Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090342

# On I-Convergent Sequence Spaces of Bounded Linear Operators Defined By A Sequence of Moduli

*Vakeel A. Khan* and Mohd Shafiq*

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Received: 12 Aug. 2014, Revised: 13 Nov. 2014, Accepted: 14 Nov. 2014
Published online: 1 May 2015

**Abstract:** In this article we introduce and study sequence spaces $\mathscr{C}^I(\mathscr{T},\mathscr{F})$, $\mathscr{C}_0^I(\mathscr{T},\mathscr{F})$ and $\mathscr{B}_\infty^I(\mathscr{T},\mathscr{F})$ on the sequence of bounded linear operators with the help of a sequence $\mathscr{F}=(f_k)$ of modulus functions. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

**Keywords:** Bounded operators, Ideal, filter, I-convergent sequence, solid and monotone space, modulus function, Lipschitz function

## 1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively.
We denote

$$\omega=\{x=(x_k):x_k\in\mathbb{R}\text{ or }\mathbb{C}\},$$

the space of all real or complex sequences.

Any subspace $\lambda$ of the linear space $\omega$ of sequences is called a sequence space. A sequence space $\lambda$ with linear topology is called a $K$-space provided each of maps $p_i:\lambda\to\mathbb{C}$ defined by $p_i(x)=x_i$ is continuous, for all $i\in\mathbb{N}$. A space $\lambda$ is called an $FK$-space provided $\lambda$ is complete linear metric space. An $FK$-space whose topology is normable is called a $BK$-space.

We denote

$$\mathscr{L}(\mathscr{T})=\left\{\mathscr{T}=(T_k):T_k:X\to Y\text{ is linear, for each k}\in\mathbb{N}\right\}, \tag{1.1}$$

the space of all linear operators from a normed space $X$ to normed space $Y$.

**Definition 1.1.** Let $X$ and $Y$ be two normed linear spaces and $T:\mathscr{D}(T)\to Y$ be a linear operator, where $\mathscr{D}(T)\subset X$. Then, the operator $T$ is said to be bounded if there exists a real $k>0$ such that

$$\| Tx \|\leq k\| x \|,\text{ for all, } x\in\mathscr{D}(T). \tag{1.2}$$

The sequences of bounded linear operators arise frequently in the abstract formulation of concrete situations, for instance, in connection with convergence problems of Fourier series or sequences of interpolation polynomials or methods of numerical integration, to name just a few. In such cases one is usually concerned with the convergence of those sequences of operators with boundedness of corresponding sequences of norms or with similar properties. We also study $I$-convergence of sequences of these operators and some related results.

Let $\mathscr{B}_\infty(\mathscr{T})$ be denote the normed space of sequences of all bounded linear operators from a normed space $X$ to a normed space $Y$ normed by

$$\| \mathscr{T} \|=\sup_k \| T_k(x) \| \ \ (\text{see}[14]). \tag{1.3}$$

$\mathscr{B}_\infty(\mathscr{T})$ is a Banach space if $Y$ is a Banach space.

Throughout, $O$ and $I$ represent zero and identity operators respectively.

It was due to J.Von Neumann(see [14,22]), the following definitions were introduced.

**Definition 1.2.** Let $X$ and $Y$ be two normed linear spaces. A sequence $(T_k)$ of operators $T_k\in\mathscr{B}_\infty(\mathscr{T})$ is said to be;
1) Uniformly convergent if $(T_k)$ converges in the norm on $\mathscr{B}_\infty(\mathscr{T})$.
2) Strongly convergent if $(T_kx)$ converges strongly in $Y$

* Corresponding author e-mail: vakhanmaths@gmail.com

for every $x \in X$.
3) Weakly convergent if $(T_k x)$ converges weakly in $Y$ for every $x \in X$.

That is, if there is a linear operator $T : X \to Y$ such that $T_k \in \mathscr{B}_\infty(\mathscr{T})$ is;
1) Uniformly convergent if $\| T_k - T \| \to 0$.
2) Strongly convergent if $\| T_k x - Tx \| \to 0$, for every $x \in X$.
3) Weakly convergent if $| f(T_k x) - f(Tx) | \to 0$, for every $x \in X$ and $f \in Y'$.

**Theorem 1.3.** Let $T_k \in \mathscr{B}_\infty(\mathscr{T})$, where k=1,2,3,... Then $T_k \to T$ if and only if for every $\varepsilon > 0$, there is an $N$, depending only on $\varepsilon$, such that for all $k > N$ and all $x \in X$ of norm 1, we have,

$$\| T_k(x) - T(x) \| < \varepsilon (\text{see}[14]). \tag{1.4}$$

Let $\mathscr{C}(\mathscr{T})$ and $\mathscr{C}_0(\mathscr{T})$ be the convergent and null sequence spaces respectively of the sequence $(T_k)$ of bounded operators defined as follows.

$$\mathscr{C}(\mathscr{T}) = \left\{ \mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : \| T_k(x) - T(x) \| \to 0, \text{ for all } x \in X \right\}$$

and

$$\mathscr{C}_0(\mathscr{T}) = \left\{ \mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : \| T_k(x) - O(x) \| \to 0 \text{ for all } x \in X \right\}, (c.f.[14],[15]).$$

Then $\mathscr{C}(\mathscr{T})$ and $\mathscr{C}_0(\mathscr{T})$ are normed spaces with norm defined as above in (1.3).

**Remark 1.4.** $\mathscr{C}_0(\mathscr{T}) \subset \mathscr{C}(\mathscr{T}) \subset \mathscr{B}_\infty(\mathscr{T})$.

As a generalisation of usual convergence, the concept of statistical convergent was first introduced by Fast [2] and also independently by Buck [1] and Schoenberg[29] for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [3], Šalát [26], Tripathy [30] and many others.

Henceforth, in this paper, $x$ is considered as an element of the normed space $X$.

**Definition 1.5.** A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) \subset \mathscr{L}(\mathscr{T})$ is said to be statistically convergent to an operator $T$ if for every $\varepsilon > 0$, we have

$$\lim_k \frac{1}{k} \left| \left\{ n \in \mathbb{N} : \| T_n(x) - T(x) \| \geq \varepsilon,\ n \leq k \right\} \right| = 0, \tag{1.5}$$

where vertical lines denote the cardinality of the enclosed set.
That is, if $\delta(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \left\{ k \in \mathbb{N} : \| T_k(x) - T(x) \| \geq \varepsilon \right\}.$$

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salǎt and Wilczyński [11,12]. Later on, it was studied by Šalát, Tripathy and Ziman [27,28], Tripathy and Hazarika [31, 32], Mursaleen and Alotaibi [16], Mursaleen and Mohiuddin [17,18,19], Mursaleen and S.K.Sharma [20], Khan *et,al* [6,7,8] and many others.

