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Classifications and Properties of (α,β)-Fuzzy Ideals in Ternary Semigroups

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Abstract: Classifications of (α,β)-fuzzy left (right and lateral) ideals of ternary semigroups are discussed. Relations between $(\in,\in\vee q)$-fuzzy left (right and lateral) ideals and $(q,\in\vee q)$-fuzzy left (right and lateral) ideals are established. Given special sets, so called t-q-set and t-$\in\vee q$-set, conditions for the t-q-set and t-$\in\vee q$-set to be left (right and lateral) ideals are considered. Conditions for a fuzzy set to be an $(\in,q)$-fuzzy left (right and lateral) ideals, a $(q,\in)$-fuzzy left (right and lateral)ideals and a $(q,\in\vee q)$-fuzzy left (right and lateral) ideals are considered. Some new characterizations of weakly regular ternary semigroups are also established. Finally, the implication-based fuzzy ideals are discussed.

Keywords: Ternary semigroup, Fuzzy point, (α,β)-fuzzy Ideal

1 Introduction

The introduction of the mathematical literature of ternary algebraic system dated back to 1930's. In 1932 Lehmer [3] investigated certain triple systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by Banach (cf. [4]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Los [4] showed that every ternary semigroup can be embedded in a semigroup. The notion of ideals created by Dedekind for the theory of algebraic numbers, was generalized by Emmy Noether for associative rings. The one- and two-sided ideals introduced by her, are still central concepts in ring theory. The concept of ideal is an interesting and important idea in many other algebraic structures as well. Sioson [9] developed the ideal theory of ternary semigroups. He has extended various well known concepts concerning ideals to ternary semigroups.

The concept of fuzzy set, introduced by Zadeh (see [12]) in his pioneering paper of 1965 was applied by Rosenfeld to the elementary theory of groupoids and groups [7]. Since then many researchers have been engaged to review various concepts and results from the realm of abstract algebra in broader framework of fuzzy setting. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [5], played a vital role to generate some different types of fuzzy subgroups, called (α,β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In ternary semigroups, the concept of (α,β)-fuzzy ideals, which is studied in the paper [6], is also important and useful generalization of the well-known concepts, called fuzzy ideals.

In this paper, we classify (α,β)-fuzzy left (right and lateral) ideals of ternary semigroups. We establish relations between $(\in,\in\vee q)$-fuzzy left (right and lateral) ideals and $(q,\in\vee q)$-fuzzy left (right and lateral) ideals. Given two special sets, so called t-q-set and t-$\in\vee q$-set, we discuss conditions for the t-q-set and t-$\in\vee q$-set to be a left (right and lateral) ideal. We provide conditions for an $(\in,\in\vee q)$-fuzzy left (right and lateral) ideal to be a $(q,\in\vee q)$-fuzzy left (right and lateral) ideal. We consider conditions for a fuzzy set to be an $(\in,q)$-fuzzy left (right and lateral) ideal, a $(q,\in)$-fuzzy left (right and lateral) ideal and a $(q,\in\vee q)$-fuzzy left (right and lateral) ideal. Some characterizations of weakly regular ternary semigroups are also established. Finally, the implication-based fuzzy ideals are discussed.

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2 Preliminaries

Definition 1. *A ternary semigroup is an algebraic structure consisting of a nonempty set S together with a ternary operation $(a,b,c) \to [abc]$ satisfying the associative law $[[abc]xy] = [a[bcx]y] = [ab[cxy]]$ for all $a,b,c,x,y \in S$.*

Example 1.(1) Any semigroup can be made into a ternary semigroup by defining the ternary product to be $[abc] = abc$.

(2) Let X be any nonempty set. The set $S(X)$ of all words of odd length form a ternary semigroup under juxtaposition as operation.

A nonempty subset A of a ternary semigroup S is called ternary subsemigroup of S if $abc \in A$ for all $a,b,c \in A$. By a left (right and lateral) ideal of a ternary semigroup S we mean a nonempty subset A of S such that $xya \in A$ ($axy \in A$ and $xay \in A$) for all $a \in A$ and $x,y \in S$. A ternary semigroup S is called left (resp. right) zero ternary semigroup if $xyz = x$ (resp. $xyz = z$) for all $x,y,z \in S$.

A fuzzy set in a set S is a function $\mu : S \to [0,1]$. A fuzzy set μ in a set S of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set S, Pu and Liu [5] introduced the symbol $x_t\alpha\mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t\, q\, \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x)+t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee q\, \mu$ (resp. $x_t \in\wedge q\, \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t\alpha\mu$ does not hold, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

A fuzzy set μ in S is called a fuzzy left (resp. right and lateral) ideal of S if and only if it satisfies the following condition:

$$\mu(xyz) \geq \mu(z) \text{ (resp. } \mu(xyz) \geq \mu(x) \text{ and } \mu(xyz) \geq \mu(y)) \tag{1}$$

for all $x,y,z \in S$.

A fuzzy set μ in a ternary semigroup S is said to be an (α,β)-*fuzzy ternary subsemigroup* of S (see [6]) where $\alpha \neq \in\wedge q$, if it satisfies the following condition:

$$x_{t_1}\alpha\mu,\ y_{t_2}\alpha\mu,\ z_{t_3}\alpha\mu \implies (xyz)_{\min\{t_1,t_2,t_3\}}\beta\mu$$

for all $x,y,z \in S$ and $t_1,t_2,t_3 \in (0,1]$. Note that a fuzzy set μ in a ternary semigroup S is an $(\in,\in\vee q)$-*fuzzy ternary subsemigroup* of S if and only if

$$(\forall x,y,z \in S)\,(\mu(xyz) \geq \min\{\mu(x),\mu(y),\mu(z),0.5\})\text{ (see [6]).}$$

Let μ, ν and λ be three fuzzy sets in a ternary semigroup S. The product $\mu\circ\nu\circ\lambda$ is defined by:

$$(\mu\circ\nu\circ\lambda)(a) = \begin{cases} \bigvee\limits_{a=xyz} \{\mu(x)\wedge\nu(y)\wedge\lambda(z)\} & \text{if } \exists\, x,y,z \in S \text{ such that } a = xyz, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set μ in a ternary semigroup S is an $(\in,\in\vee q)$-fuzzy quasi-ideal of S if and only if it satisfies:

(i) $\mu(x) \geq \min\{(\mu\circ S\circ S)(x), (S\circ\mu\circ S)(x), (S\circ S\circ\mu)(x), 0.5\}$.

(ii) $\mu(x) \geq \min\{(\mu\circ S\circ S)(x), (S\circ S\circ\mu\circ S\circ S)(x), (S\circ S\circ\mu)(x), 0.5\}$,

where S is the fuzzy set in S mapping every element of S on 1.

