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# Existence Results for a Four-Point Impulsive Boundary Value Problem Involving Fractional Differential Equation

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**Abstract:** In this paper, we discuss the existence of solutions for a class of a four-point boundary value problems involving nonlinear impulsive fractional differential equation. By use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

**Keywords:** four-point, boundary value problem, impulsive, fractional differential equation

## 1 Introduction

Fractional differential equation arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymerrheology, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equation is gaining much importance and attention. For details, see [1,2,3,5,8,9,12] and the refrences therein.

On the other hand, the study of impulsive boundary value problem involving fractional differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments, for instance, see [6,10]. Recently, in [4], the existence of three positive solution was investigated for second order four point boundary value problems for impulsive differential equation using the Leggett-Williams theorem. Ref. [11] includes results on the existence of solution of three-point boundary value problems involving nonlinear impulsive fractional differential equation. For some results on the solutions of impulsive functional differential equation with periodic boundary condition, see [7].

In this paper, we study the existence of solutions for a four-point impulsive boundary value problem involving nonlinear fractional differential equation by use of Banach's fixed point theorem and Schauder's fixed point theorem:

$$\begin{aligned} &{}^{C}D^{\theta}v(t)=h(t), \qquad 0<t<1,\ t\neq t_k, \qquad k=1,2,....p, \\ &\Delta v\Big|_{t=t_k}=I_k(v(t_k)), \quad \Delta v'\Big|_{t=t_k}=\bar{I}_k(v(t_k)), \qquad k=1,2,....p \\ &v(0)+v'(\zeta)=0, \qquad v(1)+v'(\eta)=0 \end{aligned} \tag{1}$$

Where ${}^{C}D^{\theta}$ is the Caputo fractional derivative, $\theta\in R$, $1<\theta\leq 2$, $g\colon [0,1]\times R\to R$ is a continuous function, $I_k$, $\bar{I}_k$ : $R\to R$, $\zeta$, $\eta\in(0,1)$, $\zeta\neq t_k$, $\eta\neq t_k$, $k=1,2,....p$ and $\Delta v\big|_{t=t_k}=v(t_k^+)-v(t_k^-)$, $\Delta v'\big|_{t=t_k}=v'(t_k^+)-v'(t_k^-)$, $v(t_k^+)$ and $v(t_k^-)$ represent the right-hand limit and the left-hand limit of the function $v(t)$ at $t=t_k$, and the sequences $\{t_k\}$ satisfy that $0=t_0<t_1<t_2<.....<t_p<t_{p+1}=1$, $p\in N$.

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## 2 Preliminaries

**Definition 1.***([5]) The Caputo fractional derivative of order $\theta > 0$ of function $g:\ (0,\infty) \to R$ is given by*

$$^{C}D^{\theta}g(t) = \frac{1}{\Gamma(n-\theta)}\int_0^t \frac{g^{(n)}(r)}{(t-r)^{\theta-n+1}}ds,$$

*where $n = [\theta]+1$ and $[\theta]$ denotes the integral part of number $\theta$.*

**Definition 2.***([5]) The Riemann-Liouville fractional integral of order $\theta > 0$ of function $g:\ (0,\infty) \to R$ is given by*

$$I^{\theta}g(t) = \frac{1}{\Gamma(\theta)}\int_0^t \frac{g(r)}{(t-r)^{1-\theta}}ds,$$

*provided that the integral exists.*

**Lemma 1.***([12]) Let $\theta > 0$, then the fractional differential equation*

$$^{C}D^{\theta}v(t) = 0$$

*has solution*

$$v(t) = c_0 + c_1 t + c_2 t^2 + \ldots\ldots + c_{n-1}t^{n-1},\ c_i \in R,\ i = 0,1,2,\ldots.n-1,\ n = [\theta]+1$$

