. On the other hand, C_3 is weight constant for mosquito control. The term C_3u^2 describes the cost associated with mosquito control.

The Lagrangian of this problem is given by

$$L(S_v, I_v, u) = C_1S_v + C_2I_v + \frac{1}{2}C_3u^2. \tag{10}$$

Next we form the Hamiltonian H for our problem as follows:

$$H = L(S_v,I_v,u) + \lambda_1\frac{dL_v}{dt} + \lambda_2\frac{dS_v}{dt} + \lambda_3\frac{dI_v}{dt} + \lambda_4\frac{dP}{dt} + \lambda_5\frac{dS_h}{dt} + \lambda_6\frac{dI_h}{dt} + \lambda_7\frac{dR_h}{dt},$$

where λ_i, $i = 1,\ldots,7$ are the adjoint variables or the co-state variables and can be determined by solving the following system of differential equations:

$$\lambda_1' = -\frac{\partial H}{\partial L_v} = \lambda_1[-d_1 - 2\alpha L_v - m - \gamma P] + \lambda_2 m + \lambda_4 k\gamma P, \quad (11a)$$

$$\lambda_2' = -\frac{\partial H}{\partial S_v} = \lambda_2[-c\beta\frac{(I_h+R_h)}{N_h} - (d_2+u(t))] + \lambda_3 c\beta\frac{(I_h+R_h)}{N_h}, \quad (11b)$$

$$\lambda_3' = -\frac{\partial H}{\partial I_v} = -\lambda_3(d_2+u(t)) - \lambda_5 b\beta\frac{S_h}{N_h} + \lambda_6 b\beta\frac{S_h}{N_h} \quad (11c)$$

$$\lambda_4' = -\frac{\partial H}{\partial P} = -\lambda_1\gamma L_v + \lambda_4[k\gamma L_v - d_3], \quad (11d)$$

$$\lambda_5' = -\frac{\partial H}{\partial S_h} = -\lambda_5[b\beta\frac{I_v}{N_h} - d_4] + \lambda_6[b\beta\frac{I_v}{N_h}], \quad (11e)$$

$$\lambda_6' = -\frac{\partial H}{\partial I_h} = -\lambda_2[c\beta\frac{S_v}{N_h}] + \lambda_3[c\beta\frac{S_v}{N_h}] + \lambda_5[(1-\rho)\tau] - \lambda_6(d_4+d_5+\tau) + \lambda_7(\rho\tau), \quad (11f)$$

$$\lambda_7' = -\frac{\partial H}{\partial R_h} = -\lambda_2[c\beta\frac{S_v}{N_h}] + \lambda_3[c\beta\frac{S_v}{N_h}] + \lambda_5 l_1 - \lambda_7(d_4+l_1). \quad (11g)$$

Let $\tilde{L_v}, \tilde{S_v}, \tilde{I_v}, \tilde{P}, \tilde{S_h}, \tilde{I_h}$ and $\tilde{R_h}$ be the optimum values of $L_v, S_v, I_v, P, S_h, I_h$ and R_h respectively. Also let $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_5, \tilde{\lambda}_6, \tilde{\lambda}_7\}$ be the solutions of the system (11).

We now state and prove the following theorem by following [45].

Theorem 4. *There exists optimal controls $u^* \in \Omega$ such that*

$$J(u^*) = \min_{u \in \Omega} J(u)$$

subject to the system (8).

Proof. We use [45] to prove this theorem. Here the control and the state variables are nonnegative values. The necessary convexity of the objective functional in u is satisfied for this minimizing problem. The control variable set $u \in \Omega$ is also convex and closed by definition. The optimal system is bounded which determines the compactness needed for the existence of the optimal control. In addition, the integrand in the functional (9), $C_1 S_v + C_2 I_v + \frac{1}{2} C_3 u^2$ is convex on the control set Ω and the state variables are bounded. This completes the proof of this theorem.

Since there exists an optimal control for minimizing the functional subject to equations (8) and (11), we use Pontryagin's Maximum Principle to derive the necessary conditions to find the optimal solution as follows:
If (x,u) is an optimal solution of an optimal control problem, then there exists a non trivial vector function $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ satisfying the following equalities.