**Here we give some preliminaries about the notion of I-convergence.**

**Definition 1.6.** Let $N$ be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal if
1) $I$ is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$
2) $I$ is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

**Definition 1.7.** A non-empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if
1) $\Phi \notin \pounds(I)$,
2) $\forall\, A,\ B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$,
3) $\forall\, A \in \pounds$(I) and $A \subseteq B \Rightarrow B \in \pounds(I)$.

**Definition 1.8.** An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$.

**Definition 1.9.** A non-trivial ideal $I \subseteq 2^N$ is called admissible if

$$\{\{x\} : x \in N\} \subseteq I.$$

**Definition 1.10.** A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

**Remark 1.11.** For each ideal $I$, there is a filter $\pounds(I)$ corresponding to $I$.
i.e $\pounds(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

**Definition 1.12.** A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) \subset \mathscr{L}(\mathscr{T})$ is said to be $I$-convergent to an operator $T$ if for every $\varepsilon > 0$, the set $\{k \in N : \| T_k(x) - T(x) \| \geq \varepsilon\} \in I$.
In this case, we write $I - \lim T_k = T$.

**Definition 1.13.** A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T})$ is said to be $I$-null if $T = O$. In this case, we write $I - \lim T_k = O$.

**Definition 1.14.** A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T})$ is said to be $I$-cauchy if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that $\{k \in N : \| T_k(x) - T_m(x) \| \geq \varepsilon\} \in I$.

**Definition 1.15.** A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T})$ is said to be $I$-bounded if there exists some $M > 0$ such that $\{k \in N : \| T_k(x) \| \geq M\} \in I$.

**Definition 1.16.** A sequence space $E^{\mathscr{T}}$( space of operators) said to be solid(normal) if $(\alpha_k T_k) \in E^{\mathscr{T}}$ whenever $(T_k) \in E^{\mathscr{T}}$ and for any sequence$(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

**Definition 1.17.** A sequence space $E^{\mathscr{T}}$ said to be symmetric if $(T_{\pi(k)}) \in E^{\mathscr{T}}$ whenever $T_k \in E^{\mathscr{T}}$, where $\pi$ is a permutation on $\mathbb{N}$.

**Definition 1.18.** A sequence space $E^{\mathscr{T}}$ said to be sequence algebra if $(T_k) * (S_k) = (T_k.S_k) \in E^{\mathscr{T}}$ whenever $(T_k),\ (S_k) \in E^{\mathscr{T}}$.

**Definition 1.19.** A sequence space $E^{\mathscr{T}}$ said to be convergence free if $(S_k) \in E^{\mathscr{T}}$ whenever $(T_k) \in E^{\mathscr{T}}$ and $T_k = O$ implies $S_k = O$, for all k.

**Definition 1.20.** Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 \dots\} \subset \mathbb{N}$ and $E^{\mathscr{T}}$ be a Sequence space. A $K$-step space of $E^{\mathscr{T}}$ is a sequence space $\lambda_K^{E^{\mathscr{T}}} = \{(T_{k_n}) \in \mathscr{L}(\mathscr{T}) : (T_k) \in E^{\mathscr{T}}\}$.

**Definition 1.21.** A canonical pre-image of a sequence $(T_{k_n}) \in \lambda_K^{E^{\mathscr{T}}}$ is a sequence $(T_k) \in \mathscr{L}(\mathscr{T})$ defined by

$$S_k = \begin{cases} T_k, & \text{if } k \in K, \\ O, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda_K^{E^{\mathscr{T}}}$ is a set of preimages all elements in $\lambda_K^{E^{\mathscr{T}}}$ .i.e. $\mathscr{S}$ is in the canonical preimage of $\lambda_K^{E^{\mathscr{T}}}$ iff $\mathscr{S}$ is the canonical preimage of some $\mathscr{T} \in \lambda_K^{E^{\mathscr{T}}}$.

**Definition 1.22.** A sequence space $E^{\mathscr{T}}$ is said to be monotone if it contains the canonical preimages of its step space.

**Definition 1.23.** A map $\bar{h}$ defined on a domain $D \subset X$ i.e $\bar{h} : D \subset X \rightarrow$ is said to satisfy Lipschitz condition if $|\bar{h}(x) - \bar{h}(y)| \leq K|x-y|$ where $K$ is known as the Lipschitz constant. The class of $K$-Lipschitz functions defined on D is denoted by $\bar{h} \in (D, K)$.

**Definition 1.24.** A convergence field of $I$-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of $l_\infty$ with respect to the supremum norm, $F(I) = l_\infty \cap c^I (see[27])$.

**Definition 1.25.** A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t+u) \leq f(t) + f(u)$ for all $t,\ u \geq 0$,
(3) $f$ is increasing and
(4) $f$ is continuous from the right at zero.

For any modulus function $f$, we have the inequalities $|f(x) - f(y)| \leq f(|x-y|)$ and $f(nx) \leq nf(x)$, for all $x,\ y \in [0,\infty)$.

A modulus function $f$ is said to satisfy $\Delta_2 -$Condition for all values of $u$ if there exists a constant $K > 0$ such that $f(Lu) \leq \mathrm{KL} f(u)$ for all values of $L > 1$.

The idea of modulus function was introduced by Nakano in 1953.$(See[21],\ Nakano, 1953)$.

Ruckle [23,24,25] used the idea of a modulus function $f$ to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\} = \{x = x_k : (f(|x_k|)) \in X\}.$$

This space is an $FK$-space and Ruckle [23,24,25] proved that the intersection of all such $X(f)$ spaces is $\phi$, the space of all finite sequences. The space $X(f)$ is closely related to the space $\ell_1$ which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [23,24,25] proved that, for any modulus $f$.

$$X(f) \subset \ell_1 \text{ and } L(f)^\alpha = \ell_\infty$$

where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch [5]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling [4], G.Köthe [13] and W.H.Ruckle [23,24,25].

After then E.Kolk [**?**,10] gave an extension of $X(f)$ by considering a sequence of moduli $\mathscr{F} = (f_k)$ and defined the sequence space

$$X(\mathscr{F}) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

*We used the following lemmas for establishing some results of this article.*

**Lemma(I)**. Every solid space is monotone.

**Lemma(II)**. Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

**Lemma(III)**. If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

Throughout the article $\mathscr{C}^I(\mathscr{T})$, $\mathscr{C}_0^I(\mathscr{T})$, $\mathscr{B}_\infty^I(\mathscr{T})$, $\mathscr{M}_{\mathscr{C}}^I(\mathscr{T})$ and $\mathscr{M}_{\mathscr{C}_\circ}^I(\mathscr{T})$ are considered as the classes of all I-convergent, I-null, I-bounded, bounded I-convergent and bounded I-null sequences of bounded linear operators respectively.

