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Geometrically Nonconvex Functions and Integral Inequalities

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Abstract: In this paper, we consider and investigate geometrically nonconvex functions. We derive several Hermite-Hadamard type inequalities for geometrically (GG) nonconvex (relative convex) function and geometrically arithmetically (GA) nonconvex (relative convex) functions. We also obtain some fractional Hermite-Hadamard type inequalities. It is shown that one can obtain the previously known results as special cases from our results. The ideas and techniques of this paper may inspire further research in this field.

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1 Introduction

Convexity plays an important role in different fields of pure and applied sciences. Knowing its importance, much attention has been given to this field by many researchers. Consequently the concept of convexity has been extended and generalized in different dimensions using novel and innovative techniques see [2–6, 11–20, 23, 26–29].
Youness [28] introduced a new class of convex functions with respect to an arbitrary function. This class of convex functions is called the relative convex functions or nonconvex functions. These nonconvex functions play an important role in optimization theory. Noor [15] has proved that the optimality condition for differentiable relative convex functions on relative convex sets can be characterized by a class of variational inequality which is called as general variational inequality. For the applications of relative convexity, see [13–15] and the references therein. Recently Noor et al. [18] introduced and investigated the concept of geometrically relative convex functions, which also contains the class of relative convex functions.
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For some recent extensions and generalizations of Hermite-Hadamard type inequalities, see [1, 3–6, 9, 11, 12, 17–20, 22–27, 29, 30].
Inspired and motivated and by the ongoing research we consider the class of geometrically nonconvex (relative convex) functions. Several new Hermite-Hadamard type inequalities for geometrically nonconvex functions and its variant forms are obtained.
Several special cases are discussed. The interested readers are encouraged to find the novel applications of the geometrically nonconvex functions and their variant forms in various areas of pure and applied sciences.

2 Preliminaries

In this section, we recall some previously known concepts. In the sequel of the paper, $\mathbb{R}^n$ is the finite dimensional euclidian space, whose inner product is denoted by $\langle .,. \rangle$, $\mathscr{G} = [g(a), g(b)] \subseteq \mathbb{R}_+ = (0, \infty)$ where

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$g : \mathbb{R}^n \to \mathbb{R}^n$ be arbitrary function, unless otherwise specified.

Definition 1([12]). *Let $I \subseteq \mathbb{R}_+$. A geometrically convex set is defined as*

$$x^t y^{1-t} \in I, \quad \forall x, y \in I, t \in [0,1].$$

Definition 2([12]). *A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be geometrically convex, if*

$$f(x^t y^{1-t}) \leq (f(x))^t (f(x))^{1-t}, \quad \forall x, y \in I, t \in [0,1].$$

We now define the concept of geometrically relative convex set.

Definition 3([18]). *A set $\mathscr{G}$ is said to be geometrically nonconvex (relative convex) set with respect to an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\forall x, y \in \mathbb{R}^n$ if $g(x), g(y) \in \mathscr{G}$, then*

$$(g(x))^t (g(y))^{1-t} \in \mathscr{G}, t \in [0,1].$$

Using $AM - GM$ inequality, we have

$$(g(x))^t (g(y))^{1-t} \leq tg(x) + (1-t)g(y),$$
$$\forall x, y \in \mathbb{R}^n : g(x), g(y) \in \mathscr{G}, t \in [0,1]$$

Definition 4([17, 28]). *A set $M_g \subseteq \mathbb{R}^n$ is said to be a nonconvex (relative convex) set with respect to arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$, if*

$$tg(x) + (1-t)g(y) \in M_g,$$
$$\forall x, y \in \mathbb{R}^n : g(x), g(y) \in M_g, t \in [0,1]. \tag{1}$$

It is proved in [8], that if M_g is a nonconvex set then it is possible that it may not be a classical convex set. For example, for $M_g = [-1, -\frac{1}{2}] \cup [0,1]$ and $g(x) = x^2, \forall x \in \mathbb{R}$. Clearly, this is a nonconvex set but not classical convex set. Another possibility may occur that nonconvex set may be a classical convex set, for example if $M_g = [-1,1]$ and $g(x) = \sqrt[4]{|x|}, \forall x \in \mathbb{R}$.