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H(t,x,u,\lambda)}{\partial \lambda}, \\ 0 &= \frac{\partial H(t,x,u,\lambda)}{\partial u}, \\ \lambda' &= -\frac{\partial H(t,x,u,\lambda)}{\partial x}. \end{aligned} \quad (12)$$

With the help of Pontryagin's Maximum Principle [1] we now state and prove the following theorem.

Theorem 5. *The optimal control u^* which minimizes J over the region Ω is given by*

$$u^* = \max\{\min(\tilde{u}, 1), 0\} \quad (13)$$

where

$$\tilde{u} = \frac{\tilde{\lambda}_2 \tilde{S_v} + \tilde{\lambda}_3 \tilde{I_v}}{C_3}.$$

Proof. Using the optimality condition

$$\frac{\partial H}{\partial u} = 0$$

$$u = \frac{\tilde{\lambda}_2 \tilde{S_v} + \tilde{\lambda}_3 \tilde{I_v}}{C_3} (= \tilde{u}).$$

This control is bounded with upper and lower bounds as 0 and 1 respectively, *i.e.* $u = 0$ if $\tilde{u} < 0$ and $u = 1$ if $\tilde{u} > 1$ otherwise $u = \tilde{u}$. Hence for this control (u^*), we get the optimum value of the functional J given by equation (9). Hence the theorem.

5.1 Numerical Simulation for the optimal control problem

In this section the effect of optimal control by introducing insecticide control $u(t)$ on the basic model (2) has been shown through simulation. The parameters used for the simulation purpose are as stated in Table 2, except the parameter g which is 3.3. Moreover, the time interval for which the optimal control is applied is taken as 100 days. We compared the results of optimal control model (8) with the results of model (2). The optimality system in Section 5 is solved by iterative method with the help of Runge–Kutta fourth order procedure (see Jung et al.[46], Lenhart and Workman[47], etc.). At first we solve the state equations by the forward Runge–Kutta fourth order procedure for the time interval [0, 100] starting with an initial guess for the adjoint variables. Then we use the backward Runge–Kutta fourth order procedure to solve the adjoint variables in the same time interval with the help of the solutions of the state variables and the transversal conditions. From Figure 9, it is evident that the control takes the highest value 1 in the beginning and it has to be maintained up to 20 days then the usage of insecticide can be relaxed but to be maintained at certain level then it has to be reached to the minimum value 0 at the final time $T = 100$ days. From these we conclude that the insecticide control is to be maintained at certain level up to certain days according to the duration of the optimal strategy period to get the desired optimal value for the functional (9). Figures 10-14 represent the plots of I_h, I_v, L_v, N_h, N_v and R_h with and without optimal