**Lemma 2.***([12]) Let $\theta > 0$, then*

$$I^{\theta C}D^{\theta}v(t) = v(t) + c_0 + c_1 t + c_2 t^2 + \ldots\ldots + c_{n-1}t^{n-1},\ c_i \in R,\ i = 0,1,2,\ldots.n-1,\ n = [\theta]+1$$

*for the sake of convenience, we introduce the following notation. let $J = [0,1]$, $J_0 = (0,t_1]$, $J_1 = (t_1,t_2]$, ..........., $J_{p-1} = (t_{p-1},t_p]$, $J_p = (t_p,t_1]$, $t_{p+1} = 1$, $PC(J) = \left\{v:\ [0,1] \to R \mid v \in C(J),\ v\left(t_k^{+}\right) \text{ and } v\left(t_k^{-}\right) \text{ exist and } v\left(t_k^{-}\right) = v(t_k),\ 1 \le k \le p\right\}$ obviously, $PC(J)$ is a banach space with the norm $\|v\| = \sup_{0<t<1}|v(t)|$*

**Theorem 1.***([13]) (Ascoli–Arzelà's Theorem).*

Let $E$ be a compact Hausdorff space. If $\Omega$ is an equi-continuous and point-wise bounded subset of $C(E)$, then $\Omega$ is totally bounded.

**Theorem 2.***([13]) (Schauder fixed point theorem(1930))*

Let $K$ be a nonempty, convex, and compact subset of a Banach space $E$ and $T : K \to K$ is continuous then $T$ has at least one fixed point in the set$K$.

**Theorem 3.***([13]) (Banach fixed point theorem)*

Let $(X,\rho)$ be a complete metric space and $T : X \to X$ be a contraction mapping (i.e. there exists a constant $\gamma \in (0,1)$ such that for all $x, y \in X$, $\rho(Tx,Ty) \le \gamma\rho(x,y)$). Then $T$ has a unique fixed point

**Lemma 3.***Let $h \in C[0,1]$, and $\zeta \in (t_l,t_{l+1})$, $\eta \in (t_m,t_{m+1})$, $l$, $m$ are nonnegative integer, $0 \le l,\ m \le p$, $1 < \theta \le 2$. Then the unique solution of the boundary value problem*

$$^{C}D^{\theta}v(t) = h(t), \qquad 0 < t < 1,\ t \ne t_k, \qquad k = 1,2,\ldots.p$$

$$\Delta v\ \big|_{t=t_k} = I_k(v(t_k)), \quad \Delta v'\ \big|_{t=t_k} = \bar{I}_k(v(t_k)), \qquad k = 1,2,\ldots.p$$

$$v(0) + v'(\zeta) = 0, \qquad v(1) + v'(\eta) = 0 \tag{2}$$

*is given by*

$$v(t) = \begin{cases} \frac{1}{\Gamma(\theta)}\int_0^t (t-r)^{\theta-1}h(r)\,dr - \frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}h(r)\,dr \\ -\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr - \sum_{i=1}^{l}\bar{I}_i(v(t_i)) + (1-t)D, \qquad t \in J_0, \\ \\ \frac{1}{\Gamma(\theta)}\int_{t_k}^t (t-r)^{\theta-1}h(r)\,dr + \frac{1}{\Gamma(\theta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}h(r)\,dr \\ +\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr - \frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}h(r)\,dr \\ -\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr + \sum_{i=1}^{k}(t-t_i)\bar{I}_i(v(t_i)) \\ +\sum_{i=1}^{k}I_i(v(t_i)) - \sum_{i=1}^{l}\bar{I}_i(v(t_i)) + (1-t)D, \qquad t \in J_k, k = 1,2,\ldots p \end{cases} \tag{3}$$

*where*

$$D=\frac{1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}h(r)\,dr+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr$$
$$-\frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}h(r)\,dr-\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr$$
$$+\frac{1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}h(r)\,dr+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr$$
$$+\sum_{i=1}^{p}(1-t_i)\bar{I}_i(v(t_i))+\sum_{i=1}^{p}I_i(v(t_i))-\sum_{i=1}^{l}\bar{I}_i(v(t_i))+\sum_{i=1}^{m}\bar{I}_i(v(t_i))$$