A fuzzy set μ in a ternary semigroup S is an $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S if and only if

$\mu(xuyvz) \geq \min\{\mu(x),\mu(y),\mu(z),0.5\}$

for all $u,v,x,y,z \in S$. A fuzzy set in S is an $(\in,\in\vee q)$-fuzzy bi-ideal of S if and only if it is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup and an $(\in,\in\vee q)$-fuzzy generalized bi-ideal. It is obvious that every $(\in,\in\vee q)$-fuzzy left (right and lateral) ideal is an $(\in,\in\vee q)$-fuzzy quasi-ideal and every $(\in,\in\vee q)$-fuzzy quasi-ideal is $(\in,\in\vee q)$-fuzzy bi-ideal and every $(\in,\in\vee q)$-fuzzy bi-ideal is an $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

Let μ be a fuzzy set in S. Then the set

$$U(\mu;t) := \{x \in S : \mu(x) \geq t\}$$

where $t \in [0,1]$, is called level subset of μ.

3 Classifications and properties of (α,β)-fuzzy left (right, lateral) ideals

Definition 2.([6]). *A fuzzy set μ in a ternary semigroup S is said to be an (α,β)-fuzzy* left (resp. right and lateral) ideal *of S, where $\alpha \neq \in\wedge q$, if it satisfies the following condition:*

$$z_t\alpha\mu \text{ implies } (xyz)_t\beta\mu \text{ (resp. } (zxy)_t\beta\mu \text{ and } (xzy)_t\beta\mu) \tag{2}$$

for all $x,y,z \in S$ and $t \in (0,1]$.

Example 2. Let $S = \{a,b,c,d,e\}$ and $xyz = (x*y)*z = x*(y*z)$ for all $x,y,z \in S$, where $*$ is defined by the following table:

$*$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Then S is a ternary semigroup. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.50 \text{ if } x = a, \\ 0.70 \text{ if } x = b, \\ 0.20 \text{ if } x = c, \\ 0.55 \text{ if } x = d, \\ 0.60 \text{ if } x = e. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S, but neither an $(\in, \in \vee q)$-fuzzy right ideal, nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S, since $e_{0.3} \in \mu$, but

$$(ecc)_{0.3} \overline{\in \vee q} \mu,$$

and

$$(cec)_{0.3} \overline{\in \vee q} \mu.$$

Moreover we see that:
(i) μ is not an $(\in, \in)$-fuzzy left ideal of S, since $b_{0.65} \in \mu$ but

$$(dcb)_{0.65} \overline{\in} \mu.$$

(ii) μ is not an $(\in, q)$-fuzzy left ideal of S, since $b_{0.3} \in \mu$, but

$$(ddb)_{0.3} \overline{q} \mu.$$

(iii) μ is not a (q, q)-fuzzy left ideal of S, since $b_{0.32} q \mu$, but

$$(ddb)_{0.32} \overline{q} \mu.$$

(iv) μ is not a $(q, \in)$-fuzzy left ideal of S, since $b_{0.65} q \mu$, but

$$(bad)_{0.65} \overline{\in} \mu.$$

(v) μ is not an $(\in, \in \wedge q)$-fuzzy left ideal of S, since $b_{0.31} \in \mu$, but

$$(ddb)_{0.31} \overline{q} \mu \text{ and so } (ddb)_{0.31} \overline{\in \wedge q} \mu.$$

(vi) μ is not a $(q, \in \wedge q)$-fuzzy left ideal of S, since $b_{0.33} q \mu$, but

$$(ddb)_{0.33} \overline{q} \mu \text{ and so } (ddb)_{0.33} \overline{\in \wedge q} \mu.$$

(vii) μ is not an $(\in \vee q, \in)$-fuzzy left ideal of S, since $b_{0.64} \in \vee q \mu$, but

$$(bad)_{0.64} \overline{\in} \mu.$$

(viii) μ is not an $(\in \vee q, q)$-fuzzy left ideal of S, since $b_{0.27} \in \vee q \mu$, but

$$(ddb)_{0.27} \overline{q} \mu.$$

(ix) μ is not an $(\in \vee q, \in \wedge q)$-fuzzy left ideal of S, since $b_{0.38} \in \vee q \mu$, but

$$(adb)_{0.38} \overline{q} \mu \text{ and so } (adb)_{0.38} \overline{\in \wedge q} \mu.$$

Example 3. Let $S = \{0, a, b, c, 1\}$ and $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in S$, where $*$ is defined by the following table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then S is a ternary semigroup. Define a fuzzy set λ in S as follows:

$$\lambda : S \to [0,1], \quad x \mapsto \begin{cases} 0.8 \text{ if } x = 0, \\ 0.2 \text{ if } x = a, \\ 0.5 \text{ if } x = b, \\ 0.7 \text{ if } x = c, \\ 0.2 \text{ if } x = 1. \end{cases}$$

Then simple calculations show that λ is an $(\in, \in \vee q)$-fuzzy right ideal of S, but neither an $(\in, \in \vee q)$-fuzzy left ideal nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S, since $c_{0.1} \in \lambda$, but

$$(1ac)_{0.1} \overline{\in \vee q} \lambda$$

and

$$(ac1)_{0.1} \overline{\in \vee q} \lambda.$$

Moreover we see that:
(i) λ is not an $(\in, \in)$-fuzzy right ideal of S, since $c_{0.6} \in \lambda$ but

$$(c1b)_{0.6} \overline{\in} \lambda.$$

(ii) λ is not an $(\in, q)$-fuzzy right ideal of S, since $c_{0.3} \in \lambda$, but

$$(c1b)_{0.3} \overline{q} \lambda.$$

(iii) λ is not a (q, q)-fuzzy right ideal of S, since $c_{0.31} q \lambda$, but

$$(cbb)_{0.31} \overline{q} \lambda.$$

(iv) λ is not a $(q, \in)$-fuzzy right ideal of S, since $a_{0.9} q \lambda$, but

$$(a00)_{0.9} \overline{\in} \lambda.$$

(v) λ is not an $(\in, \in \wedge q)$-fuzzy right ideal of S, since $c_{0.51} \in \lambda$, but

$$(c1b)_{0.51} \overline{\in} \lambda \text{ and so } (bca)_{0.31} \overline{\in \wedge q} \lambda.$$

(vi) λ is not a $(q, \in \wedge q)$-fuzzy right ideal of S, since $c_{0.35} q \lambda$, but

$$(cbb)_{0.35} \overline{q} \lambda \text{ and so } (cbb)_{0.35} \overline{\in \wedge q} \lambda.$$

(vii) λ is not an $(\in \vee q, \in)$-fuzzy right ideal of S, since $c_{0.59} \in \vee q \lambda$, but

$$(c1b)_{0.59} \overline{\in} \lambda.$$

(viii) λ is not an $(\in \vee q, q)$-fuzzy right ideal of S, since $c_{0.34} \in \vee q \lambda$, but

$$(abb)_{0.34} \overline{q} \lambda.$$

(ix) λ is not an $(\in \vee q, \in \wedge q)$-fuzzy right ideal of S, since $c_{0.39} \in \vee q \lambda$, but

$$(cbb)_{0.39} \overline{q} \lambda \text{ and so } (cbb)_{0.39} \overline{\in \wedge q} \lambda.$$

Lemma 1.([6]) . *A fuzzy set μ in a ternary semigroup S is an* $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal *of S, if and only if it satisfies the following condition:*

$$\mu(xyz) \geq \min\{\mu(z), 0.5\}$$

(resp. $\mu(xyz) \geq \min\{\mu(x), 0.5\}, \mu(xyz) \geq \min\{\mu(y), 0.5\}$*)*

for all $x,y,z \in S$.