## 2 Main Results

In this article, we introduce and study the following classes of sequences.

$$\mathscr{C}^I(\mathscr{T},\mathscr{F}) = \left\{\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : I - \lim_k f_k(\| T_k(x) - L \|) = 0,\ \text{for some } L\right\} \quad (2.1);$$

$$\mathscr{C}_0^I(\mathscr{T},\mathscr{F}) = \left\{\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : I - \lim_k f_k(\| T_k(x) \|) = 0\right\} \quad (2.2);$$

$$\mathscr{B}_\infty^I(\mathscr{T},\mathscr{F}) = \left\{\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : \exists K > 0 \text{ such that } \left\{k \in \mathbb{N} : f_k(\| T_k(x) \|) \geq K\right\} \in I\right\} \quad (2.3);$$

$$\mathscr{B}(\mathscr{T},\mathscr{F}) = \left\{\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}) : \sup_k f_k(\| T_k(x) \|) < \infty\right\} \quad (2.4).$$

We also denote

$$\mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F}) = \mathscr{B}_\infty(\mathscr{T},\mathscr{F}) \cap \mathscr{C}^I(\mathscr{T},\mathscr{F}) \text{ and } \mathscr{M}_{\mathscr{C}_\circ}^I(\mathscr{T},\mathscr{F}) = \mathscr{B}_\infty(\mathscr{T},\mathscr{F}) \cap \mathscr{C}_0^I(\mathscr{T},\mathscr{F}).$$

where $\mathscr{F} = (f_k)$ is a sequence of modulus functions.

**Theorem 2.1.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then, the classes of sequences $\mathscr{C}^I(\mathscr{T},\mathscr{F})$, $\mathscr{C}_0^I(\mathscr{T},\mathscr{F})$, $\mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$ and $\mathscr{M}_{\mathscr{C}_\circ}^I(\mathscr{T},\mathscr{F})$ are linear spaces.

**Proof**. We shall prove the result for the space $\mathscr{C}^I(\mathscr{T},\mathscr{F})$. For other spaces the results are similar.
For, let $(\mathscr{T}) = (T_k)$, $(\mathscr{S}) = (S_k)$ be two elements of $\mathscr{C}^I(\mathscr{T},\mathscr{F})$ and $\alpha$, $\beta$ be scalars.
Now, since $(T_k)$, $(S_k) \in \mathscr{C}^I(\mathscr{T},\mathscr{F})$. Then, there exists some $L_1$, $L_2$ such that

$$I - \lim_k f_k(\| T_k(x) - L_1 \|) = 0 \qquad (2.5)$$

and

$$I - \lim_k f_k(\| S_k(x) - L_2 \|) = 0. \qquad (2.6)$$

Then, for given $\varepsilon > 0$, the sets

$$\left\{k \in \mathbb{N} : f_k(\| T_k(x) - L_1 \|) \geq \frac{\varepsilon}{2}\right\} \in I$$

and

$$\left\{k \in \mathbb{N} : f_k(\| S_k(x) - L_2 \|) \geq \frac{\varepsilon}{2}\right\} \in I.$$

Let

$$A_1 = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - L_1 \|) < \frac{\varepsilon}{2}\right\} \in \pounds(I) \qquad (2.7)$$

and

$$A_2 = \left\{k \in \mathbb{N} : f_k(\| S_k(x) - L_2 \|) < \frac{\varepsilon}{2}\right\} \in \pounds(I) \qquad (2.8)$$

be such that $A_1^c$, $A_2^c \in I$. Now, since each $f_k(\ k \in \mathbb{N})$ is a modulus function, we have

$$A_3 = \left\{k \in \mathbb{N} : f_k\Big( \| (\alpha T_k(x) + \beta S_k(x) - (\alpha L_1 + \beta L_2) \| \Big) < \varepsilon\right\}$$

$$\supseteq \left[\left\{k \in \mathbb{N} : f_k\Big(|\alpha| \| T_k(x) - L_1 \| \Big) < \frac{\varepsilon}{2}\right\} \cap \left\{k \in \mathbb{N} : f_k\Big(|\beta| \| S_k(x)) - L_2 \| \Big) < \frac{\varepsilon}{2}\right\}\right]$$

$$\supseteq \left[\left\{k \in \mathbb{N} : f_k\Big( \| T_k(x) - L_1 \| \Big) < \frac{\varepsilon}{2}\right\} \cap \left\{k \in \mathbb{N} : f_k\Big( \| S_k(x)) - L_2 \| \Big) < \frac{\varepsilon}{2}\right\}\right]$$

Therefore,

$$A_3 = \left\{k \in \mathbb{N} : f_k\Big( \| (\alpha T_k(x) + \beta S_k(x) - (\alpha L_1 + \beta L_2) \| \Big) < \varepsilon\right\}$$

$$\supseteq \left[\left\{k \in \mathbb{N} : f_k\Big( \| T_k(x) - L_1 \| \Big) < \frac{\varepsilon}{2}\right\} \cap \left\{k \in \mathbb{N} : f_k\Big( \| S_k(x) - L_2 \| \Big) < \frac{\varepsilon}{2}\right\}\right] \qquad (2.9)$$

implies that $A_3 \in \pounds(I)$. Thus, $A_3^c = A_1^c \cup A_2^c \in I$.
Therefore, $\alpha(T_k) + \beta(S_k) \in \mathscr{C}^I(\mathscr{T},\mathscr{F})$ for all scalars $\alpha$, $\beta$ and $(T_k)$, $(S_k) \in \mathscr{C}^I(\mathscr{T},\mathscr{F})$.
Hence, the space $\mathscr{C}^I(\mathscr{T},\mathscr{F})$ is linear.

**Theorem 2.2.** The spaces $\mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$ and $\mathscr{M}_{\mathscr{C}_\circ}^I(\mathscr{T},\mathscr{F})$ are paranormed, paranormed by

$$g(\mathscr{T}) = \sup_k f_k \| T_k(x) \| .$$

**Proof**. $(P_1)$ Let $\mathscr{T} = (T_k) \in \mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$. If $\mathscr{T} = O$, then $g(\mathscr{T}) = 0$ is obvious.
$(P_2)$ Let $\mathscr{T} = (T_k) \in \mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$. Then

$$g(-\mathscr{T}) = \sup_k f_k(\| (-T_k)(x) \|) = \sup_k f_k(\| -T_k(x) \|) = \sup_k f_k(\| T_k(x) \|) = g(\mathscr{T}).$$

Therefore, $g(-\mathscr{T}) = g(\mathscr{T})$, for all $\mathscr{T} \in \mathscr{M}_{\mathscr{C}}^I(\mathscr{C},\mathscr{F})$.
$(P_3)$ Let $\mathscr{T} = (T_k)$ and $\mathscr{S} = (S_k) \in \mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$. Then,

$$g(\mathscr{T} + \mathscr{S}) = \sup_k f_k(\| (T_k + S_k)(x) \|) = \sup_k f_k(\| T_k(x) + S_k(x) \|)$$

$$\leq \sup_k f_k(\| T_k(x) \|) + \sup_k f_k(\| S_k(x) \|) = g(\mathscr{T}) + g(\mathscr{S}).$$