*Proof.*Suppose that $u$ is asolution of (1). By applying lemma 2, we have

$$v(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-r)^{\theta-1}h(r)\,dr-c_1-c_2t,\qquad t\in J_0 \tag{4}$$

for some $c_1, c_2\in R$. then, we have

$$v^{'}(t)=\frac{1}{\Gamma(\theta-1)}\int_0^t(t-r)^{\theta-2}h(r)\,dr-c_2,\qquad t\in J_0 \tag{5}$$

If $t\in J_1$. then, we have

$$v(t)=\frac{1}{\Gamma(\theta)}\int_{t_1}^t(t-r)^{\theta-1}h(r)\,dr-d_1-d_2(t-t_1), \tag{6}$$

$$v^{'}(t)=\frac{1}{\Gamma(\theta-1)}\int_{t_1}^t(t-r)^{\theta-2}h(r)\,dr-d_2, \tag{7}$$

for some $d_1, d_2\in R$. Thus,

$$v(t_1^-)=\frac{1}{\Gamma(\theta)}\int_0^{t_1}(t_1-r)^{\theta-1}h(r)\,dr-c_1-c_2t_1,$$
$$v^{'}(t_1^-)=\frac{1}{\Gamma(\theta-1)}\int_0^{t_1}(t_1-r)^{\theta-2}h(r)\,dr-c_2,$$
$$v(t_1^+)=-d_1,$$
$$v'(t_1^+)=-d_2,$$

Since we have

$$\Delta v\Big|_{t=t_{k1}}=v(t_1^+)-v(t_1^-)=I_1(v(t_1))$$

$$-d_1=\frac{1}{\Gamma(\theta)}\int_0^{t_1}(t_1-r)^{\theta-1}h(r)\,dr-c_1-c_2t_1+I_1(v(t_1)) \tag{8}$$

$$\Delta v'\Big|_{t=t_1}=v'(t_1^+)-v^{'}(t_1^-)=\bar{I}_1(v(t_1))$$

$$-d_2=\frac{1}{\Gamma(\theta-1)}\int_0^{t_1}(t_1-r)^{\theta-2}h(r)\,dr-c_2+\bar{I}_1(v(t_1)) \tag{9}$$

Substituting (8) and (9) into (6), we get

$$v(t)=\frac{1}{\Gamma(\theta)}\int_{t_1}^t(t-r)^{\theta-1}h(r)\,dr+\frac{1}{\Gamma(\theta)}\int_0^{t_1}(t_1-r)^{\theta-1}h(r)\,dr$$
$$+\frac{(t-t_1)}{\Gamma(\theta-1)}\int_0^{t_1}(t_1-r)^{\theta-2}h(r)\,dr+I_1(v(t_1))+(t-t_1)\bar{I}_1(v(t_1))-c_1-c_2t$$
$$,\ t\in J_1 \tag{10}$$

In similar way, we get

$$v(t) = \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} h(r)\, dr + \frac{1}{\Gamma(\theta)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} h(r)\, dr$$
$$+ \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{k} (t-t_i) \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr$$
$$+ \sum_{i=1}^{k} (t-t_i) \bar{I}_i(v(t_i)) + \sum_{i=1}^{k} I_i(v(t_i)) - c_1 - c_2 t$$
$$,\ t \in J_k,\ k = 1,2,....,p \tag{11}$$