Definition 5([18]). *A function $f : \mathscr{G} \to \mathbb{R}_+$ is said to be geometrically nonconvex function (GG nonconvex function) with respect to an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\forall x, y \in \mathbb{R}^n : g(x), g(y) \in \mathscr{G}, t \in [0,1]$, if*

$$f((g(x))^t (g(y))^{1-t}) \leq (f(g(x)))^t (f(g(y)))^{1-t}. \tag{2}$$

From (2), it follows that

$$\log f((g(x))^t (g(y))^{1-t}) \leq t \log f(g(x)) + (1-t) \log f(g(y)),$$
$$\forall x, y \in \mathbb{R}^n : g(x), g(y) \in \mathscr{G}, t \in [0,1].$$

Using $AM - GM$ inequality, we have

$$\begin{aligned} f((g(x))^t (g(y))^{1-t}) &\leq (f(g(x)))^t (f(g(y)))^{1-t} \\ &\leq t f(g(x)) + (1-t) f(g(y)). \end{aligned}$$

This shows that every geometrically nonconvex function (that is GG nonconvex function) is also GA nonconvex function, but the converse is not true see [12].

For $t = \frac{1}{2}$ in (2), we have Jensen geometrically nonconvex functions, that is

$$f\left(\sqrt{g(x)g(y)}\right) \leq \sqrt{f(g(x)) f(g(y))}.$$

Definition 6([18]). *Let $I = [g(a), g(b)] \subseteq \mathbb{R}_+$. Then f is geometrically nonconvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ \log g(a) & \log g(x) & \log g(b) \\ \log f(g(a)) & \log f(g(x)) & \log f(g(b)) \end{vmatrix} \geq 0,$$

where $g(a) \leq g(x) \leq g(b)$.

where $g(x) = g(a)^t g(b)^{1-t} \in I$ and $t \in [0,1]$.
For $g(x) = x$ Definition 6 reduces to the definition for geometrically convex functions, see [12].

Definition 7([18]). *A function $f : \mathscr{G} \to \mathbb{R}$ is said to be GA nonconvex function with respect to an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$, if*

$$f((g(x))^t (g(y))^{1-t}) \leq t f(g(x)) + (1-t) f(g(y)),$$
$$\forall x, y \in \mathbb{R}^n : g(x), g(y) \in \mathscr{G}, t \in [0,1]. \tag{3}$$

From Definition 3 and Definition 5, it follows that $GG \Longrightarrow GA$, but the converse is not true.
We also note that for $g(x) = e^x$ in Definition 5, we have

$$f(e^{tx+(1-t)y}) \leq t f(e^x) + (1-t) f(e^y),$$
$$\forall x, y \in \mathscr{G}, t \in [0,1]. \tag{4}$$

Again using the $AM - GM$ inequality from Definition 3, we have the following known concept of relative convex functions.

Definition 8([17, 28]). *A function f is said to be a nonconvex (relative convex) function (that is AA nonconvex function) on a nonconvex (relative convex) set M_g, there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f((1-t)g(x) + tg(y)) \leq (1-t) f(g(x)) + t f(g(y)),$$
$$\forall x, y \in \mathbb{R}^n : g(x), g(y) \in M_g, t \in [0,1]. \tag{5}$$

It is known [28] that every convex function f on a convex set is a nonconvex function, but the converse is not true. However, there are functions which are nonconvex function but may not be a convex function in the classical sense. For example, let $M_g \subset \mathbb{R}$ be given as:

$$M_g = \{(x,y) \in \mathbb{R}^2 : (x,y) = \lambda_1(0,0) + \lambda_2(0,3) + \lambda_3(2,1)\},$$

where $\lambda_i > 0$, $\sum_{i=1}^3 \lambda_i = 1$, and function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $g : (x,y) = (0,y)$, then the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x^3, & \text{if } y < 1, \\ xy^3, & \text{if } y \geq 1. \end{cases}$$

is a nonconvex function but not a convex function.