Therefore, $g(\mathscr{T} + \mathscr{S}) \leq g(\mathscr{T}) + g(\mathscr{S})$ for all, $\mathscr{T}$, $\mathscr{S} \in \mathscr{M}_{\mathscr{C}}^I(\mathscr{T},\mathscr{F})$.
$(P_4)$ Let $(\lambda_k)$ be a sequence of scalars with $(\lambda_k) \to \lambda$ $(k \to \infty)$ and
$(T_k)$, $T \in \mathscr{M}_{\mathscr{S}}^I(\mathscr{T},\mathscr{F})$ such that

$$T_k \to T\ (k \to \infty),$$

in the sense that

$$g(T_k - T) \to 0\ (k \to \infty),$$

Then, since the inequality

$$g(T_k) \leq g(T_k - T) + g(T)$$

holds by subadditivity of $g$, the sequence $\{g(T_k)\}$ is bounded.
Therefore,

$$g\left[(\lambda_k T_k - \lambda T)\right] = g\left[(\lambda_k T_k - \lambda T_k + \lambda T_k - \lambda T)\right]$$
$$= g\left[(\lambda_k - \lambda)T_k + \lambda(T_k - T)\right]$$
$$\leq g\left[(\lambda_k - \lambda)T_k\right] + g\left[\lambda(T_k - T)\right]$$
$$\leq \mid (\lambda_k - \lambda) \mid g(x_k) + \mid \lambda \mid g(x_k - T) \to 0$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous.
Hence, $\mathscr{M}^I_{\mathscr{S}}(\mathscr{T}, \mathscr{F})$ is paranormed space.
For $\mathscr{M}^I_{\mathscr{S}_\circ}(\mathscr{T}, \mathscr{F})$, the result is similar.

**Theorem 2.3.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. A sequence $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T})$ I-converges if and only if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\left\{k \in \mathbb{N} : f_k(\| T_k(x) - T_{N_\varepsilon}(x) \|) < \varepsilon\right\} \in \pounds(I). \tag{2.10}$$

**Proof**. Let us suppose that $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T}))$.
Let $L = I - \lim \mathscr{T}$. Then, the set

$$B_\varepsilon = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - L \|) < \frac{\varepsilon}{2}\right\} \in \pounds(I) \ \text{ for all } \varepsilon > 0.$$

Now, since each $f_k$, $k \in \mathbb{N}$ is a modulus function. Fix an $N_\varepsilon \in B_\varepsilon$.
Then, we have

$$f_k(\| T_k(x) - T_{N_\varepsilon}(x) \|) \leq f_k(\| T_k(x) - L \|) + f_k(\| T_{N_\varepsilon}(x) - L \|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which holds for all $k \in B_\varepsilon$.
Hence $\left\{k \in \mathbb{N} : f_k(\| T_k(x) - T_{N_\varepsilon}(x) \|) < \varepsilon\right\} \in \pounds(I)$.
Conversely, suppose that

$$\left\{k \in \mathbb{N} : f_k(\| T_k(x) - T_{N_\varepsilon}(x) \|) < \varepsilon\right\} \in \pounds(I).$$

Since, each $f_k$, $k \in \mathbb{N}$ is a modulus function and by basic norm and modulus inequalities, we have

$$\left\{k \in \mathbb{N} : \mid f_k(\| T_k(x) \|) - f_k(\| T_{N_\varepsilon}(x) \|) \mid < \varepsilon\right\} \in \pounds(I), \text{ for all } \varepsilon > 0.$$

Then, the set

$$C_\varepsilon = \left\{k \in \mathbb{N} : f_k(\| T_k(x) \|) \in \left[f_k(\| T_{N_\varepsilon}(x) \|) - \varepsilon, f_k(\| T_{N_\varepsilon}(x) \|) + \varepsilon\right]\right\}$$
$$\in \pounds(I) \ \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = \left[f_k(\| T_{N_\varepsilon}(x) \|) - \varepsilon, f_k(\| T_{N_\varepsilon}(x) \|) + \varepsilon\right]$. If we fix an $\varepsilon > 0$ then, we have $C_\varepsilon \in \pounds(I)$ as well as $C_{\frac{\varepsilon}{2}} \in \pounds(I)$. Hence $C_\varepsilon \cap C_{\frac{\varepsilon}{2}} \in \pounds(I)$. This implies that

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \phi.$$

That is

$$\{k \in \mathbb{N} : f_k(\| T_k(x) \|) \in J\} \in \pounds(I).$$

That is

$$diam J \leq diam J_\varepsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\varepsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ...........$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and
$\{k \in \mathbb{N} : f_k(\| T_k(x) \|) \in I_k\} \in \pounds(I)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim f_k(\| T_k(x) \|)$ showing that $\mathscr{T} = (T_k) \in \mathscr{B}_\infty(\mathscr{T})$ is I-convergent.
Hence, the result.

**Theorem 2.4**. Let $\mathscr{F} = (f_k)$ and $\mathscr{G} = (g_k)$ be two sequences of modulus functions and for each $k \in \mathbb{N}$, $(f_k)$ and $(g_k)$ satisfying $\Delta_2 -$ Condition, then
(a) $\mathscr{X}(\mathscr{T}, \mathscr{G}) \subseteq \mathscr{X}(\mathscr{T}, \mathscr{F} \circ \mathscr{G})$
(b) $\mathscr{X}(\mathscr{T}, \mathscr{F}) \cap (\mathscr{T}, \mathscr{G}) \subseteq \mathscr{X}(\mathscr{T}, \mathscr{F} + \mathscr{G})$
for $\mathscr{X} = \mathscr{C}^I$, $\mathscr{C}^I_\circ$, $\mathscr{M}^I_{\mathscr{C}}$ and $\mathscr{M}^I_{\mathscr{C}_\circ}$

**Proof.**(a) Let $\mathscr{T} = (T_k) \in \mathscr{C}^I_\circ(\mathscr{T}, \mathscr{G})$ be any arbitrary element. Then

$$I - \lim_k g_k(\| T_k(x) \|) = 0$$

that is, the set

$$\left\{k \in \mathbb{N} : g_k\Big( \| T_k(x) \| \Big) \geq \varepsilon\right\} \in I. \tag{2.11}$$

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that for each k $\in \mathbb{N}$, $f_k(t) < \varepsilon$, $0 \leq t \leq \delta$.
Let us denote

$$S_k = g_k\Big( \| T_k(x) \| \Big)$$

and consider

$$\lim_k f_k(S_k) = \lim_{S_k \leq \delta, k \in \mathbb{N}} f_k(S_k) + \lim_{S_k > \delta, k \in \mathbb{N}} f_k(S_k).$$