A fuzzy set μ in a ternary semigroup S is said to be an $(\in,\in\vee q)$-fuzzy two sided ideal *of S if it is an* $(\in,\in\vee q)$-fuzzy left ideal *and an* $(\in,\in\vee q)$-fuzzy right ideal *of S. By an* $(\in,\in\vee q)$-fuzzy ideal *we mean a fuzzy set μ in S which is an* $(\in,\in\vee q)$-fuzzy left ideal, an $(\in,\in\vee q)$-fuzzy right ideal *and an* $(\in,\in\vee q)$-fuzzy lateral ideal *of S.*

In considering (α,β)-fuzzy left (right and lateral) ideals in a ternary semigroup S, we have twelve different types of such structures, that is, (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in,\in\vee q)$, $(q,\in)$, (q,q), $(q,\in\wedge q)$, $(q,\in\vee q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, and $(\in\vee q,\in\vee q)$. Clearly, we have relations among these types which are described in the following theorems.

Theorem 1. *We have the following relations:*

$$\begin{tikzcd}
(\in,\in) \arrow[dr, Rightarrow] & (\in,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in,q) \arrow[dl, Rightarrow] \\
& (\in,\in\vee q) & \\
& (\in\vee q,\in\vee q) \arrow[u, Rightarrow] & \\
(\in\vee q,\in) \arrow[ur, Rightarrow] & (\in\vee q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] & (\in\vee q,q) \arrow[ul, Rightarrow]
\end{tikzcd} \tag{3}$$

and

$$\begin{tikzcd}
(q,\in) \arrow[dr, Rightarrow] & (q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q,q) \arrow[dl, Rightarrow] \\
& (q,\in\vee q) &
\end{tikzcd} \tag{4}$$

Theorem 2. *If there exists $x \in S$ such that $\mu(x) > 0.5$, then we have the following relations:*

$$\begin{tikzcd}
(\in\wedge q,\in) \arrow[dr, Rightarrow] & (\in\wedge q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in\wedge q,q) \arrow[dl, Rightarrow] \\
& (\in\wedge q,\in\vee q) & \\
& (\in,\in\vee q) \arrow[u, Rightarrow] &
\end{tikzcd}$$

Proposition 1. *In a left (resp. right) zero ternary semigroup, every fuzzy set is an $(\in,\in\vee q)$-fuzzy right (resp. left) ideal.*

Proof. Straightforward.

Remark. In a left zero ternary semigroup S, there exists a fuzzy set which is neither an $(\in,\in\vee q)$-fuzzy left ideal nor an $(\in,\in\vee q)$-fuzzy lateral ideal of S as seen in the following example.

Example 4. Let $S = \{a,b,c,d\}$ be a set with a ternary operation [ ] which is given by $[xyz] = x$ for all $x,y,z \in S$. Then S is a left zero ternary semigroup. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.32 \text{ if } x = a, \\ 0.63 \text{ if } x = b, \\ 0.79 \text{ if } x = c, \\ 0.87 \text{ if } x = d. \end{cases}$$

Then μ is neither an $(\in,\in\vee q)$-fuzzy left ideal nor an $(\in,\in\vee q)$-fuzzy lateral ideal of S, since

$$\mu(acd) \ngeq \min\{\mu(d), 0.5\}$$

and

$$\mu(acd) \ngeq \min\{\mu(c), 0.5\}.$$

Remark. In a ternary semigroup S, every $(\in,\in\vee q)$-fuzzy left (right and lateral) ideal of S is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S, but the converse is not true in general, as seen in the following examples.

Example 5. Consider the ternary semigroup S of Example 4. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.3 \text{ if } x = a, \\ 0.4 \text{ if } x = b, \\ 0.7 \text{ if } x = c, \\ 0.8 \text{ if } x = d. \end{cases}$$

Then μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S, but neither an $(\in,\in\vee q)$-fuzzy left ideal nor an $(\in,\in\vee q)$-fuzzy lateral ideal of S, since

$$\mu(acd) \ngeq \min\{\mu(d), 0.5\}$$

and

$$\mu(acd) \ngeq \min\{\mu(c), 0.5\}.$$

Example 6. Let $S = \{a,b,c,d\}$ be a set with a ternary operation [ ] which is given by $[xyz] = z$ for all $x,y,z \in S$. Then S is a right zero ternary semigroup. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.34 \text{ if } x = a, \\ 0.47 \text{ if } x = b, \\ 0.67 \text{ if } x = c, \\ 0.91 \text{ if } x = d. \end{cases}$$

Then μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S, but not an $(\in,\in\vee q)$-fuzzy right ideal of S, since

$$\mu(dba) \ngeq \min\{\mu(d), 0.5\}.$$

Now, we investigate relations between $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideals and $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideals.

Theorem 3. *Every $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S is an $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S.*

Proof. Let μ be a $(q,\in\vee q)$-fuzzy left ideal of S and let $x,y,z\in S$ and $t\in(0,1]$ be such that $z_t\in\mu$. Then $\mu(z)\geq t$. Suppose $(xyz)_t\,\overline{\in\vee q}\mu$. Then

$$\mu(xyz)<t,$$
$$\mu(xyz)+t\leq 1.$$

It follows that

$$\mu(xyz)<0.5. \tag{5}$$

Thus we have

$$\mu(xyz)<\min\{t,0.5\}$$

and so

$$\begin{aligned}1-\mu(xyz)&>1-\min\{t,0.5\}\\&=\max\{1-t,0.5\}\\&\geq\max\{1-\mu(z),0.5\}.\end{aligned}$$

Hence there exists $\delta\in(0,1]$ such that

$$1-\mu(xyz)\geq\delta>\max\{1-\mu(z),0.5\}. \tag{6}$$

From the right inequality in (6) we have $\mu(z)+\delta>1$, that is, $z_\delta q\mu$. Since μ is a $(q,\in\vee q)$-fuzzy left ideal of S, it follows that $(xyz)_\delta\in\vee q\mu$. But from the left inequality in (6) we have $\mu(xyz)+\delta\leq 1$, that is, $(xyz)_\delta\overline{q}\mu$ and $\mu(xyz)\leq 1-\delta<1-0.5=0.5<\delta$, that is, $(xyz)_\delta\overline{\in}\mu$. Hence $(xyz)_\delta\overline{\in\vee q}\mu$, a contradiction. Therefore, $(xyz)_t\in\vee q\mu$, and thus μ is an $(\in,\in\vee q)$-fuzzy left ideal of S.

