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A New Fifth Order Derivative Free Newton-Type Method for Solving Nonlinear Equations

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Abstract: The present paper deals with fifth order convergent Newton-type with and without derivative iterative methods for estimating a simple root of nonlinear equations. The error equations are used to establish the fifth order of convergence of the proposed iterative methods. Finally, various numerical comparisons are made using MATLAB to demonstrate the performence of the developed methods.

Keywords: Nonlinear equations, Newton's method, order of convergence, efficiency index, derivative free methods

1 Introduction

Many of the complex problems in science and engineering contain the functions of nonlinear and transcendental nature in the equation of the form $f(x) = 0$ in single variable. The problems arising in several branches of engineering and applied mathematics, e.g. mechanics, chemical reactor theory, kinetic theory of gases and elasticity and other areas are reduced to solve nonlinear equations. Solution of nonlinear equations plays a crucial role in information thoery also [19]. Precisely, the notion of order of convergence of an iterative method (defined in Section 2 of the present paper) has been used to introduce a new dichotomous exponential information gain function whose expected value yields a new entropy functional; an application of this entropy functional in medical diagnosis is also established (see [22] for detail). Many optimization problems also lead to such equations. The famous Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

is one of the most powerful techniques for solving nonlinear equations and minimizing functions. It is easy to implement and has a provably fast rate of convergence under fairly mild assumptions. Because of these and other nice properties, Newton's method is at the heart of many solution techniques used to solve nonlinear equations appearing in real-world problems. In practice, however, Newton's method needs to be modified to make it more robust and computationally efficient. With these modifications, Newton's method (or one of its many variations) is arguably the method of choice for a wide variety of problems in science and engineering. Recently established results regarding the improvement of the classical Newton's formula at the expense of an additional evaluation of the function, an additional evaluation of the first derivative, a change in the point of evaluation or its composition with higher order iterative schemes and making the Newton's method derivative free can be found in the literature (see [1]-[26]) and the references therein. In these works, the order of convergence and the efficiency index in the neighborhood of a simple root have been improved. The Newton's method (1) has quadratic convergence and requires one evaluation of the function and one evaluation of the first derivative per cycle. To circumvent on the derivative calculation in Newton's iteration, Steffensen first coined the following second order convergent scheme

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)} \tag{2}$$

by replacing the derivative in Newton's iteration with forward finite difference. The main advantage of this scheme is that it has quadratic convergence like Newton's method (that is, the same efficiency index 1.414) and it

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does not require a separate function for derivative calculation. Therefore, Steffensen's method referred by Jain [9] can be programmed for a generic function. Other ways to obtain derivative free methods are to replace the derivatives with suitable approximations based on divided difference, Newton interpolation, Hermite interpolation, Lagrange interpolation and rational functions.

In the present paper, firstly a fifth order Newton-type iterative method with derivative is given to approximate the simple roots α of nonlinear equations $f(x) = 0$, that is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. It is a three step method constructed by the use of two existing lower order methods, and hence it can be viewed as a variant of Newton's method. This Newton-type with derivative method is also discussed in [10,11], where it has been used in linearizing the system of nonlinear equations arised during the finite element solution of nonlinear elliptic singularly perturbed problems. It includes five evaluations of the function per iteration providing efficiency index 1.3797. Further, the present scheme is also made derivative free by replacing derivatives with suitable approximations based on divided difference. A greater efficiency index 1.4953 is achieved. The error equations are given theoretically to show that the proposed with and without derivative Newton-type iterative methods have fifth-order of convergence. Finally, various numerical comparisons are made using MATLAB to demonstrate the performence of the developed methods.

This paper is organized as follows: In section 2, we construct the fifth order convergent Newton-type iterative method with derivative using the combination of two existing lower order methods and establish the formula for its order of convergence. Section 3 contains the derivative free Newton-type iterative method together with its proof of order of convergence five, using error equations and divided differences. The performence of each method is demonstrated with the help of several examples using MATLAB in Section 4. Finally, Section 5 contains the concluding remark for this research paper.