Definition 9([10]). *The left sided and right sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of function $f(x)$ are defined as:*

$$({}_HJ^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ a < x \leq b,$$

and

$$({}_HJ^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ a \leq x < b,$$

where $\Gamma(.)$ is the gamma function.

Lemma 1([21]). *For $0 < \sigma \leq 1$ and $0 \leq a < b$, we have* $|a^{\sigma} - b^{\sigma}| \leq (b-a)^{\sigma}$.

Lemma 2([21]). *For all $\lambda, \upsilon, \omega > 0$, then for any $t > 0$, we have*

$$\int_0^1 (t-s)^{\upsilon-1}s^{\lambda-1}e^{\omega s}ds$$
$$\leq \max\{1, 2^{1-\upsilon}\}\Gamma(\lambda)\left(1+\frac{\lambda}{\upsilon}\right)\omega^{\lambda}t^{\upsilon-1}.$$

3 Main Results

In this section, we derive our main results.

Theorem 1. *Let $f : [g(a), g(b)] \to \mathbb{R}_+$ be geometrically nonconvex function. Then*

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))}{g(x)}dg(x)$$
$$\leq L(g(b), g(a)) \leq A(g(a), g(b)).$$

Proof. Let f be geometrically nonconvex function. Then

$$f((g(a))^t(g(b))^{1-t}) \leq (f(g(a)))^t(f(g(b)))^{1-t}$$
$$= f(g(b))\left[\frac{f(g(a))}{f(g(b))}\right]^t.$$

Integrating with respect t on $[0,1]$, we have

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))}{g(x)}dg(x)$$
$$\leq \frac{f(g(b)) - f(g(a))}{\ln f(g(b)) - \ln f(g(a))}$$
$$= L(f(g(b)), f(g(a)))$$
$$\leq \frac{f(g(a)) + f(g(b))}{2}$$
$$= A(f(g(a)), f(g(b))).$$

This completes the proof. $\square$

Theorem 2. *Let $f, w : [g(a), g(b)] \to \mathbb{R}_+$ be geometrically nonconvex functions. Then*

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$\leq L(g(b)g(b), g(a)g(a)) \leq A(g(a)g(a), g(b)g(b)).$$

Proof. Let f and w be geometrically nonconvex functions. Then

$$f((g(a))^t(g(b))^{1-t})w((g(a))^t(g(b))^{1-t})$$
$$\leq (f(g(a)))^t(f(g(b)))^{1-t}(w(g(a)))^t(w(g(b)))^{1-t}$$
$$= f(g(b))w(g(b))\left[\frac{f(g(a))w(g(a))}{f(g(b))w(g(b))}\right]^t.$$

Integrating with respect t on $[0,1]$, we have

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$\leq \frac{f(g(b))w(g(b)) - f(g(a))w(g(a))}{\ln f(g(b))w(g(b)) - \ln f(g(a))w(g(a))}$$
$$= L(f(g(b))w(g(b)), f(g(a))w(g(a)))$$
$$\leq \frac{f(g(a))w(g(a)) + f(g(b))w(g(b))}{2}$$
$$= A(f(g(a))w(g(a)), f(g(b))w(g(b))).$$

This completes the proof. $\square$

Theorem 3. *Let $f, w : [g(a), g(b)] \to \mathbb{R}_+$ be geometrically nonconvex functions. If $\alpha + \beta = 1$, then*

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$\leq \alpha\frac{f(g(a)) + f(g(b))}{2}\left[L_{(\frac{1}{\alpha}-1)}(f(g(b)), f(g(a)))\right]^{\frac{1-\alpha}{\alpha}}$$
$$+\beta\frac{w(g(a)) + w(g(b))}{2}\left[L_{(\frac{1}{\beta}-1)}(w(g(b)), w(g(a)))\right]^{\frac{1-\beta}{\beta}}.$$