By (4), (11), we have

$$v(0) = -c_1 \tag{12}$$

$$v(1) = \frac{1}{\Gamma(\theta)} \int_{tp}^{1} (1-r)^{\theta-1} h(r)\, dr + \frac{1}{\Gamma(\theta)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} h(r)\, dr$$
$$+ \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{p} (1-t_i) \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr$$
$$+ \sum_{i=1}^{p} (1-t_i) \bar{I}_i(v(t_i)) + \sum_{i=1}^{p} I_i(v(t_i)) - c_1 - c_2 \tag{13}$$

$$v'(\zeta) = \frac{1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} h(r)\, dr + \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr$$
$$+ \sum_{i=1}^{l} \bar{I}_i(v(t_i)) - c_2 \tag{14}$$

$$v'(\eta) = \frac{1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} h(r)\, dr + \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr$$
$$+ \sum_{i=1}^{m} \bar{I}_i(v(t_i)) - c_2 \tag{15}$$

By the boundary condition

$$v(0) + v'(\zeta) = 0$$

$$c_1 = \frac{1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} h(r)\, dr + \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr + \sum_{i=1}^{l} \bar{I}_i(v(t_i)) - c_2 \tag{16}$$

By the boundary condition

$$v(1) + v'(\eta) = 0$$

$$\frac{1}{\Gamma(\theta)} \int_{tp}^{1} (1-r)^{\theta-1} h(r)\, dr + \frac{1}{\Gamma(\theta)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} h(r)\, dr$$
$$+ \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{p} (1-t_i) \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr$$
$$+ \sum_{i=1}^{p} (1-t_i) \bar{I}_i(v(t_i)) + \sum_{i=1}^{p} I_i(v(t_i)) - c_1 - c_2$$
$$\frac{1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} h(r)\, dr + \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} h(r)\, dr + \sum_{i=1}^{m} \bar{I}_i(v(t_i)) - c_2 = 0 \tag{17}$$

Substituting (16) into (17), we have

$$
\begin{aligned}
c_2 = & \frac{1}{\Gamma(\theta)}\int_{tp}^{1}(1-r)^{\theta-1}h(r)\,dr+\frac{1}{\Gamma(\theta)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}h(r)\,dr \\
& +\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr \\
& -\frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}h(r)\,dr-\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr \\
& +\frac{1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}h(r)\,dr+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}h(r)\,dr \\
& +\sum_{i=1}^{p}(1-t_i)\bar{I}_i(v(t_i))+\sum_{i=1}^{p}I_i(v(t_i)) \\
& +\sum_{i=1}^{m}\bar{I}_i(v(t_i))-\sum_{i=1}^{l}\bar{I}_i(v(t_i))
\end{aligned} \tag{18}
$$

Substituting (18) into (16), we have

$$
\begin{aligned}
c_1 = & -\frac{1}{\Gamma(q)}\int_{tp}^{1}(1-r)^{q-1}y(r)\,dr-\frac{1}{\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{q-1}y(r)\,dr \\
& -\frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-r)^{q-2}y(r)\,dr \\
& +\frac{2}{\Gamma(q-1)}\int_{t_l}^{\zeta}(\zeta-r)^{q-2}y(r)\,dr+\frac{2}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{q-2}y(r)\,dr \\
& -\frac{1}{\Gamma(q-1)}\int_{t_m}^{\eta}(\eta-r)^{q-2}y(r)\,dr-\frac{1}{\Gamma(q-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-r)^{q-2}y(r)\,dr \\
& -\sum_{i=1}^{p}(1-t_i)\bar{I}_i(v(t_i))-\sum_{i=1}^{p}I_i(v(t_i)) \\
& -\sum_{i=1}^{m}\bar{I}_i(v(t_i))+2\sum_{i=1}^{l}\bar{I}_i(v(t_i))
\end{aligned} \tag{19}
$$

Substituting (18) and (19) into (4), (11) respectively, we get (3)