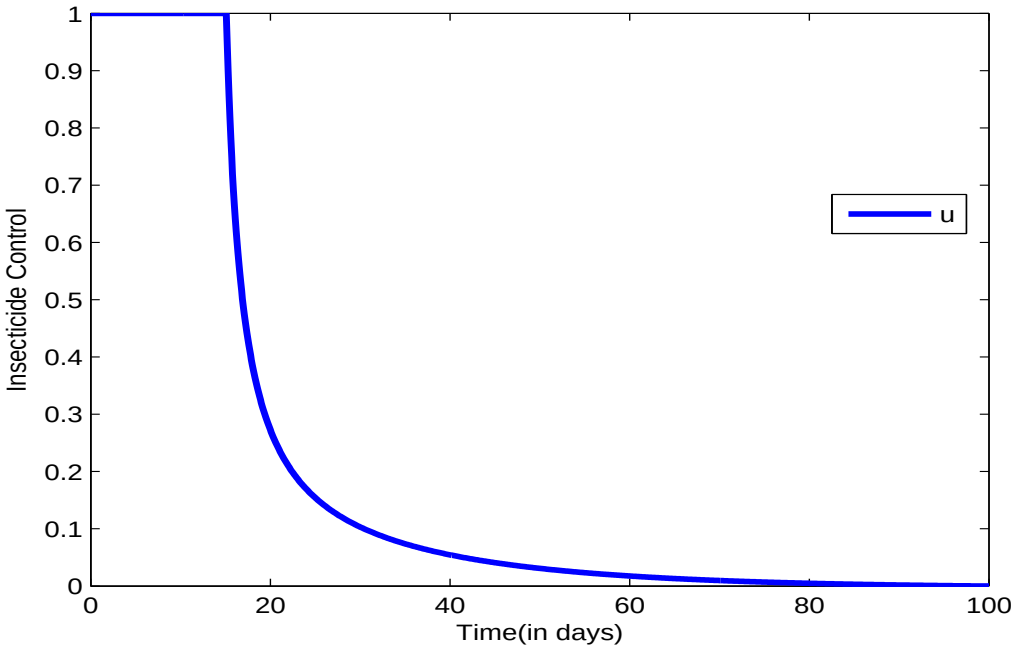

Fig. 9: Control profile of the parameter $u(t)$ (insecticide control) of the model (8) for the parameter values $g = 3.3$, $\beta = 0.3$ and all the other parameter values are as given in Table 2.

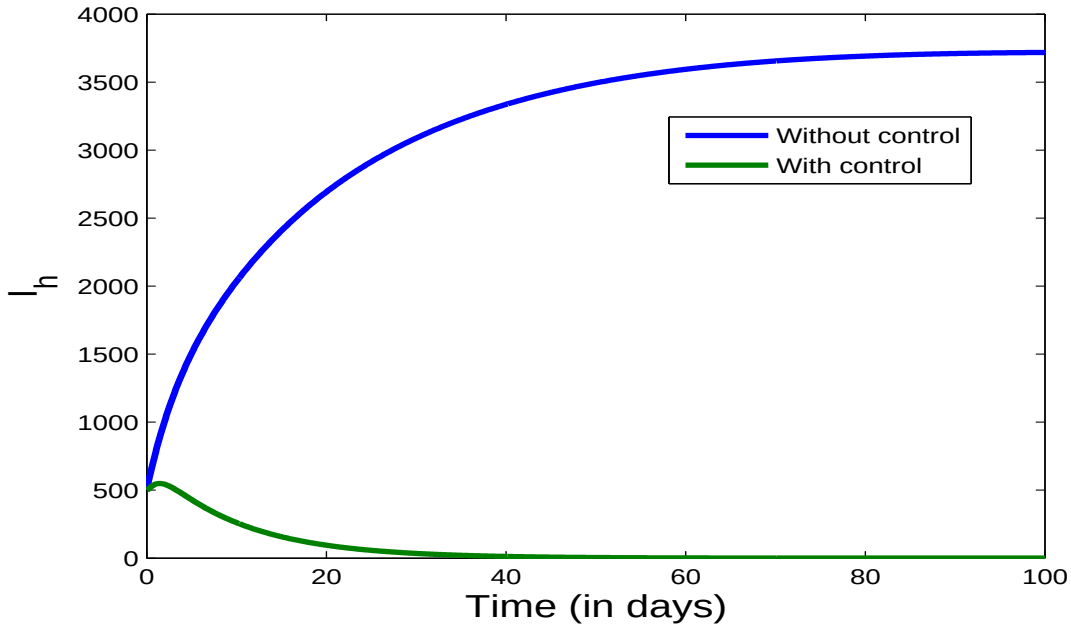

Fig. 10: Simulations of the malaria model showing the effect of the optimal control strategy on I_h.