Proof. Let f and w be geometrically nonconvex functions. Using inequality,

$$xy \leq \alpha x^{\frac{1}{\alpha}} + \beta y^{\frac{1}{\beta}}, \quad \alpha, \beta > 0, \alpha + \beta = 1,$$

we have

$$\frac{1}{\ln g(b) - \ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$= \int_0^1 f((g(a))^t(g(b))^{1-t})w((g(a))^t(g(b))^{1-t})dt$$

$$\leq \int_0^1 \left\{ \alpha(f((g(a))^t(g(b))^{1-t})^{\frac{1}{\alpha}} + \beta(w((g(a))^t(g(b))^{1-t})^{\frac{1}{\beta}} \right\} dt$$

$$\leq \int_0^1 \left\{ \alpha[(f(g(a)))^t(f(g(b)))^{1-t}]^{\frac{1}{\alpha}} + \beta[(w(g(a)))^t(w(g(b)))^{1-t}]^{\frac{1}{\beta}} \right\} dt$$

$$= \alpha(f(g(b)))^{\frac{1}{\alpha}} \int_0^1 \left(\frac{f(g(a))}{f(g(b))}\right)^{\frac{t}{\alpha}} dt + \beta(w(g(b)))^{\frac{1}{\beta}} \int_0^1 \left(\frac{w(g(a))}{w(g(b))}\right)^{\frac{t}{\beta}} dt$$

$$= \alpha^2(f(g(b)))^{\frac{1}{\alpha}} \int_0^{\frac{1}{\alpha}} \left(\frac{f(g(a))}{f(g(b))}\right)^{u} du + \beta^2(w(g(b)))^{\frac{1}{\beta}} \int_0^{\frac{1}{\beta}} \left(\frac{w(g(a))}{w(g(b))}\right)^{v} dv$$

$$= \alpha^2 \frac{(f(g(b)))^{\frac{1}{\alpha}} - (f(g(a)))^{\frac{1}{\alpha}}}{\log f(g(b)) - \log f(g(a))} + \beta^2 \frac{(w(g(b)))^{\frac{1}{\beta}} - (w(g(a)))^{\frac{1}{\beta}}}{\log w(g(b)) - \log w(g(a))}$$

$$= \alpha^2 \frac{(f(g(b)))^{\frac{1}{\alpha}} - (f(g(a)))^{\frac{1}{\alpha}}}{f(g(b)) - f(g(a))} L(f(g(b)), f(g(a))) + \beta^2 \frac{(w(g(b)))^{\frac{1}{\alpha}} - (w(g(a)))^{\frac{1}{\beta}}}{w(g(b)) - w(g(a))} L(w(g(b)), w(g(a)))$$

$$= \alpha \left[L_{(\frac{1}{\alpha}-1)}(f(g(b)), f(g(a))) \right]^{\frac{1-\alpha}{\alpha}} L(f(g(b)), f(g(a))) + \beta \left[L_{(\frac{1}{\beta}-1)}(w(g(b)), w(g(a))) \right]^{\frac{1-\beta}{\beta}} L(w(g(b)), w(g(a)))$$

$$\leq \alpha \frac{f(g(a)) + f(g(b))}{2} \left[L_{(\frac{1}{\alpha}-1)}(f(g(b)), f(g(a))) \right]^{\frac{1-\alpha}{\alpha}} + \beta \frac{w(g(a)) + w(g(b))}{2} \left[L_{(\frac{1}{\beta}-1)}(w(g(b)), w(g(a))) \right]^{\frac{1-\beta}{\beta}}.$$