Similarly we can prove the case of right ideal and lateral ideal of S.

Combining Theorem 3 and (4) in Theorem 1, we have the following relations.

$$\begin{tikzcd}
(q,\in) \arrow[dr, Rightarrow] & (q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q,q) \arrow[dl, Rightarrow] \\
& (q,\in\vee q) \arrow[d, Rightarrow] & \\
& (\in,\in\vee q) &
\end{tikzcd} \tag{7}$$

We now provide conditions for an $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal to be a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal.

Theorem 4. *Assume that every fuzzy point has the value t in $(0,0.5]$. Then every $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S is a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S.*

Proof. Let μ be an $(\in,\in\vee q)$-fuzzy left ideal of S. Let $x,y,z\in S$ and $t\in(0,0.5]$ be such that $z_t q\mu$. Then $\mu(z)+t>1$. It follows that $\mu(z)>1-t\geq t$, that is, $z_t\in\mu$. By hypothesis it follows that $(xyz)_t\in\vee q\mu$. Therefore μ is a $(q,\in\vee q)$-fuzzy left ideal of S.

The case of right ideal and lateral ideal can be proved like wise.

Corollary 1. *Let μ be an (α,β)-fuzzy left (resp. right and lateral) ideal of S where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, and $(\in\vee q,\in\vee q)$. If every fuzzy point has the value t in $(0,0.5]$, then μ is a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S.*

Lemma 2. *If S is a ternary semigroup, then a nonempty subset A of S is a right (resp. lateral) ideal of S if and only if the characteristic function χ_A of A is an $(\in,\in\vee q)$-fuzzy right (resp. lateral) ideal of S.*

Proof. Let A be a right ideal of S and let χ_A be the characteristic function of A. Let $x\in S$. If $x\notin A$, then $\chi_A(x)=0$ and so $\chi_A(xyz)\geq\min\{\chi_A(x),0.5\}$. If $x\in A$, then $\chi_A(x)=1$. Since A is a right ideal of S, so $xyz\in A$ and $\chi_A(xyz)=1$. It follows that

$$\chi_A(xyz)\geq\min\{\chi_A(x),0.5\}.$$

Thus χ_A is an $(\in,\in\vee q)$-fuzzy right ideal of S.

Conversely assume that the characteristic function χ_A of A is an $(\in,\in\vee q)$-fuzzy right ideal of S. Let $x\in ASS$. Then $x=auv$ for some $u,v\in S$ and $a\in A$. Therefore $\chi_A(a)=1$. It follows that

$$\chi_A(x)=\chi_A(auv)\geq\min\{\chi_A(a),0.5\}=0.5.$$

This implies that $\chi_A(x)=1$. Thus $x\in A$. Therefore A is a right ideal of S.

Theorem 5.([6]) *A fuzzy set μ in S is an $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S if and only if the set*

$$U(\mu;t):=\{x\in S\mid\mu(x)\geq t\}$$

is a left (resp. right and lateral) ideal of S for all $t\in(0,0.5]$.

Corollary 2. *Let μ be an (α,β)-fuzzy left (resp. right and lateral) ideal of S where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, and $(\in\vee q,\in\vee q)$. Then the set*

$$U(\mu;t):=\{x\in S\mid\mu(x)\geq t\}$$

is a left (resp. right and lateral) ideal of S for all $t\in(0,0.5]$.

For a fuzzy set μ in S and $t\in(0,1]$, consider the t-q-set S_q^t and t-$\in\vee q$-set with respect to t (briefly, t-q-set and t-$\in\vee q$-set, respectively) as follows:

$$S_q^t := \{x\in X \mid x_t\, q\, \mu\} \text{ and } S_{\in\vee q}^t := \{x\in X \mid x_t \in\vee q\, \mu\}.$$

Note that, for any $t,r\in(0,1]$, if $t\geq r$ then every r-q-set is contained in the t-q-set, that is, $S_q^r\subseteq S_q^t$, and $U(\mu;t)\cup S_q^r\subseteq U(\mu;r)\cup S_q^t$. Obviously, $S_{\in\vee q}^t=U(\mu;t)\cup S_q^t$.

Theorem 6. *If μ is an $(\in,\in)$-fuzzy left (resp. right and lateral) ideal of S, then the t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0,1]$, whenever it is nonempty.*

Proof. Let $x,y,z\in S$ and $t\in(0,1]$ be such that $z\in S_q^t$. Then $z_t q\mu$, that is, $\mu(z)+t>1$. It follows that

$$\mu(xyz)+t\geq\mu(z)+t>1$$

and so $(xyz)_t\, q\mu$. Hence $xyz\in S_q^t$, and therefore S_q^t is a left ideal of S.

Similarly we can prove the cases of right ideal and lateral ideal of S.

Theorem 7. *For a fuzzy set μ in S, if the t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$, then μ is an $(\in,q)$-fuzzy left (resp. right and lateral) ideal of S.*

Proof. Let $x,y,z\in S$ and $t\in(0.5,1]$ be such that $z_t\in\mu$. Then $\mu(z)\geq t$. It follows that $\mu(z)+t>0.5+0.5=1$, that is, $z_t q\mu$ and so $z\in S_q^t$. By hypothesis we have $xyz\in S_q^t$ and so $(xyz)_t\, q\mu$. Therefore μ is an $(\in,q)$-fuzzy left ideal of S.

Similarly we can prove the cases of right ideal and lateral ideal of S.

Theorem 8. *For a fuzzy set μ in S, if the t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0,0.5]$, then μ is a $(q,\in)$-fuzzy left (resp. right and lateral) ideal of S.*

Proof. Let $x,y,z\in S$ and $t\in(0,0.5]$ be such that $z_t q\mu$. Then $z\in S_q^t$. By hypothesis we have $xyz\in S_q^t$ and so $(xyz)_t\, q\mu$. It follows that $\mu(xyz)+t>1$, that is, $\mu(xyz)>1-t\geq t$, and so $\mu(xyz)\geq t$. Hence $(xyz)_t\in\mu$. Therefore μ is a $(q,\in)$-fuzzy left ideal of S.

In a similar fashion we can prove the case of right ideal and lateral ideal.

Theorem 9. *If μ is a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S, then the t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$, whenever it is nonempty.*

Proof. Let $x,y,z\in S$ and $t\in(0.5,1]$ be such that $z\in S_q^t$. Then $z_t q\mu$. Since μ is a $(q,\in\vee q)$-fuzzy left ideal of S, we have $(xyz)_t\in\vee q\mu$, that is, $(xyz)_t\in\mu$ or $(xyz)_t\, q\mu$. If $(xyz)_t\, q\mu$, then $xyz\in S_q^t$. If $(xyz)_t\in\mu$, then $\mu(xyz)\geq t>1-t$, since $t>0.5$. Hence $(xyz)_t\, q\mu$ and so $xyz\in S_q^t$. Therefore S_q^t is a left ideal of S.