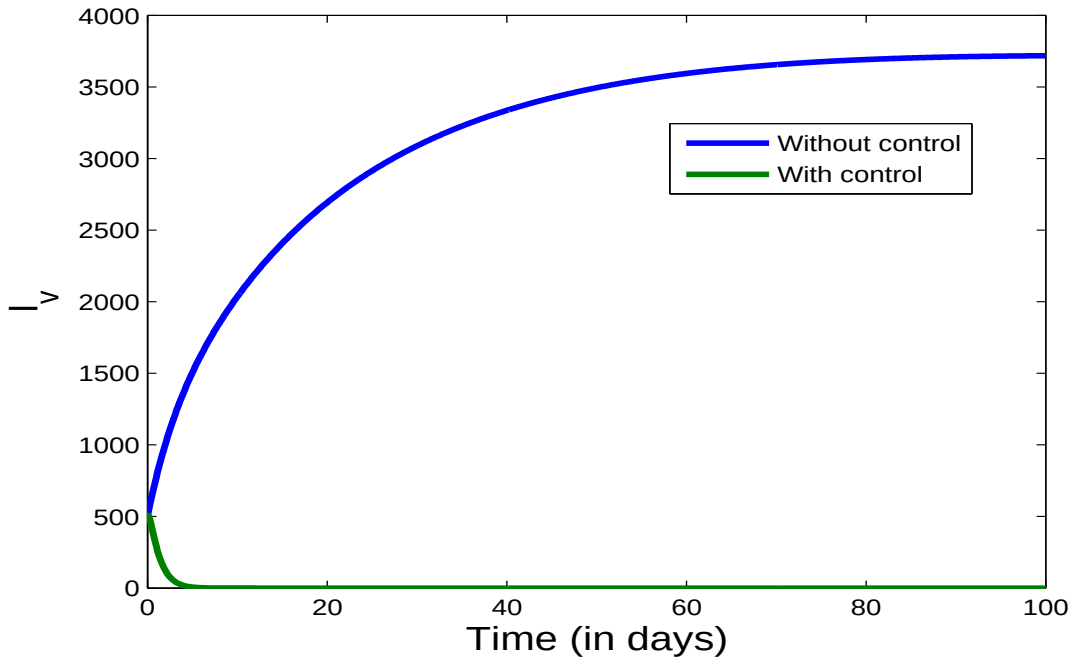

Fig. 11: Simulations of the malaria model showing the effect of the optimal control strategy on I_v.

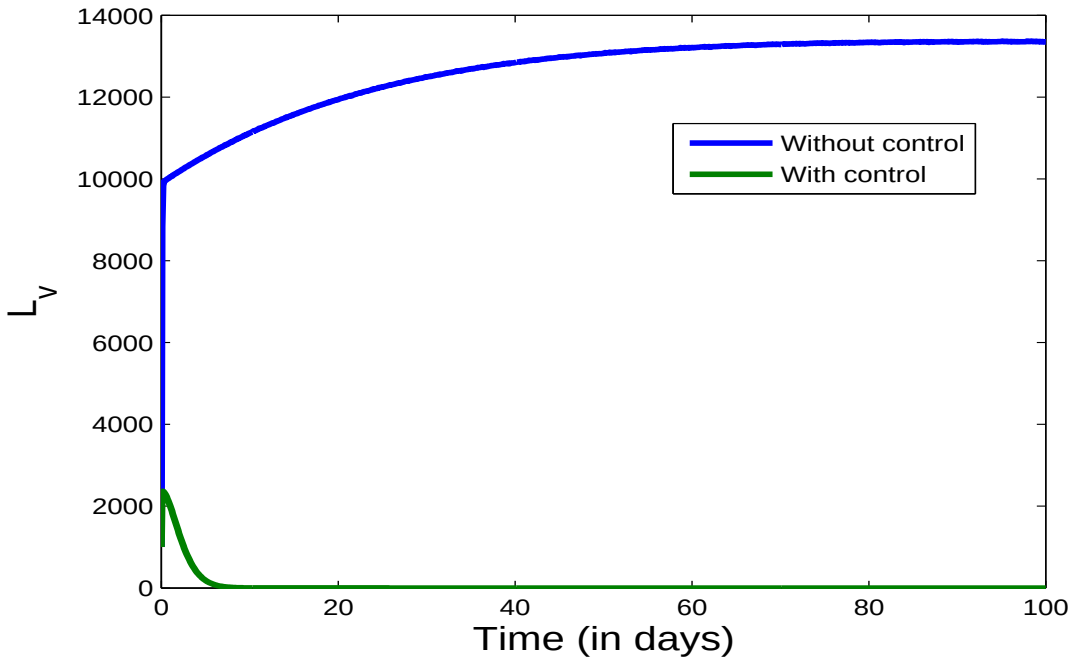

Fig. 12: Simulations of the malaria model showing the effect of the optimal control strategy on L_v.