Now, since $f_k$ for each k $\in \mathbb{N}$ is an modulus function , we have

$$\lim_{S_k \leq \delta, k \in \mathbb{N}} f_k(S_k) \leq f_k(2) \lim_{S_k \leq \delta, k \in \mathbb{N}} (S_k). \tag{2.12}$$

For $S_k > \delta$, we have

$$S_k < \frac{S_k}{\delta} < 1 + \frac{S_k}{\delta}.$$

Now, since each $f_k$ is non-decreasing and modulus, it follows that

$$f_k(S_k) < f_k(1 + \frac{S_k}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2S_k}{\delta}).$$

Again, since $f_k$ satisfies $\Delta_2$ – Condition, we have

$$f_k(S_k) < \frac{1}{2}K\frac{(S_k)}{\delta}f_k(2) + \frac{1}{2}K\frac{(S_k)}{\delta}f_k(2)$$

Thus, $f_k(S_k) < K\frac{(S_k)}{\delta}f_k(2)$.
Hence,

$$\lim_{S_k>\delta, k\in\mathbb{N}} f_k(S_k) \leq \max\{1, K\delta^{-1}f_k(2) \lim_{S_k>\delta, k\in\mathbb{N}} (S_k). \tag{2.13}$$

Therefore, from (2.11), (2.12) and (2.13), we have, $\mathscr{T} = (T_k) \in \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F} \circ \mathscr{G})$.
Thus, $\mathscr{C}_\circ^I(\mathscr{T}, \mathscr{G}) \subseteq \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F} \circ \mathscr{G})$. Hence, $\mathscr{X}(\mathscr{T}, \mathscr{G}) \subseteq \mathscr{X}(\mathscr{T}, \mathscr{F} \circ \mathscr{G})$ for $\mathscr{X} = \mathscr{C}_\circ^I$.
For $\mathscr{X} = \mathscr{C}^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$ the inclusions can be established similarly.
(b) Let $\mathscr{T} = (T_k) \in \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F}) \cap \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{G})$.

$$I - \lim_k f_k(\| T_k(x) \|) = 0 \tag{2.14}$$

and

$$I - \lim_k g_k(\| T_k(x) \|) = 0. \tag{2.15}$$

Therefore, from (2.14) and (2.15), we have

$$I - \lim_k \Big[ (f_k + g_k)(\| T_k(x) \|) \Big] = 0$$

implies that $\mathscr{T} = (T_k) \in \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F} + \mathscr{G})$.
Thus, $\mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F}) \cap \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{G}) \subseteq \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F} + \mathscr{G})$.
Hence, $\mathscr{X}(\mathscr{T}, \mathscr{F}) \cap \mathscr{X}(\mathscr{T}, \mathscr{G}) \subseteq \mathscr{X}(\mathscr{T}, \mathscr{F} + \mathscr{G})$ for $\mathscr{X} = \mathscr{C}_\circ^I$.
For $\mathscr{X} = \mathscr{C}^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$ the inclusions are similar.
*For $g_k(x) = x$ and $f_k(x) = f(x)$, $\forall\, x \in [0, \infty)$, we have the following corollary.*

**Corollary 2.5**. $\mathscr{X}(\mathscr{T}) \subseteq \mathscr{X}(\mathscr{T}, \mathscr{F})$ for $\mathscr{X} = \mathscr{C}^I$, $\mathscr{C}_\circ^I$, $\mathscr{M}_\mathscr{C}^I$ and $\mathscr{M}_{\mathscr{C}_\circ}^I$.

**Theorem 2.6.** Let $I$ be an admissible ideal and $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then, the following are equivalent.
(a) $(T_k) \in \mathscr{C}^I(\mathscr{T}, \mathscr{F})$;
(b) there exists $(S_k) \in \mathscr{C}(\mathscr{T}, \mathscr{F})$ such that $T_k = S_k$, for a.a.k.r.$I$;
(c) there exists $(S_k) \in \mathscr{C}(\mathscr{T}, \mathscr{F})$ and $(U_k) \in \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F})$ such that $T_k = S_k + U_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f_k(\| S_k(x) - L \|) \geq \varepsilon\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2 < k_3 ...\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} f_k(\| T_{k_n}(x) - L \|) = 0$.

**Proof**. (a) implies (b). Let $(T_k) \in \mathscr{C}^I(\mathscr{T}, \mathscr{F})$. Then, for any $\varepsilon > 0$, there exists some $L$ such that the set

$$\{k \in \mathbb{N} : f_k(\| T_k(x) - L \|) \geq \varepsilon\} \in I.$$

Let $(m_t)$ be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : f_k(\| T_k(x) - L \|) \geq t^{-1}\} \in I.$$

Define a sequence $(S_k)$ as

$$S_k = T_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$S_k = \begin{cases} T_k, \text{ if } f_k(\| T_k(x) - L \|) < t^{-1}, \\ L, \text{otherwise}. \end{cases}$$

Then, $(S_k) \in \mathscr{C}(\mathscr{T}, \mathscr{F})$ and from the following inclusion

$$\{k \leq m_t : T_k \neq S_k\} \subseteq \{k \leq m_t : f_k(\| T_k(x) - L \|) \geq \varepsilon\} \in I.$$

We get $T_k = S_k$, for a.a.k.r.$I$.
(b) implies (c). For $(T_k) \in \mathscr{C}^I(\mathscr{T}, \mathscr{F})$. Then, there exists $(S_k) \in \mathscr{C}(\mathscr{T}, \mathscr{F})$ such that $T_k = S_k$, for a.a.k.r.$I$. Let $K = \{k \in \mathbb{N} : T_k \neq S_k\}$, then $K \in I$.
Define a sequence $(U_k)$ as

$$U_k = \begin{cases} T_k - S_k, & \text{if } k \in K, \\ O, & \text{otherwise}. \end{cases}$$

Then $U_k \in \mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F})$ and $S_k \in \mathscr{C}(\mathscr{T}, \mathscr{F})$.
(c) implies (d). Let $P_1 = \{k \in \mathbb{N} : f_k(\| U_k(x) \|) \geq \varepsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have $\lim_{n\to\infty} f_k(\| T_{k_n}(x) - L \|) = 0$.
(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and

$$\lim_{n\to\infty} f_k(\| T_{k_n}(x) - L \|) = 0.$$

Then, for any $\varepsilon > 0$, and by Lemma (II), we have

$$\{k \in \mathbb{N} : f_k(\| T_k(x) - L \|) \geq \varepsilon\} \subseteq K^c \cup \{k \in K : f_k(\| T_k(x) - L \|) \geq \varepsilon\}.$$

Thus, $(T_k) \in \mathscr{C}^I(\mathscr{T}, \mathscr{F})$.
Hence the result.

**Theorem 2.7.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then, the inclusions $\mathscr{C}_0^I(\mathscr{T}, \mathscr{F}) \subset \mathscr{C}^I(\mathscr{T}, \mathscr{F}) \subset \mathscr{B}_\infty^I(\mathscr{T}, \mathscr{F})$ hold.