The same argument leads to the proof of others.

Corollary 3. *If μ is an (α,β)-fuzzy left (resp. right and lateral) ideal of S where (α,β) is one of $(q,\in)$, (q,q) and $(q,\in\wedge q)$, then the t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$, whenever it is nonempty.*

Lemma 3([6]). *For a left (resp. right and lateral) ideal A of S, let μ be a fuzzy set in S such that*

(1) $\mu(x)\geq 0.5$ for all $x\in A$,
(2) $\mu(x)=0$ for all $x\in S\setminus A$.
Then μ is a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S.

Using Theorem 9 and Lemma 3, we have the following result.

Theorem 10. *For a left (resp. right and lateral) ideal A of S, if μ is a fuzzy set in S such that*

(1) $\mu(x)\geq 0.5$ for all $x\in A$,
(2) $\mu(x)=0$ for all $x\in S\setminus A$,
then the nonempty t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$.

Theorem 11. *For a fuzzy set μ in S, if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a left (resp. right and lateral) ideal of S for all $t\in(0,1]$, then μ is a $(q,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S.*

Proof. We prove only for left ideal. Others follow in an analogous way. Let $x,y,z\in S$ and $t\in(0,1]$ be such that $z_t q\mu$. Then $z\in S_q^t\subseteq S_{\in\vee q}^t$. From the hypothesis it follows that $xyz\in S_{\in\vee q}^t$. Hence $(xyz)_t\in\vee q\mu$. Therefore μ is a $(q,\in\vee q)$-fuzzy left ideal of S.

One naturally asks the following interesting question: **Question:** If μ is an $(\in,\in\vee q)$-fuzzy fuzzy left (resp. right and lateral) ideal of S, then is the t-q-set S_q^t a left (resp. right and lateral) ideal of S?

The answer to the above question is negative (for $t\leq 0.5$) as seen in the following example:

Example 7. Consider the ternary semigroup S of Example 2. Define a fuzzy set μ in S as follows:

$$\mu: S\to[0,1],\ \ x\mapsto\begin{cases}0.87 \text{ if } x=a,\\ 0.74 \text{ if } x=b,\\ 0.25 \text{ if } x=c,\\ 0.62 \text{ if } x=d,\\ 0.41 \text{ if } x=e.\end{cases}$$

Then simple calculations show that μ is an $(\in,\in\vee q)$-fuzzy left ideal of S, but the set

$$S_q^{0.27}=\{a,b\}$$

is not a left ideal of S because $aab=d\notin S_q^{0.27}$.

But the following theorem answers the above question affirmatively:

Theorem 12. *If μ is an $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S, then the nonempty t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$.*

Proof. Assume that $S_q^t\neq\emptyset$ for $t\in(0.5,1]$. Let $x,y\in S$ and let $z\in S_q^t$. Then $z_t q\mu$, that is, $\mu(z)+t>1$. It follows that

$$\begin{aligned}\mu(xyz)+t &\geq \min\{\mu(z),0.5\}+t\\ &=\min\{\mu(z)+t,0.5+t\}\\ &>1.\end{aligned}$$

So $(xyz)_t\, q\mu$. Hence $xyz\in S_q^t$ and therefore S_q^t is a left ideal of S.

The same argument leads to the proof of the cases of right and lateral ideal.

Corollary 4. *If μ is an (α,β)-fuzzy left (resp. right and lateral) ideal of S where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, $(\in\vee q,\in\vee q)$, $(q,\in)$, (q,q), $(q,\in\vee q)$, and $(q,\in\wedge q)$, then the nonempty t-q-set S_q^t is a left (resp. right and lateral) ideal of S for all $t\in(0.5,1]$.*

Theorem 13. *Let μ be a fuzzy set in S. Then μ is an $(\in,\in\vee q)$-fuzzy left (resp. right and lateral) ideal of S, if and only if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a left (resp. right and lateral) ideal of S for all $t\in(0,1]$.*

Proof. Assume that μ is an $(\in,\in\vee q)$-fuzzy left ideal of S. Let $z\in S_{\in\vee q}^t$. Then $z_t\in\vee q\mu$, that is, $\mu(z)\geq t$ or $\mu(z)+t>1$. Since μ is an $(\in,\in\vee q)$-fuzzy left ideal of S, we have

$$\mu(xyz)\geq\min\{\mu(z),0.5\}.$$

Case I. $\mu(z)\geq t$. If $t>0.5$, then

$$\mu(xyz)\geq\min\{\mu(z),0.5\}=0.5$$

and hence $(xyz)_t\, q\mu$. If $t\leq 0.5$, then

$$\mu(xyz)\geq\min\{\mu(z),0.5\}\geq t$$

and so $(xyz)_t\in\mu$. Hence $(xyz)_t\in\vee q\mu$.

Case II. Let $\mu(z)+t>1$. If $t>0.5$, then $1-t<0.5<t$ and

$$\begin{aligned}\mu(xyz) &\geq \min\{\mu(z),0.5\}\\ &=\begin{cases}\mu(z) & \text{if } \mu(z)<0.5\\ 0.5 & \text{if } \mu(z)\geq 0.5\end{cases}\\ &>1-t,\end{aligned}$$

and hence $(xyz)_t\, q\mu$. If $t\leq 0.5$, then

$$\mu(xyz)\geq\min\{\mu(z),0.5\}\geq\min\{1-t,0.5\}=0.5\geq t,$$

and so $(xyz)_t\in\mu$. Thus $(xyz)_t\in\vee q\mu$. Hence in any case, we have $(xyz)_t\in\vee q\mu$. Therefore $S_{\in\vee q}^t$ is a left ideal of S.

Conversely let μ be a fuzzy set in S and $t\in(0,1]$ be such that $S_{\in\vee q}^t$ is a left ideal of S. If possible, let

$$\mu(xyz)<t\leq\min\{\mu(z),0.5\}$$

for some $t\in(0,0.5)$ and $x,y,z\in S$. Then $z\in U(\mu;t)\subseteq S_{\in\vee q}^t$, which implies that $xyz\in S_{\in\vee q}^t$. Hence $\mu(xyz)\geq t$ or $\mu(xyz)+t>1$, a contradiction. It follows that

$$\mu(xyz)\geq\min\{\mu(z),0.5\}$$

for all $x,y,z\in S$. Therefore μ is an $(\in,\in\vee q)$-fuzzy left ideal of S.

The cases of right ideal and lateral ideal of S can be proved like wise.