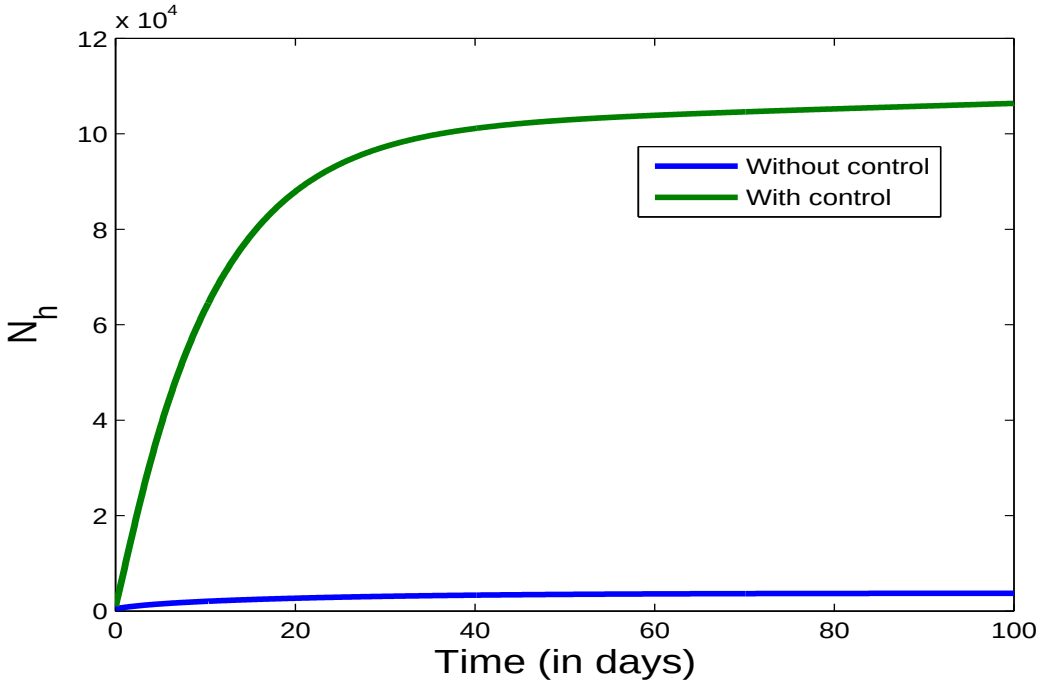

Fig. 13: Simulations of the malaria model showing the effect of the optimal control strategy on N_h.

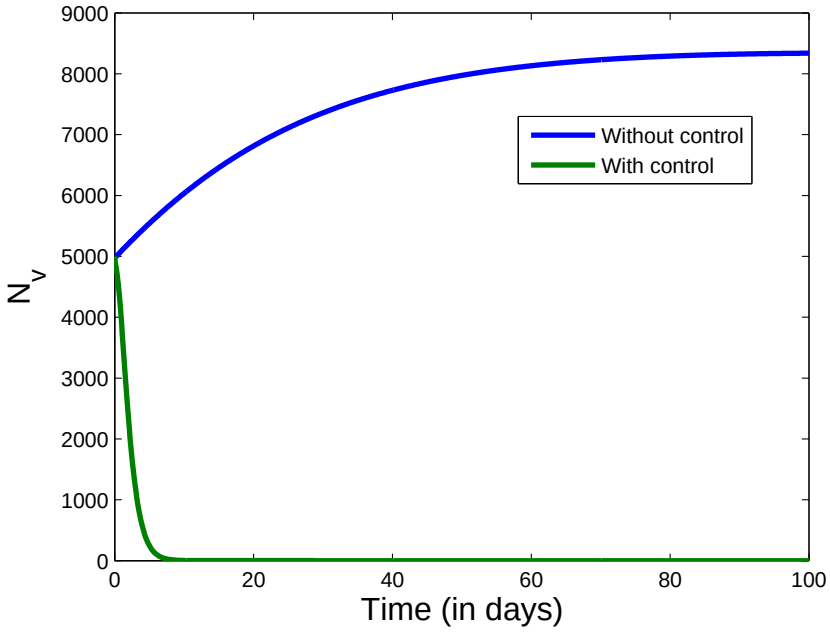

Fig. 14: Simulations of the malaria model showing the effect of the optimal control strategy on N_v.