**Proof**. Let $(T_k) \in \mathscr{C}^I(\mathscr{T}, \mathscr{F})$. Then, there exists some $L$ such that the set

$$\big\{k \in \mathbb{N} : f_k(\| T_k(x) - L \|) \geq \varepsilon\big\} \in I.$$

Since, each $f_k$ is modulus, we have

$$f_k(\| T_k(x) \|) = f_k(\| T_k(x) - L + L \|) \leq f_k(\| T_k(x) - L \|) + f_k(\| L \|).$$

Taking supremum over k on both sides, we get $(T_k) \in \mathscr{B}_\infty^I(\mathscr{T}, \mathscr{F})$.
The inclusion $\mathscr{C}_0^I(\mathscr{T}, \mathscr{F}) \subset \mathscr{C}^I(\mathscr{T}, \mathscr{F})$ is obvious.
Hence, $\mathscr{C}_\circ^I(\mathscr{T}, \mathscr{F}) \subset \mathscr{C}^I(\mathscr{T}, \mathscr{F}) \subset \mathscr{B}_\infty^I(\mathscr{T}, \mathscr{F})$.

**Theorem 2.8.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions with $f_k(x) = x$ for all $x \in [0, \infty]$ and $k \in \mathbb{N}$. Then, the function $\bar{h} : \mathscr{M}_\mathscr{C}^I(\mathscr{T}, \mathscr{F}) \to \mathbb{R}$ defined by

$\bar{h}(\mathscr{T}) = \| I - \lim \mathscr{T} \|$, where $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F}) = \mathscr{B}^I_\infty(\mathscr{T},\mathscr{F}) \cap \mathscr{C}^I(\mathscr{T},\mathscr{F})$, is a Lipschitz function and hence uniformly continuous.

**Proof**. First of all, we show that the function $\bar{h}$ is well defined.
For, let $\mathscr{T}$, $\mathscr{S}$ be two elements of $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ such that

$$\mathscr{T} = \mathscr{S} \Rightarrow I - \lim \mathscr{T} = I - \lim \mathscr{S}$$

$$\Rightarrow \| I - \lim \mathscr{T} \| = \| I - \lim \mathscr{S} \| \Rightarrow \bar{h}(\mathscr{T}) = \bar{h}(\mathscr{S}).$$

Thus, $\bar{h}$ is well defined.
Next, let $\mathscr{T} = (T_k)$, $\mathscr{S} = (S_k) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$, $\mathscr{T} \neq \mathscr{S}$. Then, the sets

$$A_{\mathscr{T}} = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - \bar{h}(\mathscr{T}) \|) \geq \| \mathscr{T} - \mathscr{S} \|_* \right\} \in I.$$

$$A_{\mathscr{S}} = \left\{k \in \mathbb{N} : f_k(\| S_k(x) - \bar{h}(\mathscr{S}) \|) \geq \| \mathscr{T} - \mathscr{S} \|_* \right\} \in I.$$

where $\| \mathscr{T} - \mathscr{S} \|_* = \sup_k f_k \| (T_k - S_k)(x) \|$.
Thus, the sets

$$B_{\mathscr{T}} = \left\{k \in \mathbb{N} : \| T_k(x) - \bar{h}(\mathscr{T}) \| < \| \mathscr{T} - \mathscr{S} \|_* \right\} \in \pounds(I).$$

$$B_{\mathscr{S}} = \left\{k \in \mathbb{N} : \| S_k(x) - \bar{h}(\mathscr{S}) \| < \| \mathscr{T} - \mathscr{S} \|_* \right\} \in \pounds(I).$$

Hence, $B = B_{\mathscr{T}} \cap B_{\mathscr{S}} \in \pounds(I)$, so that $B \neq \emptyset$.
Now taking $k \in B$, we have

$$| \bar{h}(\mathscr{T}) - \bar{h}(\mathscr{S}) | \leq \| \bar{h}(\mathscr{T}) - T_k(x) \| + \| T_k(x) - S_k(x) \|$$

$$+ \| S_k(x) - \bar{h}(\mathscr{S}) \| \leq 3 \| (\mathscr{T}) - (\mathscr{S}) \|_* .$$

Thus, $\bar{h}$ is Lipschitz function and hence uniformly continuous.

**Theorem 2.9.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions with $f_k(x) = x$ for all $x \in [0,\infty]$ and $k \in \mathbb{N}$. If $\mathscr{T} = (T_k), \mathscr{S} = (S_k) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ with $T_kS_k(x) = T_k(x).S_k(x)$, then $(\mathscr{T}.\mathscr{S}) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ and $\bar{h}(\mathscr{T}.\mathscr{S}) = \bar{h}(\mathscr{T})\bar{h}(\mathscr{S})$, where $\bar{h} : \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F}) \to \mathbb{R}$ defined by $\bar{h}(\mathscr{T}) = \| I - \lim \mathscr{T} \|$.

**Proof**. For $\varepsilon > 0$, the sets

$$B_{\mathscr{T}} = \left\{k \in \mathbb{N} : \| T_k(x) - \bar{h}(\mathscr{T}) \| < \| \mathscr{T} - \mathscr{S} \|_* \right\} \in \pounds(I) \tag{2.16}$$

$$B_{\mathscr{S}} = \left\{k \in \mathbb{N} : \| S_k(x) - \bar{h}(\mathscr{S}) \| < \| \mathscr{T} - \mathscr{S} \|_* \right\} \in \pounds(I). \tag{2.17}$$

Now,

$$\| T_kS_k(x) - \bar{h}(\mathscr{T})\bar{h}(\mathscr{S}) \| =$$

$$\| T_k(x)S_k(x) - T_k(x)\bar{h}(\mathscr{S}) + T_k(x)\bar{h}(\mathscr{S}) - \bar{h}(\mathscr{T})\bar{h}(\mathscr{S}) \|$$

$$\leq \| T_k(x) \| \| S_k(x) - \bar{h}(\mathscr{S}) \| + |\bar{h}(\mathscr{S})| \| T_k(x) - \bar{h}(\mathscr{T}) \| . \tag{2.18}$$

As $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F}) \subseteq \mathscr{B}_\infty(\mathscr{T},\mathscr{F})$, there exists an $M \in \mathbb{R}$ such that $\| T_k(x) \| < M$.

Therefore, from (2.16), (2.17) and (2.18), we have

$$\| T_kS_k - \bar{h}(\mathscr{T})\bar{h}(\mathscr{S}) \| < M\varepsilon + |\bar{h}(\mathscr{S})|\varepsilon = \varepsilon_1(say)$$

for all $k \in B_{\mathscr{T}} \cap B_{\mathscr{S}} \in \pounds(I)$. Hence $(\mathscr{T}.\mathscr{S}) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ and $\bar{h}(\mathscr{T}\mathscr{S}) = \bar{h}(\mathscr{T})\bar{h}(\mathscr{S})$.