This completes the proof. □

Theorem 4. *Let $f, w : [g(a), g(b)] \to \mathbb{R}_+$ be increasing and geometrically nonconvex functions. Then*

$$\frac{1}{\ln g(b) - \ln g(a)} \int_{g(a)}^{g(b)} f(g(x))dg(x) L[w(g(a)), w(g(b))] + \frac{1}{\ln g(b) - \ln g(a)} \int_{g(a)}^{g(b)} w\left(\frac{g(a)g(b)}{g(x)}\right) dg(x) \times L[f(g(a)), f(g(b))]$$
$$\leq \frac{1}{\ln g(b) - \ln g(a)} \int_{g(a)}^{g(b)} f(g(x)) w\left(\frac{g(a)g(b)}{g(x)}\right) dg(x) + L[f(g(a))w(g(a)), f(g(b))w(g(b))].$$

Proof. Let f and w be geometrically nonconvex functions. Then we have

$$f((g(a))^{1-t}(g(b))^t) \leq [f(g(a))]^{1-t}[f(g(b))]^t$$
$$w((g(a))^t(g(b))^{1-t}) \leq [w(g(a))]^t[w(g(b))]^{1-t}.$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2 < x_3 < x_4$, we have

$$f((g(a))^{1-t}(g(b))^t)[w(g(a))]^t[w(g(b))]^{1-t} + w((g(a))^t(g(b))^{1-t})[f(g(a))]^{1-t}[f(g(b))]^t$$
$$\leq f((g(a))^{1-t}(g(b))^t) w((g(a))^t(g(b))^{1-t}) + [f(g(a))]^{1-t}[f(g(b))]^t[w(g(a))]^t[w(g(b))]^{1-t}.$$

Integrating above inequalities with respect to t on $[0,1]$, we have

$$\int_0^1 f((g(a))^{1-t}(g(b))^t)[w(g(a))]^t[w(g(b))]^{1-t} dt + \int_0^1 w((g(a))^t(g(b))^{1-t})[f(g(a))]^{1-t}[f(g(b))]^t dt$$
$$\leq \int_0^1 f((g(a))^{1-t}(g(b))^t) w((g(a))^t(g(b))^{1-t}) dt + \int_0^1 [f(g(a))]^{1-t}[f(g(b))]^t[w(g(a))]^t[w(g(b))]^{1-t} dt.$$

Now, since f and w are increasing, we have

$$\int_0^1 f((g(a))^{1-t}(g(b))^t) dt \int_0^1 [w(g(a))]^t[w(g(b))]^{1-t} dt + \int_0^1 w((g(a))^t(g(b))^{1-t}) dt \int_0^1 [f(g(a))]^{1-t}[f(g(b))]^t dt$$
$$\leq \int_0^1 f((g(a))^{1-t}(g(b))^t) w((g(a))^t(g(b))^{1-t}) dt$$

$$+\int_0^1 [f(g(a))]^{1-t}[f(g(b))]^t[w(g(a))]^t[w(g(b))]^{1-t}dt.$$

This implies that

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)L[w(g(a)),w(g(b))]$$
$$+\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} w\left(\frac{g(a)g(b)}{g(x)}\right)dg(x)$$
$$\times L[f(g(a)),f(g(b))]$$
$$\leq \frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} f(g(x))w\left(\frac{g(a)g(b)}{g(x)}\right)dg(x)$$
$$+L[f(g(a))w(g(a)),f(g(b))w(g(b))].$$

This completes the proof. □

Theorem 5. *Let f and w be two GA-nonconvex functions. If f and w are similarly ordered then the product fw is again a GA-nonconvex function.*

Proof. The proof is obvious. □

Theorem 6. *Let $f,w:[g(a),g(b)]\to\mathbb{R}$ be similarly ordered GA-nonconvex functions. Then we have*

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$\leq \frac{f(g(a))w(g(a))+f(g(b))w(g(b))}{2}.$$

Proof. The proof directly follows from integrating inequality (6) with respect to t on $[0,1]$. □

Theorem 7. *Let $f,w:[g(a),g(b)]\to\mathbb{R}$ be GA-nonconvex functions. Then*

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$\leq \frac{1}{8}\left(A^2+B^2\right),$$

where

$$A=f(g(a))+w(g(a)),$$

and

$$B=f(g(b))+w(g(b)),$$

respectively.