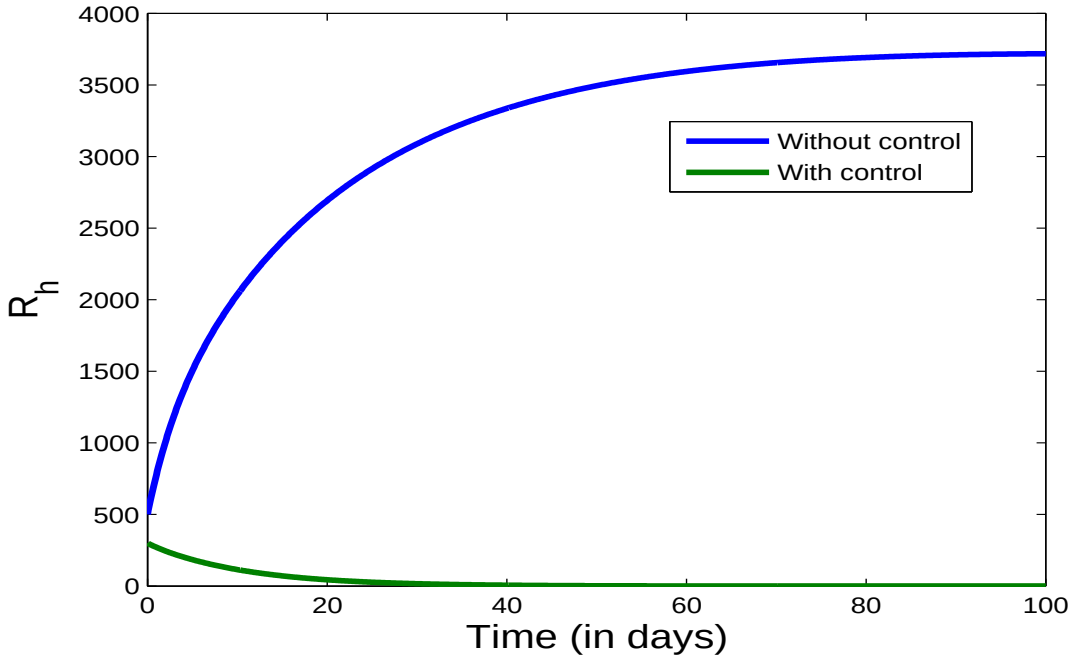

Fig. 15: Simulations of the malaria model showing the effect of the optimal control strategy on R_h.

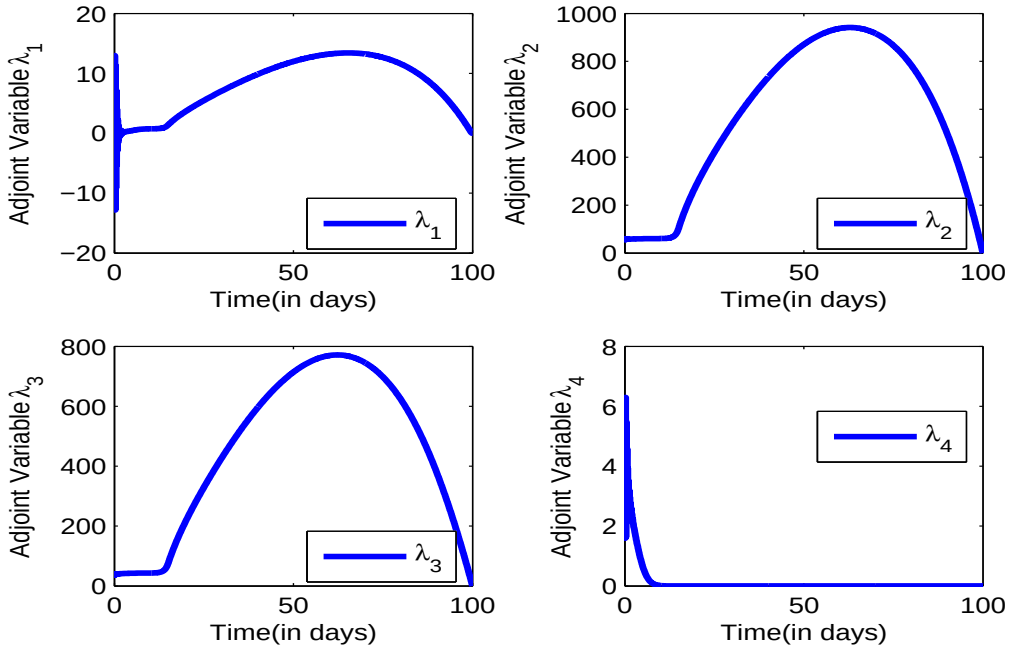

Fig. 16: Plots of the adjoint variables λ_j, $j = 1, 2 \ldots 4$.