## 3 Main result

Let $\zeta \in (t_l, t_{l+1})$, $\eta \in (t_m, t_{m+1})$, $l$, $m$ are nonnegative integer, $0 \leq l$, $m \leq p$. Define an operator $T : PC(J) \to PC(J)$ by

$$
\begin{aligned}
(Tv)(t) &= \frac{1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} g(r, v(r))\, dr + \frac{1}{\Gamma(\theta)} \sum_{0< t_k < t} \int_{t_{k-1}}^{t_k} (t_k - r)^{\theta-1} g(r, v(r))\, dr \\
&+ \frac{1}{\Gamma(\theta-1)} \sum_{0< t_k < t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - r)^{\theta-2} g(r, v(r))\, dr \\
&- \frac{1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} g(r, v(r))\, dr - \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} g(r, v(r))\, dr \\
&+ \sum_{0< t_k < t} (t-t_k) \bar{I}_k(v(t_k)) + \sum_{0< t_k < t} I_k(v(t_k)) - \sum_{i=1}^{l} \bar{I}_i(v(t_i)) \\
&+ (1-t) \left\{
\begin{array}{c}
\frac{1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} g(r, v(r))\, dr \\
+ \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{p} (1-t_i) \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} g(r, v(r))\, dr \\
- \frac{1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} g(r, v(r))\, dr \\
- \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} g(r, v(r))\, dr \\
+ \frac{1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} g(r, v(r))\, dr \\
+ \frac{1}{\Gamma(\theta-1)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} g(r, v(r))\, dr \\
+ \sum_{i=1}^{p} (1-t_i) \bar{I}_i(v(t_i)) + \sum_{i=1}^{p} I_i(v(t_i)) \\
- \sum_{i=1}^{l} \bar{I}_i(v(t_i)) + \sum_{i=1}^{m} \bar{I}_i(v(t_i))
\end{array}
\right\}
\end{aligned}
$$

Clearly, the fixed point of the operator $T$ are solution of problem (1). Our first result is based on Banach's fixed point theorem.

**Theorem 4.** *Assume that*:

$(C_1)$ There exists a constant $L_1 > 0$ such that $|g(t,x) - g(t,y)| \leq L_1 |x-y|$, for each $t \in J$ and all $x, y \in R$.

$(C_2)$ There exists a constant $L_2, L_3 > 0$ such that $|I_k(x) - I_k(y)| \leq L_2 |x-y|$, $\left|\bar{I}_k(x) - \bar{I}_k(y)\right| \leq L_3 |x-y|$, for each $t \in J$ and all $x, y \in R$, $k = 1, 2, \ldots p$.

If

$$
L_1 \left( \frac{(1+2p)}{\Gamma(q+1)} + \frac{(p-1)}{\Gamma(q)} \right) + p(2L_2 + L_3) < 1
$$

then the problem (1) has a unique solution.

*Proof.*Let $x,y\in PC(J)$. Then, for each $t\in J$, we have

$$\begin{aligned}
|(Tx)(t)-(Ty)(t)| \leq{}& \frac{L_1\|x-y\|}{\Gamma(\theta)}\int_{t_k}^{t}(t-r)^{\theta-1}dr+\frac{L_1\|x-y\|}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}dr \\
&+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\
&-\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}dr-\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\
&+\sum_{i=1}^{p}L_3|x-y|+\sum_{i=1}^{p}L_2|x-y|-\sum_{i=1}^{p}L_3|x-y| \\
&+\frac{L_1\|x-y\|}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}dr+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\
&-\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}dr-\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\
&+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}dr+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\
&+\sum_{i=1}^{p}L_3|x-y|+\sum_{i=1}^{p}L_2|x-y|-\sum_{i=1}^{p}L_3|x-y|+\sum_{i=1}^{p}L_3|x-y|
\end{aligned}$$

$$\begin{aligned}
={}&\frac{L_1\|x-y\|}{\Gamma(\theta)}\int_{t_k}^{t}(t-r)^{\theta-1}dr-\frac{2L_1\|x-y\|}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}dr \\
&+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}dr+\frac{2L_1\|x-y\|}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}dr \\
&+\frac{L_1\|x-y\|}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr+2\sum_{i=1}^{p}L_2|x-y|+\sum_{i=1}^{p}L_3|x-y| \\
&\leq\left[L_1\left(\frac{p-1}{\Gamma(\theta)}+\frac{2p+1}{\Gamma(\theta+1)}\right)+p(2L_2+L_3)\right]\|x-y\|
\end{aligned}$$