**Theorem 2.10.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then, the spaces $\mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\mathscr{T},\mathscr{F})$ are solid and monotone.

**Proof**. We shall prove the result for $\mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$. For $\mathscr{M}^I_{\mathscr{C}_\circ}(\mathscr{T},\mathscr{F})$, the result can be established similarly.
For, let $(T_k) \in \mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$. Then

$$\left\{k \in \mathbb{N} : f_k(\| T_k(x) \|) \geq \varepsilon \right\} \in I. \tag{2.19}$$

Let $(\alpha_k)$ be a sequence of scalars with $| \alpha_k | \leq 1$, for all, $k \in \mathbb{N}$. then, the result follows from (2.19) and the following inequality

$$f_k(\| \alpha_k T_k x \|) \leq | \alpha_k | f_k(\| T_k x \|) \leq f_k(\| T_k x \|), \text{for all } k \in \mathbb{N}.$$

That the space is monotone follows from lemma (I).

**Theorem 2.11.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. If $\mathscr{T} = (T_k), \mathscr{S} = (S_k) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ with $T_kS_k(x) = T_k(x).S_k(x)$. Then, the spaces $\mathscr{C}^I(\mathscr{T},\mathscr{F})$ and $\mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$ are sequence algebra.

**Proof**. For, let $(T_k), (S_k) \in \mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$, then

$$I - \lim_k f_k(\| T_k(x) \|) = 0 \tag{2.20}$$

and

$$I - \lim_k f_k(\| S_k(x) \|) = 0. \tag{2.21}$$

Then, from (2.20) and (2.21), we have

$$I - \lim_k f_k(\| T_k.S_k(x) \|) = I - \lim_k f_k(\| T_k(x).S_k(x) \|) = 0$$

implies that $(T_k.S_k) \in \mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$.
Hence, $\mathscr{C}^I_\circ(\mathscr{T},\mathscr{F})$ is a sequence algebra.
For $\mathscr{C}^I(\mathscr{T},\mathscr{F})$, the result is similar.

**Theorem 2.12.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ is closed subspace of $\mathscr{B}_\infty(\mathscr{T},\mathscr{F})$.

**Proof.** Let $(T_k^{(n)})$ be a Cauchy sequence in $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ such that $T_k^{(n)} \to T$.
We show that $T \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$.
Since $(T_k^{(n)}) \in \mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$, then there exists $A_n$ such that

$$\left\{k \in \mathbb{N} : f_k(\| T_k^{(n)}(x) - A_n \|) \geq \varepsilon \right\} \in I.$$

We need to show that
(1) $(A_n)$ converges to $A$.
(2) If $U = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - A \|) < \varepsilon\right\}$, then $U^c \in I$.
(1) Since $(T_k^{(n)})$ is Cauchy sequence in $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F}) \Rightarrow$ for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k f_k(\| T_k^{(n)}(x) - T_k^{(q)}(x) \|) < \frac{\varepsilon}{3},$$

for all $n,\ q \geq k_0$.
For a given $\varepsilon > 0$, we have

$$B_{nq} = \{k \in \mathbb{N} : f_k(\| T_k^{(n)}(x) - T_k^{(q)}(x) \|) < \frac{\varepsilon}{3}\};$$

$$B_q = \{k \in \mathbb{N} : f_k(\| T_k^{(q)}(x) - A_q \|) < \frac{\varepsilon}{3}\};$$

$$B_n = \{k \in \mathbb{N} : f_k(\| T_k^{(n)}(x) - A_n \|) < \frac{\varepsilon}{3}\}.$$

Then, $B^c_{nq}, B^c_q, B^c_n \in I$.
Let $B^c = B^c_{nq} \cup B^c_q \cup B^c_n$, where

$$B = \left\{k \in \mathbb{N} : f_k(\| A_q - A_n \|) < \varepsilon\right\}.$$

Then, $B^c \in I$. We choose $k_0 \in B^c$, then for each $n, q \geq k_0$, we have

$$\left\{k \in \mathbb{N} : f_k(\| A_q - A_n \|) < \varepsilon\right\} \supseteq \left[\left\{k \in \mathbb{N} : f_k(\| A_q - T_k^{(q)}(x) \|) < \frac{\varepsilon}{3}\right\} \cap \left\{k \in \mathbb{N} : f_k(\| T_k^{(q)}(x) - T_k^{(n)}(x) \|) < \frac{\varepsilon}{3}\right\} \cap \left\{k \in \mathbb{N} : f_k(\| T_k^{(n)}(x) - A_n \|) < \frac{\varepsilon}{3}\right\}\right].$$

Then $(A_n)$ is a Cauchy sequence in $Y$ and $Y$ is complete, so there exists an element $A$ in $Y$ such that $A_n \to A$ as $n \to \infty$.
(2) Let $0 < \delta < 1$ be given. Then we show that if

$$U = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - A \|) < \delta\right\},$$

then $U^c \in I$.
Since $T_k^{(n)} \to T$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \left\{k \in \mathbb{N} : f_k(\| T_k^{(q_0)}(x) - T_k(x) \| < \frac{\delta}{3}\right\} \tag{2.22}$$

implies $P^c \in I$.
The number $q_0$ can be chosen that together with (2.22), we have

$$Q = \left\{k \in \mathbb{N} : f_k(\| A_{q_0} - A \|) < \frac{\delta}{3}\right\}$$

such that $Q^c \in I$. Since

$$\left\{k \in \mathbb{N} : f_k(\| T_k^{(q_0)}(x) - A_{q_0} \|) \geq \delta\right\} \in I.$$

Then we have a subset $S$ of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{k \in \mathbb{N} : f_k(\| T_k^{(q_0)}(x) - A_{q_0} \|) < \frac{\delta}{3}\right\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \left\{k \in \mathbb{N} : f_k(\| T_k(x) - A \|) < \delta\right\}$.
Therefore, for each $k \in U^c$, we have

$$\left\{k \in \mathbb{N} : f_k(\| T_k(x) - A \|) < \delta\right\} \supseteq \left[\left\{k \in \mathbb{N} : f_k(\| T_k(x) - T_k^{(q_0)}(x) \|) < \frac{\delta}{3}\right\} \cap \left\{k \in \mathbb{N} : f_k(\| T_k^{(q_0)}(x) - A_{q_0} \|) < \frac{\delta}{3}\right\} \cap \left\{k \in \mathbb{N} : f_k(\| A_{q_0} - A \|) < \frac{\delta}{3}\right\}\right].$$

Then the result follows.
Since the inclusions $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F}) \subset \mathscr{B}_\infty(\mathscr{T},\mathscr{F})$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\mathscr{T},\mathscr{F}) \subset \mathscr{B}_\infty(\mathscr{T},\mathscr{F})$ are strict so in view of Theorem (2.11) we have the following result.