Corollary 5. *If μ is an (α,β)-fuzzy left (resp. right and lateral) ideal of S where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, $(\in\vee q,\in\vee q)$, $(q,\in)$, (q,q), $(q,\in\vee q)$, and $(q,\in\wedge q)$, then the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a left (resp. right and lateral) ideal of S for all $t\in(0,1]$.*

4 Weakly regular ternary semigroups

In this section we characterize right weakly regular ternary semigroups in terms of $(\in,\in\vee q)$-fuzzy right ideals, $(\in,\in\vee q)$-fuzzy two sided ideals and $(\in,\in\vee q)$-fuzzy generalized bi-ideals.

Definition 3.([8]). *A ternary semigroup S is said to be right (resp. left) weakly regular, if $x\in(xSS)^3$ $\left(\text{resp. } x\in(SSx)^3\right)$ for all $x\in S$.*

Lemma 4.([8]). *A ternary semigroup S is right weakly regular if and only if $R\cap I=RII$, for every right ideal R and every two sided ideal I of S.*

For a fuzzy set μ in S, we define $\mu^-(x)=\mu(x)\wedge 0.5$ for all $x\in S$.

Theorem 14. *For a ternary semigroup S, the following assertions are equivalent:*

(1) S is right weakly regular;

(2) $(\mu\wedge\nu)^-=(\mu\circ\nu\circ\nu)^-$ for every $(\in,\in\vee q)$-fuzzy right ideal μ and every $(\in,\in\vee q)$-fuzzy two sided ideal ν of S;

(3) $(\mu\wedge\nu)^-=(\mu\circ\nu\circ\nu)^-$ for every $(\in,\in\vee q)$-fuzzy right ideal μ and every $(\in,\in\vee q)$-fuzzy ideal ν of S.

Proof. $(1)\Rightarrow(2)$: Let μ be an $(\in,\in\vee q)$-fuzzy right ideal and ν an $(\in,\in\vee q)$-fuzzy two sided ideal of S. Now for any $a\in S$, there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}(\mu\circ\nu\circ\nu)^-(a) &= (\mu\circ\nu\circ\nu)(a)\wedge 0.5\\ &=\left\{\bigvee_{a=pqr}\{\mu(p)\wedge\nu(q)\wedge\nu(r)\}\right\}\wedge 0.5\\ &\leq\left\{\begin{array}{c}\bigvee_{a=pqr}\{\mu(pqr)\wedge 0.5\}\wedge\nu(q)\wedge\\ \{\nu(pqr)\wedge 0.5\}\end{array}\right\}\wedge 0.5\\ &\leq\left\{\bigvee_{a=pqr}\{\mu(pqr)\wedge\nu(pqr)\}\right\}\wedge 0.5\\ &\leq\{\mu(a)\wedge\nu(a)\}\wedge 0.5=(\mu\wedge\nu)^-(a)\end{aligned}$$

Thus $(\mu\circ\nu\circ\nu)^{-}\leq(\mu\wedge\nu)^{-}$. On the other hand

$$\begin{aligned}(\mu\wedge\nu)^{-}(a) &= (\mu\wedge\nu)(a)\wedge 0.5\\ &= \{\mu(a)\wedge\nu(a)\wedge\nu(a)\}\wedge 0.5\\ &\leq \{\mu(as_1t_1)\wedge\nu(as_2t_2)\wedge\nu(as_3t_3)\}\\ &\quad\wedge 0.5\\ &\leq \left\{\bigvee_{a=xyz}(\mu(x)\wedge\nu(y)\wedge\nu(z))\right\}\wedge 0.5\\ &= (\mu\circ\nu\circ\nu)(a)\wedge 0.5=(\mu\circ\nu\circ\nu)^{-}(a).\end{aligned}$$

Thus $(\mu\wedge\nu)^{-}\leq(\mu\circ\nu\circ\nu)^{-}$. Consequently $(\mu\wedge\nu)^{-}=(\mu\circ\nu\circ\nu)^{-}$.

$(2)\Rightarrow(3)$: This is obvious because every $(\in,\in\vee q)$-fuzzy ideal is an $(\in,\in\vee q)$-fuzzy two sided ideal of S.

$(3)\Rightarrow(1)$: Let R be a right ideal and I an ideal of S. Then by Lemma 2, χ_R and χ_I are $(\in,\in\vee q)$-fuzzy right ideal and $(\in,\in\vee q)$-fuzzy ideal of S, respectively. By hypothesis it follows that

$$\begin{aligned}(\chi_R\wedge\chi_I)^{-} &= (\chi_R\circ\chi_I\circ\chi_I)^{-}\\ \chi_{R\cap I}^{-} &= (\chi_{RII})^{-}.\end{aligned}$$

Thus $R\cap I=RII$. Hence by Lemma 4, S is right weakly regular.

Corollary 6. *If $(\mu\wedge\nu)^{-}=(\mu\circ\nu\circ\nu)^{-}$ for every (α,β)-fuzzy right ideal μ and every (α,β)-fuzzy two sided ideal ν of S, where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, and $(\in\vee q,\in\vee q)$, then S is a right weakly regular ternary semigroup.*

Theorem 15. *For a ternary semigroup S, the following assertions are equivalent:*

(1) S is right weakly regular;

(2) Each $(\in,\in\vee q)$-fuzzy right ideal μ of S is idempotent.

Proof. $(1)\Rightarrow(2)$: Let μ be an $(\in,\in\vee q)$-fuzzy right ideal of S. Now for any $a\in S$, there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}(\mu\circ\mu\circ\mu)^{-}(a) &= (\mu\circ\mu\circ\mu)(a)\wedge 0.5\\ &= \left\{\bigvee_{a=pqr}\{\mu(p)\wedge\mu(q)\wedge\mu(r)\}\right\}\wedge 0.5\\ &\leq \left\{\bigvee_{a=pqr}\{\{\mu(pqr)\wedge 0.5\}\wedge\mu(q)\wedge\mu(r)\}\right\}\\ &\quad\wedge 0.5\\ &\leq \bigvee_{a=pqr}\mu(pqr)\wedge 0.5\\ &\leq \mu(a)\wedge 0.5=\mu^{-}(a).\end{aligned}$$

Thus $(\mu\circ\mu\circ\mu)^{-}\leq\mu^{-}$. On the other hand

$$\begin{aligned}\mu^{-}(a) &= \mu(a)\wedge 0.5\\ &= \{\mu(a)\wedge\mu(a)\wedge\mu(a)\}\wedge 0.5\\ &\leq \left\{\begin{array}{c}\{\mu(as_1t_1)\wedge 0.5\}\wedge\{\mu(as_2t_2)\wedge 0.5\}\\ \wedge\{\mu(as_3t_3)\wedge 0.5\}\end{array}\right\}\wedge 0.5\\ &\leq \left\{\bigvee_{a=pqr}\{\mu(p)\wedge\mu(q)\wedge\mu(r)\}\right\}\wedge 0.5\\ &= (\mu\circ\mu\circ\mu)(a)\wedge 0.5=(\mu\circ\mu\circ\mu)^{-}(a).\end{aligned}$$

Thus $\mu^{-}\leq(\mu\circ\mu\circ\mu)^{-}$. Hence $\mu^{-}=(\mu\circ\mu\circ\mu)^{-}$.