2 Newton-type method with derivative

In this section, the fifth-order Newton-type method with derivative is constructed firstly and then its order of convergence is established using the following crucial definitions ([10], [11], cf. [23]):

Definition 1: Let $f \in C^m(D)$be a real-valued function defined on an open set D and let there exists a simple root α of the nonlinear equation $f(x) = 0$. An iterative method $\psi(x_n) = y_n$ is said to have an integer order of convergence m as $n \to \infty$, if there is a sequence$\{x_n\}$of real numbers such that

$$y_n - \alpha = K(x_n - \alpha)^m + O\left((x_n - \alpha)^{m+1}\right), \tag{3}$$

where $K \neq 0$ is a constant, $x_n \subset U(\alpha)$ and $U(\alpha)$is a neighbourhood of α.

Definition 2: The efficiency of an iterative method $\psi(x)$ is measured by the concept of efficiency index which is defined by $\rho^{\frac{1}{\beta}}$, where ρ is the order of the iterative method $\psi(x)$ and β is the total number of function evaluations per iteration.

Potra and Ptak [20] proposed the following modification of Newton's method with third-order of convergence:

$$z_{n+1} = x_n - \frac{f(x_n) + f(x_n - f(x_n)/f'(x_n))}{f'(x_n)} \tag{4}$$

Using (1) and (4), a new Newton-type iterative scheme can be written as:

$$x_{n+1} = x_n - \frac{f(x_n) + f(x_n - f(x_n)/f'(x_n))}{f'(x_n)}$$
$$- \frac{f\left(x_n - \frac{f(x_n)+f\left(x_n - f(x_n)/f'(x_n)\right)}{f'(x_n)}\right)}{f'(x_n - f(x_n)/f'(x_n))}. \tag{5}$$

For practical applications, the scheme (5) can also be expressed as the following three step iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6}$$

$$z_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)}, \tag{7}$$

$$x_{n+1} = z_{n+1} - \frac{f(z_{n+1})}{f'(y_n)}. \tag{8}$$

Convergence of the Method

Let $f(x) \in C^3(D)$be a real-valued function having a simple root $\alpha \in D$, where D is an open interval in $\mathbb{R}$. Since the two iterative formulae $z_n(x)$ and $y_n(x)$ have an order of convergence 3 and 2, respectively, in $U(\alpha)$, we have:

$$z_{n+1} - \alpha = K_1(z_n - \alpha)^3 + ... + K_4(z_n - \alpha)^6 + O(z_n - \alpha)^7 \tag{9}$$

$$y_n - \alpha = K_5(x_n - \alpha)^2 + O(x_n - \alpha)^3, \tag{10}$$

where $K_i \neq 0\,(i = 1,2,...,5)$ are constants.
Further, let us suppose that $\frac{f''(\alpha)}{f'(\alpha)} = M \neq 0$ and the error is expressed as $e_n = x_n - \alpha = z_n - \alpha = y_n - \alpha$ as $n \to \infty$. Using Taylor's expansion of $f(z_{n+1})$ and $f'(y_n)$ about α, we have:
$f(z_{n+1})$

$$= f'(\alpha)(z_{n+1} - \alpha) + \frac{f''(\alpha)}{2}(z_{n+1} - \alpha)^2 + O(z_{n+1} - \alpha)^3$$
$$= f'(\alpha)\left[(z_{n+1} - \alpha) + \frac{f''(\alpha)}{2f'(\alpha)}(z_{n+1} - \alpha)^2 + O(z_{n+1} - \alpha)^3\right]$$
$$= f'(\alpha)\left[K_1 e_n^3 + ... + \frac{M}{2}K_1^2 e_n^6 + O(e_n^7)\right],$$

$$f'(y_n) = f'(\alpha) + f''(\alpha)(y_n-\alpha) + O(y_n-\alpha)^2$$
$$= f'(\alpha)\left[1 + \tfrac{f''(\alpha)}{f'(\alpha)}(y_n-\alpha) + O(y_n-\alpha)^2\right]$$
$$= f'(\alpha)\left[1 + MK_5e_n^2 + O(e_n^3)\right].$$