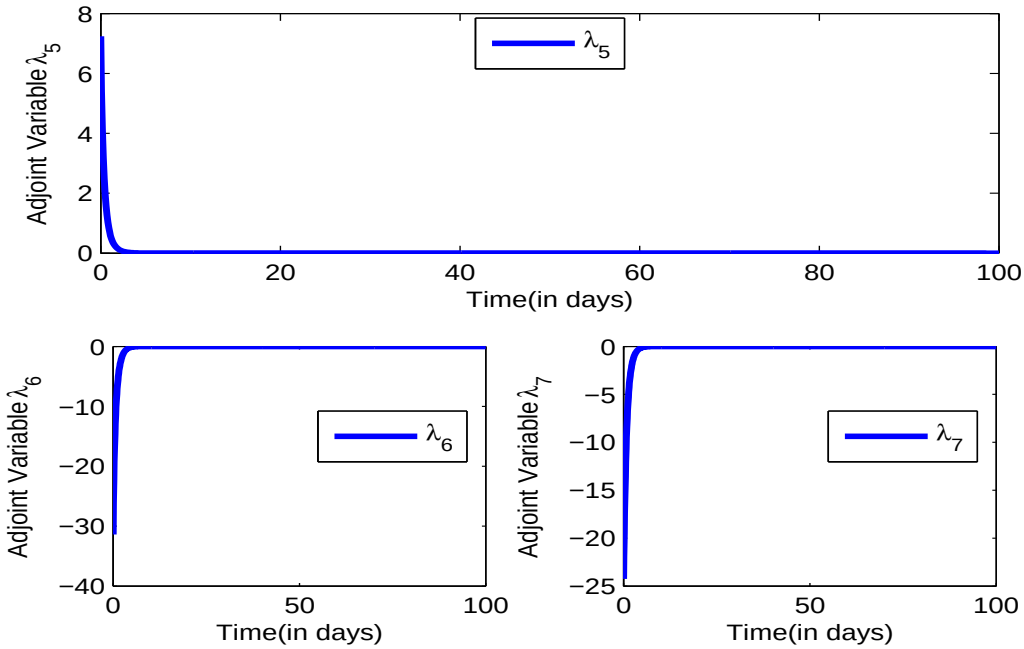

Fig. 17: Plots of the adjoint variables λ_j, $j = 5, 6, 7$.

control respectively. From these figures it is evident that the implementation of optimal control strategies produces better results in the sense that it decreases the infected/infectious population.

From Figures 16 & 17 it is evident that the adjoint variables are directly related to the change of the value of the Hamiltonian as the time derivatives of the adjoint variables are negative of the corresponding partial derivatives of the Hamiltonian, H with respect to the state variables.

6 Conclusion

Here a nonlinear mathematical model for malaria is formulated and analyzed by incorporating the effect of introduction of larvivorous fish as biological control agent. The expression for the basic reproduction number R_0 is computed and equilibria of the model are found. The stability of these equilibria are discussed in detail. It is observed that system may exhibit backward bifurcation under some restriction on parameters. This fact is also demonstrated numerically. Here the bifurcation parameter β is involved in the disease transmission but it is found that the rate of treatment also plays an important role in the occurrence of backward bifurcation. In fact the rate of treatment has positive impact on the elimination of malaria as increase in it forces R_0^c to move towards 1, leading to disappearance of backward bifurcation. And when there is no backward bifurcation, system exhibits only forward bifurcation and in this case reducing R_0 below 1 becomes sufficient to eliminate the disease from the population.

Further the basic model is extended to an optimal control problem to study the dynamics of the disease by introducing the insecticide control parameter. Pontryagin's maximum principle is used to solve this optimal control problem. Later, numerical simulation is performed to see the effect of optimal control on the dynamics of this disease. Simulation results predict that the optimal control model gives better results compared to the model without optimal control.

Acknowledgement

The authors thank the handling editor and anonymous referees for their valuable comments and suggestions which led to an improvement of our original manuscript.

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