Thus

$$\|Tx-Ty\|\leq\left[L_1\left(\frac{p-1}{\Gamma(\theta)}+\frac{2p+1}{\Gamma(\theta+1)}\right)+p(2L_2+L_3)\right]\|x-y\|\leq\|x-y\|$$

Since

$$L_1\left(\frac{(1+2p)}{\Gamma(\theta+1)}+\frac{(p-1)}{\Gamma(\theta)}\right)+p(2L_2+L_3)<1$$

consequently $T$ is a contraction. As a consequence of Banach's fixed point theorem, we deduce that $T$ has a fixed point which is a solution of problem (1).

**Theorem 5.***Assume that*:

$(C_3)$ The function $g::[0,1]\times R\to R$ is continuous, and there exists a constant $M_1>0$ such that $|g(t,v)|\leq M_1$, for each $t\in J$ and all $v\in R$.

$(C_4)$ The function $I_k,\bar{I_k}:R\to R$ are continuous, and there exists a constant $M_{2,}\ M_3>0$ such that $|I_k|\leq M_2,\left|\bar{I_k}\right|\leq M_3$, for all $v\in R$, $k=1,2,.....p$.

Then problem (1) has at least one solution.

*Proof.*We shall use Schauder's fixed point theorem to prove that $T$ has a fixed point. The proof will be given in four steps

**Step 1**: $T$ is continuous.

Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $PC(J)$.

$$
\begin{aligned}
|(Tv_n)(t)-(Tv)(t)| \leq & \frac{1}{\Gamma(\theta)}\int_{t_k}^{t}(t-r)^{\theta-1}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{ti}(t_i-r)^{\theta-1}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&-\frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&-\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right| \\
&+\sum_{i=1}^{p}|I_i(v_n(t_i))-I_i(v(t_i))|-\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right| \\
&+\frac{1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}|g(r,v_n(r))-g(r,v(r))|dr
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&-\frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&-\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right|+\sum_{i=1}^{p}|I_i(v_n(t_i))-I_i(v(t_i))| \\
&-\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right|+\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right|
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{1}{\Gamma(\theta)}\int_{t_k}^{t}(t-r)^{\theta-1}|g(r,v_n(r))-g(r,v(r))|dr \\
&-\frac{2}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{2}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{ti}(t_i-r)^{\theta-1}|g(r,v_n(r))-g(r,v(r))|dr \\
&+\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}|g(r,v_n(r))-g(r,v(r))|dr \\
&+2\sum_{i=1}^{p}|I_i(v_n(t_i))-I_i(v(t_i))|+\sum_{i=1}^{p}\left|\bar{I}_i(v_n(t_i))-\bar{I}_i(v(t_i))\right|
\end{aligned}
$$

Since $g$, $I$ and $\overline{I}$ are continuous function, then we have

$\|Tv_n - Tv\| \to 0$ as $n \to \infty$

**Step2**: $T$ maps bounded sets into bounded sets.

Indeed, it is enough to show that, for any $\rho > 0$, there exists a positive constant $L$ such that, for each $v \in \Omega_p = \{v \in PC(J) \mid \|v\| \leq \rho\}$, we have $\|Tv\| \leq L$. By $(C_3)$ and $(C_4)$, we have, for each $t \in J$,

$$\begin{aligned}
|(Tv)(t)| \leq & \frac{M_1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} dr + \frac{M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& - \frac{M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr - \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_2 - \sum_{i=1}^{p} M_3 \\
& + \frac{M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& - \frac{M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr - \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} dr + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_2 - \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_3
\end{aligned}$$

$$\begin{aligned}
= & \frac{M_1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} dr - \frac{2M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} dr + \frac{2M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr + 2\sum_{i=1}^{p} M_2 + \sum_{i=1}^{p} M_3
\end{aligned}$$

$$\leq \left[ M_1 \left( \frac{p-1}{\Gamma(\theta)} + \frac{2p+1}{\Gamma(\theta+1)} \right) + p(2M_2 + M_3) \right]$$