Substituting the values of $f(z_{n+1})$, $f'(y_n)$ and z_{n+1} in (8), we have

$$x_{n+1}-\alpha = [K_1e_n^3 + \ldots + O(e_n^7)]$$
$$-[K_1e_n^3 + \ldots + (K_4 + \tfrac{M}{2}K_1^2)e_n^6 + O(e_n^7)]\left(1 + MK_5e_n^2 + \ldots\right)^{-1}$$
$$= K_1K_5Me_n^5 + O(e_n^6),$$

which implies that the Newton-type iterative method defined by (6-8) has fifth-order of convergence.

3 Newton-type method without derivative

In this section, we shall make the scheme (6-8) derivative free by using divided difference [23]. Therefore the derivatives in (6-8) are replaced by

$$f'(x_n) \approx f[x_n,w_n] = \frac{f(w_n)-f(x_n)}{w_n-x_n} = \frac{f(w_n)-f(x_n)}{f(x_n)}, \tag{11}$$

$$f'(y_n) \approx \frac{f[y_n,x_n].f[y_n,w_n]}{f[x_n,w_n]};\ w_n = x_n + f(x_n). \tag{12}$$

Substituting (11) and (12) into (6-8), we get

$$y_n = x_n - \frac{f(x_n)}{f[w_n,x_n]}, \tag{13}$$

$$z_{n+1} = x_n - \frac{f(x_n)+f(y_n)}{f[w_n,x_n]}, \tag{14}$$

$$x_{n+1} = z_{n+1} - \frac{f(z_{n+1}).f[w_n,x_n]}{f[x_n,y_n]\,f[w_n,y_n]}. \tag{15}$$

Convergence of the Method

Now, we shall prove that the derivative free method defined by (13-15) also has fifth-order of convergence.
Let α be a simple root of $f(x)$, that is, $f(\alpha) = 0$and $f'(\alpha) \neq 0$. Then using Taylor's expansion, we have:

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \ldots, \tag{16}$$

where $c_k = \frac{f^k(\alpha)}{k!}$; $k = 1, 2, 3, \ldots$ and the error function is expressed as $e_n = x_n - \alpha$.
The expansion of the particular terms used in the iterative scheme (13-15) is given as:

$$f[w_n,x_n] = c_1 + (2c_2 + c_1c_2)e_n + (3c_3 + 3c_1c_3$$
$$+c_1^2c_3 + c_2^2)e_n^2 + \ldots, \tag{17}$$

$$y_n = x_n - \frac{f(x_n)}{f[w_n,x_n]} = \alpha + \left(\frac{c_2}{c_1}\right)(c_1+1)e_n^2 + \ldots, \tag{18}$$

$$f(y_n) = c_2(c_1+1)e_n^2$$
$$+\frac{(c_1^3c_3 - 2c_2^2 + 3c_1^2c_3 + 2c_1c_3 - c_1^2c_2^2 - 2c_1c_2^2)}{c_1}e_n^3 + \ldots \tag{19}$$

Substituting appropriate expressions in (14), we obtain

$$z_{n+1} - \alpha = e_n - \left(\frac{f(x_n)+f(y_n)}{f[w_n,x_n]}\right) = \left(\frac{3c_1c_2^2+2c_2^2+c_1^2c_2^2}{c_1^2}\right)e_n^3 + \ldots,$$

$$f(z_{n+1}) = \left(\frac{3c_1c_2^2 + 2c_2^2 + c_1^2c_2^2}{c_1}\right)e_n^3 + \ldots \tag{20}$$