Proof. Let f and w be GA-nonconvex functions. Using inequality

$$xy\leq\frac{1}{4}(x+y)^2 \quad \forall x,y\in\mathbb{R},$$

we have

$$\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)} \frac{f(g(x))w(g(x))}{g(x)}dg(x)$$
$$=\int_0^1 f((g(a))^t(g(b))^{1-t})w((g(a))^t(g(b))^{1-t})dt$$
$$\leq\frac{1}{4}\int_0^1\Big[(f((g(a))^t(g(b))^{1-t}))$$
$$+(w((g(a))^t(g(b))^{1-t}))\Big]^2dt$$
$$\leq\frac{1}{4}\int_0^1\Big[tf(g(a))+(1-t)f(g(b))$$
$$+tw(g(a))+(1-t)w(g(b))\Big]^2dt$$
$$=\frac{1}{4}\int_0^1\Big[t\{f(g(a))+w(g(a))\}$$
$$+(1-t)\{f(g(b))+w(g(b))\}\Big]$$
$$=\frac{1}{4}\int_0^1\Big[tA+(1-t)B\Big]^2dt$$
$$=\frac{1}{4}\int_0^1\Big[t^2A^2+(1-t)^2B^2+2t(1-t)AB\Big]dt$$
$$=\frac{1}{12}\Big[A^2+B^2+AB\Big]\leq\frac{1}{8}\left(A^2+B^2\right).$$

This completes the proof. □

Now we prove some Hermite-Hadamard type inequalities via fractional integrals. First of all, we present some results which play a key role in proving our next results. Using essentially the technique of [24], one can prove the following results.

Lemma 3. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$. Suppose $f'\in L[g(a),g(b)]$, then*

$$\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))^{\alpha}}\left[{}_HJ^{\alpha}_{g(a)^+}f(g(b))+{}_HJ^{\alpha}_{g(b)^-}f(g(a))\right]$$
$$-f\left(\frac{g(a)+g(b)}{2}\right)$$
$$=\frac{g(b)-g(a)}{2}\int_0^1\psi(t)f'(tg(a)+(1-t)g(b))dt$$

$$-\frac{\ln g(b)-\ln g(a)}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]e^{t\ln g(a)+(1-t)\ln g(b)}$$
$$\times f'(e^{t\ln g(a)+(1-t)\ln g(b)})dt,$$

where

$$\psi(t)=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1.\end{cases}$$

Lemma 4. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$. Suppose $f'\in L[g(a),g(b)]$, then then following identity holds:*

$$\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))^{\alpha}}\left[{}_HJ^{\alpha}_{g(a)^+}f(g(b))+{}_HJ^{\alpha}_{g(b)^-}f(g(a))\right]$$
$$-f\sqrt{g(a)g(b)}$$
$$=\frac{\ln g(b)-\ln g(a)}{2}\left[\int_0^1\psi(t)e^{\ln g(b)-t(\ln g(b)-\ln g(a))}\right.$$
$$\times f'(e^{\ln g(b)-t(\ln g(b)-\ln g(a))})dt$$
$$-\int_0^1[(1-t)^{\alpha}-t^{\alpha}]e^{\ln g(b)-t(\ln g(b)-\ln g(a))}$$
$$\left.\times f'(e^{\ln g(b)-t(\ln g(b)-\ln g(a))})dt\right],$$

where

$$\psi(t)=\begin{cases}1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1.\end{cases}$$