**Theorem 2.13.** Let $\mathscr{F} = (f_k)$ be a sequence of modulus functions. Then, the spaces $\mathscr{M}^I_{\mathscr{C}}(\mathscr{T},\mathscr{F})$ and $\mathscr{M}^I_{\mathscr{C}_\circ}(\mathscr{T},\mathscr{F})$ are nowhere dense subsets of $\mathscr{B}_\infty(\mathscr{T},\mathscr{F})$.

## Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

## References

[1] Buck,R.C.:*Generalized asymptotic density,* Amer.,J.Math.75,335-346(1953).
[2] Fast,H.: *Sur la convergence statistique*,Colloq.Math.2(1951),241-244.
[3] Fridy,J.A. :*On statistical convergence*,Analysis.5(1985).301-313.
[4] Garling,D.J.H.:*On Symmetric Sequence Spaces*,Proc.London. Math.Soc.16(1966), 85-106.
[5] Gramsch,B.:*Die Klasse metrisher linearer Raume L($\phi$)*,Math.Ann.171(1967), 61-78.

[6] Khan,V.A. and Ebadullah,K.,Ayhan Esi and Mohd Shafiq: *On some Zweier I-convergent sequence spaces defined by a modulus function*. Africa Mathematika,Journal of the African Mathematical Soc. and Springer Verlag Berlin Heidelberg,2013.
[7] Khan,V.A. and Ebadullah,K. and Suthep Suantai: *On a new I-convergent sequence space.*,Analysis,32,199-208(2012).
[8] Khan,V.A.,Suthep Suantai and Ebadullah,K.:*On some I-convergent sequence spaces defined by a sequence of moduli*,J.of Nonlinear Analysis and Optimization, Vol.3,No.2(2012),145-152.
[9] Kolk,E.: *On strong boundedness and summability with respect to a sequence of moduli*, Acta Comment.Univ.Tartu.,960(1993),41-50.
[10] Kolk,E.:*Inclusion theorems for some sequence spaces defined by a sequence of modulii*, Acta Comment.Univ. Tartu., 970(1994), 65-72.
[11] Kostyrko,P.,Mačaj,M.,and Šalát,T.: *Statistical convergence and I-convergence*.Real Analysis Exchange.
[12] Kostyrko,P.,Šalát,T.and Wilczyński,W.: *I-convergence*,Raal Analysis Analysis Exchange.26(2),669-686(2000).
[13] Köthe,G.: *Topological Vector spaces*.1.(Springer,Berlin,1970.)
[14] Kreyszig,E.: *Introductory functional analysis with applications*,John Wiley and Sons,Inc.New York-Chichester-Brisbane-Toronto,1978.
[15] Limaye,B.V.: *Functional analysis*,2nd Edition,New age international Ltd.(1996).
[16] Mursaleen.M and Alotaibi.A, *On I-convergence in random 2-normed spaces,Math.Slovaca*, **61(6)**(2011),933-940.
[17] Mursaleen.M , Mohiuddine,S.A, and Edely,Osama H. H., On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces, *Comput. Math. Appl.,* **59** (2010):603-611.
[18] Mursaleen.M , and Mohiuddine,S.A, On the ideal convergence of double sequences in Probabilistic normed spaces,Math.Reports, 12(64)(4)(2010): 359- 371.
[19] Mursaleen.M , and Mohiuddine,S.A,On ideal convergence in probabilistic normed spaces,*Math.Slovaca*, **62**(2012), 49-62.
[20] Mursaleen.M and Sharma,S.K, Spaces of Ideal Convergent sequences,Hindawi Publishing Corporation, Thee Scientific World Journal Volume 2014, Article ID 134534, 6 pages http://dx.doi.org/10.1155/2014/134534.
[21] Nakano,H.: *Concave modulars.* J. Math Soc. Japan,5(1953)29-49.
[22] Neumann,J.Von.: *Zur Algebra der funktional-operationen und Theorie der normalen Operationen.* Math.Annalen,102.370-427(1929-30b).
[23] Ruckle,W.H.: *On perfect Symmetric BK-spaces.*,Math.Ann.175(1968)121-126.
[24] Ruckle,W.H.: *Symmetric coordinate spaces and symmetric bases*,Canad.J.Math.19(1967) 828-838.
[25] Ruckle,W.H.: *FK-spaces in which the sequence of coordinate vectors is bounded*,Canad.J. Math.25(5)(1973)973-975.
[26] Šalát,T.: *On statistical convergent sequences of real numbers*,Math, Slovaca 30(1980).
[27] Šalát,T.,Tripathy,B.C.,Ziman,M.: *On some properties of I-convergence* Tatra Mt. Math. Publ.28(2004) 279-286.
[28] Šalát,T.,Tripathy,B.C.,Ziman,M: *On I-convergence field.*,Ital.J.Pure Appl. Math. 17: 45-54(2005).
[29] Schoenberg,I.J.: *The integrability of certain functions and related summability methods*,Amer.Math.Monthly.66,361-375(1959).
[30] Tripathy,B.C.: *On statistical convergence*,Proc.Estonian Acad.Sci..Phy.Math.Analysis. (1998),299-303.
[31] Tripathy,B.C. and Hazarika,B.: *Some I-convergent sequence spaces defined by Orlicz function.*,Acta Mathematicae Applicatae Sinica.27(1)149-154.(2011).
[32] Tripathy,B.C.and Hazarika,B.: *Paranorm I-convergent sequence spaces*, Math. Slovaca 59 (4) (2009) 485-494.

**Vakeel A.Khan** received the M.Phil. and Ph.D. degrees in Mathematics from Aligarh Muslim University, Aligarh,India. Currently he is a senior Assistant Professor at Aligarh Muslim University, Aligarh, India . A vigorous researcher in the area of Sequence Spaces , he has published a number of research papers in reputed national and international journals, including Journal of Indian Mathematical Society, Numerical Functional analysis and Optimization(Taylor's and Francis),Information Sciences(Elsevier), Applied Mathematics A Journal of Chinese Universities and Springer- Verlag(CHINA), Rendiconti del Circolo Matematico di Palermo Springer- Verlag(ITALY), Journal of Mathematics and Applications(POLAND),Acta Math.Univ.Comenianae (Bratislava, Slovak Republic ), Kyungpook Mathematical Journal (KOREA), Vietnam Journal of Mathematics(VIETNAM), Studia Mathematica (POLAND), Thai Journal of mathematics (THAILAND), Southeast Asian Bull.Math(CHINA), Mathematicki Vesnik (SERBIA), Applied Sciences (ROMANIA).

**Mohd Shafiq** did M.Sc. in Mathematics from University of Jammu, Jammu and Kashmir, India. Currently he is a Ph.D. scholar at Aligarh Muslim University, Aligarh,India . His research interests are Functional Analysis, sequence spaces, I-convergence, invariant means, zweier sequences, intreval numbers theory, bicomplex numbers .