In order to evaluate the essential terms of (13-15), we expand term by term

$$f[y_n,x_n] = c_1 + c_2e_n\left(\frac{c_1c_3 + c_2^2 + c_1c_2^2}{c_1}\right)e_n^2 + \ldots \tag{21}$$

$$f[y_n,w_n]$$
$$= c_1 + (c_2 + c_1c_2)e_n + \left(\frac{3c_1^2c_3 + c_2^2 + 2c_1c_2^2 + c_1^3c_3}{c_1}\right)e_n^2 + \ldots \tag{22}$$

$$\frac{f[x_n,w_n]}{f[y_n,x_n].f[y_n,w_n]}$$
$$= \frac{1}{c_1} + \left(\frac{c_1^2c_3 - 3c_2^2 - 3c_1c_2^2 + c_1c_3}{c_1^3}\right)e_n^2 + \ldots \tag{23}$$

Equation (15) may also be written as

$$e_{n+1} = x_{n+1} - \alpha = (z_{n+1}-\alpha) - \frac{f(z_{n+1}).f[x_n,w_n]}{f[y_n,x_n]\,f[y_n,w_n]}. \tag{24}$$

Simplifying (24) with the help of (20) and (23), we obtain the error equation
$$e_{n+1}$$
$$= \left(\frac{3c_1c_2^2 + 2c_2^2 + c_1^2c_2^2}{c_1^2}\right)\left(\frac{c_1^2c_3 - 3c_2^2 - 3c_1c_2^2 + c_1c_3}{c_1^2}\right)e_n^5 + \ldots \tag{25}$$

The expression (25) establishes the asymptotic error constant for the fifth-order of convergence for the new fifth-order derivative free method defined by (13-15).

4 Numerical Implementation

We do the computation with the following nonlinear numerical examples to test the performance of our iterative methods. The exact root α of each nonlinear test function is also listed in front of each up to 15 decimal places when such roots are not integers. In our numerical examples, we have used the stopping criterian $|x_{n+1} - x_n| \leq 10^{-15}$.

$$f_1(x) = x^2 - e^x - 3x + 2 \quad \alpha = 0.257530285439860,$$

$$f_2(x) = x^3 + 4x^2 - 10 \quad \alpha = 1.365230013414097,$$
$$f_3(x) = e^x.\sin x - 2x - 5 \quad \alpha = -2.523245230732555,$$
$$f_4(x) = \log x + \sqrt{x} - 5 \quad \alpha = 8.309432694231571,$$
$$f_5(x) = \sqrt{x} - \frac{1}{x} - 3 \quad \alpha = 8.309432694231571,$$
$$f_6(x) = e^{-x} - \cos x \quad \alpha = 1.292695719373398,$$
$$f_7(x) = \cos(x)^2 - \frac{x}{5} \quad \alpha = 1.085982678007472,$$
$$f_8(x) = x^{10} - x^3 - x - 1 \quad \alpha = -0.674177935277052,$$
$$f_9(x) = \sin x - x + 1 \quad \alpha = 1.934563210752024,$$
$$f_{10}(x) = e^{(-x^2+x+2)} - \cos x + x^3 + 1 \quad \alpha = -1.0,$$
$$f_{11}(x) = \sin^{-1}(x^2-1) - x/2 + 1 \quad \alpha = 0.594810968398369,$$
$$f_{12}(x) = \tanh x - \tan x \quad \alpha = 7.068582745628732,$$
$$f_{13}(x) = (x-1)^3 - 1, \quad \alpha = 2.0,$$
$$f_{14}(x) = \cos x - x, \quad \alpha = 0.739085133215160,$$
$$f_{15}(x) = \sin^2 x - x^2 + 1,, \quad \alpha = 1.404491648215341,$$
$$f_{16}(x) = (x+2)e^x - 1, \quad \alpha = -0.442854401002389.$$