Thus,

$$\|Tv\| \leq \left[ M_1 \left( \frac{p-1}{\Gamma(\theta)} + \frac{2p+1}{\Gamma(\theta+1)} \right) + p(2M_2 + M_3) \right] = L$$

**Step 3**: $T$ maps bounded sets into equicontinuous sets.

Let $\Omega_p$ be a bounded set of $PC(J)$ as in step 2, and let $v \in \Omega_p$ for each $t \in J_k$, $0 \le k \le p$, we have

$$\begin{aligned}\left|(Tv)'(t)\right| \le & \frac{M_1}{\Gamma(\theta-1)}\int_{t_k}^{t}(t-r)^{\theta-2}dr \\ & +\frac{M_1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr+\sum_{i=1}^{p}M_3 \\ & -\frac{M_1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}dr-\frac{M_1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\ & +\frac{M_1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}dr+\frac{M_1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\ & -\frac{M_1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}dr-\frac{M_1}{\Gamma(\theta-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}dr \\ & -\sum_{i=1}^{p}M_3-\sum_{i=1}^{p}M_2+\sum_{i=1}^{p}M_3-\sum_{i=1}^{p}M_3\end{aligned}$$

$$\begin{aligned}&=\frac{M_1}{\Gamma(\theta-1)}\int_{t_k}^{t}(t-r)^{\theta-2}dr+\frac{M_1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}dr \\ &-\frac{M_1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}dr-\frac{M_1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}dr-\sum_{i=1}^{p}M_2 \\ &\le \frac{M_1(1-p)}{\Gamma(\theta)}-pM_2 :=H\end{aligned}$$

Hence, leting $t'', t' \in J_k$, $t' < t''$, $0 \le k \le p$, we have
$\left|(Tv)\left(t''\right)-(Tv)\left(t'\right)\right| \le \int_{t'}^{t''}\left|(Tv)'(r)\,dr\right| \le H\left(t''-t'\right)$.
So, $T(\Omega_p)$ is equicontinuous on all $J_k$.We can conclude that $T: PC(J) \to PC(J)$ is completely continuous.
**Step 4**: Now it remains to show that the set $\Omega = \{v \in PC(J) \mid v = \lambda T v \text{ for some } 0 < \lambda < 1\}$ is bounded.
Let $v \in \Omega$, then $v = \lambda T v$ for some $0 < \lambda < 1$. thus, for each $t \in J$, we have

$$\begin{aligned}v(t) = & \frac{\lambda}{\Gamma(\theta)}\int_{t_k}^{t}(t-r)^{\theta-1}g(r,v(r))\,dr+\frac{\lambda}{\Gamma(\theta)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-r)^{\theta-1}g(r,v(r))\,dr \\ & +\frac{\lambda}{\Gamma(\theta-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-r)^{\theta-2}g(r,v(r))\,dr \\ & -\frac{\lambda}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}g(r,v(r))\,dr-\frac{\lambda}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}g(r,v(r))\,dr \\ & +\lambda\sum_{0<t_k<t}(t-t_k)\bar{I}_k(v(t_k))+\lambda\sum_{0<t_k<t}I_k(v(t_k))-\lambda\sum_{i=1}^{l}\bar{I}_i(v(t_i)) \\ & +(1-t)\lambda\left\{\begin{array}{c}\frac{1}{\Gamma(\theta)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-1}g(r,v(r))\,dr \\ +\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}g(r,v(r))\,dr \\ -\frac{1}{\Gamma(\theta-1)}\int_{t_l}^{\zeta}(\zeta-r)^{\theta-2}g(r,v(r))\,dr \\ -\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}g(r,v(r))\,dr \\ +\frac{1}{\Gamma(\theta-1)}\int_{t_m}^{\eta}(\eta-r)^{\theta-2}g(r,v(r))\,dr \\ +\frac{1}{\Gamma(\theta-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-r)^{\theta-2}g(r,v(r))\,dr \\ +\sum_{i=1}^{p}(1-t_i)\bar{I}_i(v(t_i))+\sum_{i=1}^{p}I_i(v(t_i)) \\ -\sum_{i=1}^{l}\bar{I}_i(v(t_i))+\sum_{i=1}^{m}\bar{I}_i(v(t_i))\end{array}\right\}\end{aligned}$$