$(2)\Rightarrow(1)$: Let $x\in S$, we show that $x\in(xSS)^3$. Let $A=x\cup xSS$ be the right ideal generated by x and let χ_A be the characteristic function of A. Then by Lemma 2, χ_A is an $(\in,\in\vee q)$-fuzzy right ideal of S. Thus by hypothesis

$$\begin{aligned}\chi_A^{-} &= (\chi_A\circ\chi_A\circ\chi_A)^{-}\\ &= \chi_{A^3}^{-}.\end{aligned}$$

This implies that $A=A^3$.

Since $x\in A$, it follows that $x\in A^3$. This implies that $x\in A^3=(x\cup xSS)^3$. Thus

$$\begin{aligned}x &\in (x\cup xSS)(x\cup xSS)(x\cup xSS)\\ &= xxx\cup xxSSx\cup xSSxx\cup xSSxSSx\\ &\quad\cup xxxSS\cup xxSSxSS\cup xSSxxSS\cup xSSxSSxSS\end{aligned}$$

implies $x\in(xSS)^3$. Therefore S is right weakly regular.

Corollary 7. *If every (α,β)-fuzzy right ideal μ of S is idempotent, where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, and $(\in\vee q,\in\vee q)$, then S is a right weakly regular ternary semigroup.*

Theorem 16. *For a ternary semigroup S, the following assertions are equivalent:*

(1) S is right weakly regular;

(2) $(\mu\wedge\nu\wedge\lambda)^{-}=(\mu\circ\nu\circ\lambda)^{-}$ for every $(\in,\in\vee q)$-fuzzy bi-ideal μ, every $(\in,\in\vee q)$-fuzzy two sided ideal ν and every $(\in,\in\vee q)$-fuzzy right ideal λ of S;

(3) $(\mu\wedge\nu\wedge\lambda)^{-}=(\mu\circ\nu\circ\lambda)^{-}$ for every $(\in,\in\vee q)$-fuzzy quasi-ideal μ, every $(\in,\in\vee q)$-fuzzy two sided ideal ν and every $(\in,\in\vee q)$-fuzzy right ideal λ of S.

Proof. $(1)\Rightarrow(2)$: Let μ be an $(\in,\in\vee q)$-fuzzy bi-ideal, ν an $(\in,\in\vee q)$-fuzzy two sided ideal and λ an $(\in,\in\vee q)$-fuzzy right ideal of S. Now for $a\in S$, there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)=a(s_1t_1as_2t_2)(as_3t_3)$. Now

$$\begin{aligned}(\mu\wedge\nu\wedge\lambda)^{-}(a) &= (\mu\wedge\nu\wedge\lambda)(a)\wedge 0.5\\ &= \{\mu(a)\wedge\nu(a)\wedge\lambda(a)\}\wedge 0.5\\ &\leq \left\{\begin{array}{c}\{\mu(a)\wedge\nu(s_1t_1as_2t_2)\wedge 0.5\}\wedge\\ \{\lambda(as_3t_3)\wedge 0.5\}\end{array}\right\}\wedge 0.5\\ &= \left\{\bigvee_{a=pqr}\{\mu(p)\wedge\nu(q)\wedge\lambda(r)\}\right\}\wedge 0.5\\ &= (\mu\circ\nu\circ\lambda)(a)\wedge 0.5=(\mu\circ\nu\circ\lambda)^{-}(a).\end{aligned}$$

Thus $(\mu \wedge \nu \wedge \lambda)^- \leq (\mu \circ \nu \circ \lambda)^-$.

$(2) \Rightarrow (3)$: This is obvious because every $(\in, \in \vee q)$-fuzzy quasi-ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

$(3) \Rightarrow (1)$: Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Take $\lambda = \nu$. Since every $(\in, \in \vee q)$-fuzzy right ideal is also $(\in, \in \vee q)$-fuzzy quasi-ideal. Thus by hypothesis

$(\mu \wedge \nu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$, which implies $(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$. But $(\mu \circ \nu \circ \nu)^- \leq (\mu \wedge \nu)^-$ is straightforward. Hence $(\mu \wedge \nu)^- = (\mu \circ \nu \circ \nu)^-$. Therefore by Theorem 14, S is right weakly regular.

Corollary 8. *If $(\mu \wedge \nu \wedge \lambda)^- = (\mu \circ \nu \circ \lambda)^-$ for every (α, β)-fuzzy bi-ideal (quasi-ideal) μ, every (α, β)-fuzzy two sided ideal ν and every (α, β)-fuzzy right ideal λ of S, where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$, then S is a right weakly regular ternary semigroup.*

Theorem 17. *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *$(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal μ and every $(\in, \in \vee q)$-fuzzy two sided ideal ν of S;*

(3) *$(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal μ and every $(\in, \in \vee q)$-fuzzy two sided ideal ν of S.*

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy bi-ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Now for any $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}(\mu \wedge \nu)^-(a) &= (\mu \wedge \nu)(a) \wedge 0.5 \\ &= \{\mu(a) \wedge \nu(a) \wedge \nu(a)\} \wedge 0.5 \\ &\leq \left\{ \begin{array}{c} \{\mu(a) \wedge \nu(s_1t_1as_2t_2) \wedge 0.5\} \wedge \\ \{\nu(as_3t_3) \wedge 0.5\} \end{array} \right\} \wedge 0.5 \\ &= \left\{ \bigvee_{a=lmn} \{\mu(l) \wedge \nu(m) \wedge \nu(n)\} \right\} \wedge 0.5 \\ &= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 = (\mu \circ \nu \circ \nu)^-(a).\end{aligned}$$

Thus $(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$.

$(2) \Rightarrow (3)$: This is obvious because every $(\in, \in \vee q)$-fuzzy quasi-ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

$(3) \Rightarrow (1)$: Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Since every $(\in, \in \vee q)$-fuzzy right ideal is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Thus by hypothesis $(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$. Also

$$\begin{aligned}(\mu \circ \nu \circ \nu)^-(a) &= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 \\ &= \left\{ \bigvee_{a=lmn} \{\mu(l) \wedge \nu(m) \wedge \nu(n)\} \right\} \wedge 0.5 \\ &\leq \left\{ \begin{array}{c} \bigvee_{a=lmn} \{\mu(lmn) \wedge 0.5\} \wedge \nu(m) \wedge \\ \{\nu(lmn) \wedge 0.5\} \end{array} \right\} \wedge 0.5 \\ &\leq \bigvee_{a=lmn} \mu(lmn) \wedge \nu(lmn) \wedge 0.5 \\ &= (\mu \wedge \nu)^-(a).\end{aligned}$$

Thus $(\mu \circ \nu \circ \nu)^- \leq (\mu \wedge \nu)^-$. Hence $(\mu \wedge \nu)^- = (\mu \circ \nu \circ \nu)^-$. Therefore by Theorem 14, S is right weakly regular.