We firstly employ our fifth-order derivative method and fifth-order derivative free method to solve the nonlinear equations and compared with Newton's method, Noor (NR [16]), Abbasbandy (AB [1]), Grau (GR [7]), Noor et al. (NN [17]) methods, and Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]) and Heydari et al. (HHL [8]) methods. All computations are done by Matlab version 7.8.0.347 (R2009a), 64 bit (win 64). The computation results are shown in the following Table1, Table 2, Table 3 and Table 4, respectively. We find that these results are consistent to the iterative property, namely, the iterative number of the fifth-order derivative free method is less than or comparable to that of fifth-order derivative method, Newton's method, Noor (NR [16]), Abbasbandy (AB [1]), Grau (GR [7]), Noor et al. (NN [17]) methods, and Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]), Heydari et al. (HHL [8]) and Ghanbari (GH [5]) and Thukral (TH [24]) methods. Note that some of the iterative methods: Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]) and Heydari et al. (HHL [8]) are some higher order iterative methods than our fifth order method. The following Table 1 includes the comparison between number of iterations (N) and running time (t) in seconds of CPU for Newton's method, fifth-order Newton-type with and without derivative methods. We have used "Div.", when the iteration diverges for the considered initial guess x_0. Table 2, Table 3 and Table 4 include the comparison of computational results obtained by Noor (NR [16]), Abbasbandy (AB [1]), Grau (GR [6]), Noor et al. (NN [17]) methods, and Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]), Heydari et al. (HHL [8]), Ghanbari (GH [5]) and Thukral (TH [24]) methods, respectively and our method.

Table 1: Comparison with Newton's methods required such that $|x_{n+1} - x_n| \leq 10^{-15}$

Function	**Root**	**Initial value** (x_0)	**Newton's method**	**Fifth order method**	
				With derivative	**Without derivative**
$x^2 - e^x - 3x + 2$	0.257530285439860	-1.0	N = 6 t=0.00384	N = 3 t=0.00431	N = 3 t=0.01007
		0.5	N = 5 t=0.00278	N = 3 t=0.00443	N = 2 t=0.00666
$x^3 + 4x^2 - 10$	1.365230013414097	-1.0	N = 25 t=0.01447	N = 46 t=0.06208	N = 9 t=0.02838
		0.5	N = 8 t=0.00444	N = 6 t=0.00811	N = 5 t=0.01533
$e^x.\sin x - 2x - 5$	-2.523245230732555	-1.0	N = 5 t=0.00318	N = 3 t=0.00423	N = 2 t=0.00668
		0.0	N = 6 t=0.00338	N = 4 t=0.00579	N = 2 t=0.00649
$\log x + \sqrt{x} - 5$	8.309432694231571	15	N = 7 t=0.00391	N = 4 t=0.00563	N = 3 t=0.00921
		20	N = 8 t=0.00437	N = 5 t=0.00684	N = 3 t=0.00880
$\sqrt{x} - \frac{1}{x} - 3$	9.633595562832696	20	N = 7 t=0.00392	N = 4 t=0.00734	N = 3 t=0.00904
		25	N = 7 t=0.00393	N = 4 t=0.00561	N = 3 t=0.00892
$e^{-x} - \cos x$	1.292695719373398	-1.0	Div.	N = 3 t=0.00368	N = 3 t=0.00814
		1.5	N = 5 t=0.00260	N = 3 t=0.00384	N = 2 t=0.00559
$\cos(x)^2 - \frac{x}{5}$	1.085982678007472	0.5	N = 6 t=0.00340	N = 3 t=0.00421	N = 2 t=0.00639
		1.0	N = 5 t=0.00277	N = 3 t=0.00413	N = 2 t=0.00622
$x^{10} - x^3 - x - 1$	-0.674177935277052	0.25	N = 7 t=0.00395	N = 5 t=0.00682	N = 3 t=0.00929
		0.50	N = 7 t=0.00405	N = 4 t=0.00548	N = 3 t=0.00911
$\sin x - x + 1$	1.934563210752024	2.0	N = 5 t=0.00243	N = 3 t=0.00374	N = 2 t=0.00572
		20	N = 80 t=0.04007	N = 5 t=0.00615	N = 2 t=0.00594
$e^{(-x^2+x+2)} - \cos x + x^3 + 1$	-1.000000000000000	-0.9	N = 5 t=0.00392	N = 3 t=0.00504	N = 2 t=0.00738
		-0.6	N = 5 t=0.00346	N = 3 t=0.00498	N = 3 t=0.01078
$\sin^{-1}(x^2 - 1) - x/2 + 1$	0.594810968398369	0.3	N = 5 t=0.00321	N = 3 t=0.00462	N = 2 t=0.00852
		0.9	N = 5 t=0.00311	N = 3 t=0.00453	N = 2 t=0.00673
$\tanh x - \tan x$	7.068582745628732	5.0	N = 9 t=0.00510	N = 3 t=0.00402	N = 3 t=0.00813
		7.1	N = 5 t=0.01128	N = 5 t=0.00709	N = 2 t=0.00572