This implies by $(C_3)$ and $(C_4)$ that, for each $t \in J$, we have

$$\begin{aligned}
|v(t)| \leq & \frac{M_1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} N_1 dr + \frac{M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& - \frac{M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr - \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_2 - \sum_{i=1}^{p} M_3 \\
& + \frac{M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& - \frac{M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr - \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} dr + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr \\
& + \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_2 - \sum_{i=1}^{p} M_3 + \sum_{i=1}^{p} M_3
\end{aligned}$$

$$\begin{aligned}
= & \frac{M_1}{\Gamma(\theta)} \int_{t_k}^{t} (t-r)^{\theta-1} dr - \frac{2M_1}{\Gamma(\theta-1)} \int_{t_l}^{\zeta} (\zeta - r)^{\theta-2} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \int_{t_m}^{\eta} (\eta - r)^{\theta-2} dr + \frac{2M_1}{\Gamma(\theta)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-1} dr \\
& + \frac{M_1}{\Gamma(\theta-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - r)^{\theta-2} dr + 2\sum_{i=1}^{p} M_2 + \sum_{i=1}^{p} M_3 \\
& \leq M_1 \left( \frac{p-1}{\Gamma(\theta)} + \frac{2p+1}{\Gamma(\theta+1)} \right) + p(2M_2 + M_3)
\end{aligned}$$

Thus, for every $t \in J$, we have

$$\|v\| \leq M_1 \left( \frac{p-1}{\Gamma(\theta)} + \frac{2p+1}{\Gamma(\theta+1)} \right) + p(2M_2 + M_3).$$

This shows that the set $\Omega$ is bounded. As a consequence of Schauder's fixed point theorem, we deduce that $T$ has a fixed point which is a solution of problem (1).

## 4 Example

In this section, we consider an example to illustrate our results.

*Example 1.*Let $\theta = 1.5$, $\zeta = 0.5$, $\eta = 0.7$, $a = 0.4$, $b = 0.5$, $p = 1$. We consider the following boundary value problem:

$$\begin{aligned}
& {}^{C}D^{\theta} v(t) = g(t, v(t)).\ t \neq \frac{1}{3}, \quad t \in (0,1) \\
& \Delta v \Big|_{t=\frac{1}{3}} = I\left(v\left(\frac{1}{3}\right)\right), \quad \Delta v' \Big|_{t=\frac{1}{3}} = \bar{I}\left(v\left(\frac{1}{3}\right)\right), \qquad (20) \\
& v(0) + v'(\zeta) = 0, \qquad v(1) + v'(\eta) = 0 \\
& \qquad v(0) = av(\zeta) \qquad v(1) = bv(\eta)
\end{aligned}$$

where

$$g(t,v(t)) = \frac{1+tv^2\sin^4 v}{1+t^2+v^4}, \qquad I(v) = \frac{3+2v^2}{1+v^2}, \qquad \bar{I}(v) = \frac{5v^2}{1+v^2}$$

Obviously, $g$ , $I$ and $I$ are continuous function, and

(1) $|g(t,v)| \leq 1$, for each $t \in (0,1)$ and all $v \in R$.

(2) $|I(v)| \leq 3$, $\left|\bar{I}(v)\right| \leq 5$, for all $v \in R$.

So conditions $(C_3)$ and $(C_4)$ hold, by Theorem 5, the BVP (20) has at least one solution.

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