Corollary 9. *If $(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$ for every (α, β)-fuzzy bi-ideal (quasi-ideal) μ and every (α, β)-fuzzy two sided ideal ν of S, where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$, then S is a right weakly regular ternary semigroup.*

5 Implication-based fuzzy ideals

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables, and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

$$[x \in \mu] = \mu(x), \tag{8}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi],\ [\Psi]\}, \tag{9}$$

$$[\Phi \rightarrow \Psi] = \min\{1,\ 1 - [\Phi] + [\Psi]\}, \tag{10}$$

$$[\forall x\ \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{11}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{12}$$

The truth valuation rules given in (10) are those in the Łuckasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a section of them in the following
(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a, b) = \begin{cases} 1 \text{ if } a \leq b, \\ 0 \text{ otherwise} \end{cases}$$

(b) Gödel implication operator (I_G):

$$I_G(a,b) = \begin{cases} 1 \text{ if } a \leq b, \\ b \text{ otherwise} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{cG}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1-a & \text{otherwise} \end{cases}$$

for all $a,b \in [0,1]$. Ying [11] introduced the concept of fuzzifying topology. We can expand his/her idea to ternary semigroups, and we define fuzzifying left (right and lateral) ideals as follows:

Definition 4. *A fuzzy set μ in S is called a fuzzifying left (resp. right and lateral) ideal of S if it satisfies the following condition:*

$$\models [z \in \mu] \to [xyz \in \mu] \tag{13}$$

(resp. $\models [z \in \mu] \to [zxy \in \mu]$ and $\models [z \in \mu] \to [xzy \in \mu]$)

for all $x,y,z \in S$.

Obviously, condition (13) is equivalent to (1). Therefore a fuzzifying left (resp. right and lateral) ideal is an ordinary fuzzy left (resp. right and lateral) ideal. In [10] the concept of t-tautology is introduced, that is,

$$\models_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.}$$

Now we extend the concept of implication-based fuzzy left (right and lateral) ideal in the following way:

Definition 5. *Let μ be a fuzzy set in S and $t \in (0,1]$. Then μ is called a t-implication-based fuzzy left (resp. right and lateral) ideal of S if it satisfies the following condition:*

$$\models_t [z \in \mu] \to [xyz \in \mu] \tag{14}$$

(resp. $\models_t [z \in \mu] \to [zxy \in \mu]$ and $\models_t [z \in \mu] \to [xzy \in \mu]$)

for all $x,y,z \in S$.

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy left (resp. right and lateral) ideal of S if and only if it satisfies:

$$I(\mu(z), \mu(xyz)) \geq t \tag{15}$$

$$(\text{resp. } I(\mu(z), \mu(zxy)) \geq t \text{ and } I(\mu(z), \mu(xzy)) \geq t)$$

Theorem 18. *For any fuzzy set μ in S, if $I = I_{GR}$, then μ is a t-implication-based fuzzy left ideal of S if and only if μ is fuzzy left ideal of S for all $t \in (0,1]$.*

Proof. Assume that $I = I_{GR}$ and μ is a t-implication-based fuzzy left ideal of S. Then

$$I_{GR}(\mu(z), \mu(xyz)) \geq t$$

which implies that

$$I_{GR}(\mu(z), \mu(xyz)) = 1.$$

Hence $\mu(xyz) \geq \mu(z)$. Therefore μ is a fuzzy left ideal of S.

Conversely suppose that μ is a fuzzy left ideal of S. Then $\mu(xyz) \geq \mu(z)$ and so

$$I_{GR}(\mu(z), \mu(xyz)) = 1 \geq t,$$

for all $t \in (0,1]$. Hence μ is a t-implication-based fuzzy left ideal of S.

Corollary 10. *For any fuzzy set in S, if $I = I_{GR}$, then every t-implication-based fuzzy left ideal of S is an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Theorem 19. *For any fuzzy set μ in S, if $I = I_G$, then μ is a 0.5-implication-based fuzzy left ideal of S if and only if μ is an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Proof. Assume that μ is a 0.5-implication-based fuzzy left ideal of S. Then for all $x,y,z \in S$

$$I_G(\mu(z), \mu(xyz)) \geq 0.5.$$

Thus $\mu(xyz) \geq \mu(z)$ or $\mu(z) > \mu(xyz) \geq 0.5$. It follows that $\mu(xyz) \geq \min\{\mu(z),\ 0.5\}$. Hence μ is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Conversely suppose that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S. Then for all $x,y,z \in S$,

$$\mu(xyz) \geq \min\{\mu(z),\ 0.5\}.$$

Case I. If $\min\{\mu(z),\ 0.5\} = \mu(z)$, then $\mu(xyz) \geq \mu(z)$ and so

$$I_G(\mu(z), \mu(xyz)) = 1 \geq 0.5.$$

Case II. If $\min\{\mu(z),\ 0.5\} = 0.5$, then $\mu(xyz) \geq 0.5$ and so

Case II-1. If $\mu(xyz) \geq \mu(z)$, then clearly

$$I_G(\mu(z), \mu(xyz)) = 1 \geq 0.5.$$

Case II-2. If $\mu(xyz) < \mu(z)$, then

$$I_G(\mu(z), \mu(xyz)) = \mu(xyz) \geq 0.5.$$

Hence μ is a 0.5-implication-based fuzzy left ideal of S.

Theorem 20. *For any fuzzy set μ in S, if $I = I_{cG}$, then μ is a 0.5-implication-based fuzzy left ideal of S if and only if*

$$\max\{\mu(xyz), 0.5\} \geq \mu(z) \tag{16}$$

for all $x,y,z \in S$.

Proof. Assume that μ is a 0.5-implication-based fuzzy left ideal of S. Then

$$I_{cG}(\mu(z), \mu(xyz)) \geq 0.5$$

and so $\mu(z) \leq \mu(xyz)$ or $1 - \mu(z) \geq 0.5$, that is $\mu(z) \leq 0.5$. It follows that

$$\max\{\mu(xyz), 0.5\} \geq \mu(z)$$

for all $x, y, z \in S$.

Conversely suppose that (16) is valid. If $\max\{\mu(xyz), 0.5\} = \mu(xyz)$, then

$$\mu(xyz) \geq \mu(z)$$

and so

$$I_{cG}(\mu(z), \mu(xyz)) = 1 \geq 0.5.$$

If $\max\{\mu(xyz), 0.5\} = 0.5$, then $\mu(z) \leq 0.5$ and $\mu(xyz) \leq 0.5$. If $\mu(xyz) \geq \mu(z)$, then

$$I_{cG}(\mu(z), \mu(xyz)) = 1 \geq 0.5.$$

If $\mu(xyz) < \mu(z)$, then

$$I_{cG}(\mu(z), \mu(xyz)) = 1 - \mu(z) \geq 0.5.$$

Hence μ is a 0.5-implication-based fuzzy left ideal of S.

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