Table 2: Comparison with Noor (NR [16]), Abbasbandy (AB [1]), Grau (GR [7]) and Noor et al. (NN [17]) methods required such that $|x_{n+1} - x_n| \leq 10^{-15}$

Function		$x^2 - e^x - 3x + 2$, Initial value $x_0 = 2.0$	
Method by		Iterations	Root
Noor (NR[16])		N=3	0.257530285439860
Abbasbandy (AB [1])		N=5	0.257530285439860
Grau (GR [7])		N=4	0.257530285439860
Noor et al. (NN [17])		N=4	0.257530285439860
Fifth order	**With derivative**	N=3 t=0.002030	0.257530285439860
	Without derivative	N=3 t=0.004680	0.257530285439860
Function		$\cos x - x$, Initial value $x_0 = 1.7$	
Method by		Iterations	Root
Noor (NR [16])		N=4	0.739085133215160
Abbasbandy (AB [1])		N=4	0.739085133215160
Grau (GR [7])		N=2	0.739085133215160
Noor et al. (NN [17])		N=3	0.739085133215160

Fifth order	**With derivative**	N=3 t=0.002180	0.739085133215160
	Without derivative	N=2 t=0.005510	0.739085133215160
Function		$(x-1)^3-1$, Initial value $x_0=3.5$	
Method by		Iterations	Root
Noor (NR [16])		N=3	2.0
Abbasbandy (AB [1])		N=5	2.0
Grau (GR[7])		N=3	2.0
Noor et al. (NN [17])		N=5	2.0
Fifth order	**With derivative**	N=5 t=0.003120	2.0
	Without derivative	N=5 t=0.008400	2.0

Table 3: Comparison with Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]) and Heydari et al. (HHL [8]) methods required such that $|x_{n+1}-x_n| \leq 10^{-15}$

Function		$\sin^2 x - x^2 + 1$, Initial value $x_0 = 1.6$	
Method by		Iterations	Root
Chun et al. (CH [2])		N=4	1.404491648215341
Kou et al. (KLW [13])		N=3	1.404491648215341
Kung et al. (KT [14])		N=3	1.404491648215341
Heydari et al. (HHL [8])		N=3	1.404491648215341
Fifth order	**With derivative**	N=3 t=0.002020	1.404491648215341
	Without derivative	N=3 t=0.004370	1.404491648215341
Function		$(x+2)e^x - 1$, Initial value $x_0 = 0.0$	
Method by		Iterations	Root
Chun et al. (CH [2])		N=4	-0.442854401002388
Kou et al. (KLW [13])		N=4	-0.442854401002388
Kung et al. (KT [14])		N=3	-0.442854401002388
Heydari et al. (HHL [8])		N=3	-0.442854401002388
Fifth order	**With derivative**	N=4 t=0.002660	-0.442854401002389
	Without derivative	N=3 t=0.004830	-0.442854401002389
Function		$e^{(-x^2+x+2)} - \cos x + x^3 + 1$, Initial value $x_0 = -0.7$	
Method by		Iterations	Root
Chun et al. (CH [2])		N=3	-1.0
Kou et al. (KLW [13])		N=3	-1.0
Kung et al. (KT [14])		N=3	-1.0
Heydari et al. (HHL [8])		N=3	-1.0
Fifth order	**With derivative**	N=3 t=0.002500	-1.0
	Without derivative	N=3 t=0.005310	-1.0

Table 4: Comparison with Ghanbari (GH [5]) and Thukral (TH [24]) methods required such that $|x_{n+1}-x_n| \leq 10^{-15}$

Function		$x^2 - e^x - 3x + 2$, Initial value $x_0 = 2.0$	
Method by		Iterations	Root
Ghanbari (GH [5]) Method given by Eq. 12		N=4 t=0.010933	0.257530285439861
Ghanbari (GH [5]) Method given by Eq. 13		N=6 t=0.015572	0.257530285439861
Thukral (TH [23]) Method given by Eq. (2-4)		N=3 t=0.0185839	Div.
Fifth order	**With derivative**	N=3 t=0.002030	0.257530285439860
	Without derivative	N=3 t=0.004680	0.257530285439860
Function		$\cos x - x$, Initial value $x_0 = 1.7$	
Method by		Iterations	Root
Ghanbari (GH [5]) Method given by Eq. 12		N=3 t=0.008075	0.739085133215161
Ghanbari (GH [5]) Method given by Eq. 13		N=5 t=0.012721	0.739085133215161
Thukral (TH [23]) Method given by Eq. (2-4)		N=4 t=0.019152	0.739085133215161
Fifth order	**With derivative**	N=3 t=0.002180	0.739085133215160
	Without derivative	N=2 t=0.005510	0.739085133215160
Function		$(x-1)^3-1$, Initial value $x_0 = 3.5$	
Method by		Iterations	Root
Ghanbari (GH [5]) Method given by Eq. 12		N=5 t=0.014848	2.0
Ghanbari (GH [5]) Method given by Eq. 13		N=8 t=0.021382	2.0
Thukral (TH [23]) Method given by Eq.(2-4)		N=4 t=0.022975	Div.
Fifth order	**With derivative**	N=5 t=0.003120	2.0
	Without derivative	N=5 t=0.008400	2.0

5 Concluding Remark

In this research work, we have constructed the fifth-order Newton-type iterative with and without derivative formulae having efficiency indices 1.3797 and 1.4953, respectively. From the analysis done in Section 3 and the computational results shown in the above Tables, we observe that the new proposed derivative free Newton-type method is free from any derivative evaluation of the function in computing process, showing the efficiency of the method where the computational cost of the derivative is expensive. We can also observe that the iterative number and running time of the fifth-order derivative free method is less than or comparable to that of fifth-order derivative method, Newton's method, Noor (NR [16]), Abbasbandy (AB [1]), Grau (GR [7]), Noor et al. (NN [17]), Ghanbari (GH [5]) and Thukral (TH [24]) methods, and some higher order iterative methods such as Chun et al. (CH [2]), Kou et al. (KLW [13]), Kung et al. (KT [14]) and Heydari et al. (HHL [8]) methods. Moreover, it is also an efficient method for finding out the simple root of the nonlinear equations involving inverse and hyperbolic functions. Moreover, the inclusion of fifth order Newton type iterative method gives more rapid result than other approximation methods in linearizing the system of nonlinear equations arised during the finite element solution of nonlinear singularly perturbed Ginzberg Landau equation [10] and Schrodinger's equation [11]. Consequently, this new iterative formula can be used as an alternative to the existing methods or in some cases, where existing methods are not successful. Thus, the superiority of the presented derivative free method can be corroborated on the basis of the ease of the programming of the method and the numerical results displayed in the above Tables. Moreover, In case of multistep iterative methods, a fast convergence can be attained if starting points are sufficiently close to the sought roots; for this reason, a special attention should be paid to finding suitable starting points [cf. [25]]. Finally, we conclude that the Newton-type methods constructed in this paper are some efficient fifth-order convergent iterative methods and like all other iterative methods, these methods have its own domain of validity.

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