Lemma 5. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$. Suppose $f'\in L[g(a),g(b)]$, then*

$$\Gamma(\alpha+1)[{}_HJ^{\alpha}_{g(b)^-}f(g(a))+{}_HJ^{\alpha}_{g(a)^+}f(g(b))]$$
$$-[f(g(a))(\ln g(x)-\ln g(a))^{\alpha}+f(g(b))(\ln g(b)-\ln g(x))^{\alpha}]$$
$$=(\ln g(b)-\ln g(x))^{\alpha+1}\int_0^1(t^{\alpha}-1)e^{t\ln g(x)+(1-t)\ln g(b)}$$
$$\times f'(e^{t\ln g(x)+(1-t)\ln g(b)})dt$$
$$-(\ln g(x)-\ln g(a))^{\alpha+1}\int_0^1(t^{\alpha}-1)e^{t\ln g(x)+(1-t)\ln g(a)}$$
$$\times f'(e^{t\ln g(x)+(1-t)\ln g(a)})dt$$

Now using above results, we derive our results via fractional integrals

Theorem 8. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$ where g is any arbitrary function. If $\alpha\in(0,1]$, $f'\in L[g(a),g(b)]$ and is nondecreasing, then*

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))}\left[{}_HJ^{\alpha}_{g(a)^+}f(g(b))+{}_HJ^{\alpha}_{g(b)^-}f(g(a))\right]\right.$$
$$\left.-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\le\frac{|f'(g(b))|}{2}\left[A_1+A_2+A_3\right],$$

where $A_1=g(b)-g(a)$, $A_2=\frac{g(b)(\alpha+2)}{\alpha+1}$ and $A_3=\frac{\sqrt{g(a)g(b)}(\ln g(b)-\ln g(a))}{2(\alpha+1)}$ respectively.

Proof. Using Lemma 3 and the fact that f' is nondecreasing, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))}\left[{}_HJ^{\alpha}_{g(a)^+}f(g(b))+{}_HJ^{\alpha}_{g(b)^-}f(g(a))\right]\right.$$
$$\left.-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\le\frac{g(b)-g(a)}{2}\int_0^1|f'(tg(a)+(1-t)g(b))|dt+\frac{\ln g(b)-\ln g(a)}{2}$$
$$\times\int_0^1|(1-t)^{\alpha}-t^{\alpha}|e^{t\ln g(a)+(1-t)\ln g(b)}|f'(e^{t\ln g(a)+(1-t)\ln g(b)})|dt$$
$$\le\frac{g(b)-g(a)}{2}|f'(g(b))|$$
$$+\frac{\ln g(b)-\ln g(a)}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|e^{t\ln g(a)+(1-t)\ln g(b)}|f'(g(b))|dt$$
$$=\frac{g(b)-g(a)}{2}|f'(g(b))|$$
$$+\frac{\ln g(b)-\ln g(a)}{2}\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]e^{t\ln g(a)+(1-t)\ln g(b)}|f'(g(b))|dt$$
$$+\frac{\ln g(b)-\ln g(a)}{2}\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]e^{t\ln g(a)+(1-t)\ln g(b)}|f'(g(b))|dt$$
$$=\frac{g(b)-g(a)}{2}|f'(g(b))|$$
$$+\frac{g(b)(\ln g(b)-\ln g(a))}{2}|f'(g(b))|(\mathscr{C}_1+\mathscr{C}_2), \tag{7}$$

where

$$\mathscr{C}_1=\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]e^{-t(\ln g(b)-\ln g(a))}dt.$$
$$\mathscr{C}_2=\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}]e^{-t(\ln g(b)+\ln g(a))}dt.$$

Now

$$\mathscr{C}_1=\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]e^{-t(\ln g(b)-\ln g(a))}dt$$
$$\le\int_0^{\frac{1}{2}}(1-2t)^{\alpha}e^{-t(\ln g(b)-\ln g(a))}dt$$
$$=\frac{1}{2}\int_0^{\frac{1}{2}}(1-s)^{(\alpha+1)-1}e^{-\frac{1}{2}(\ln g(b)-\ln g(a))s}ds$$

$$\leq \frac{1}{2}\max\{1,2^{-\alpha}\}\left(1+\frac{1}{\alpha+1}\right)\left(\frac{\ln g(b)-\ln g(a)}{2}\right)^{-1}$$

$$\leq \frac{\alpha+2}{(\alpha+1)(\ln g(b)-\ln g(a))}, \tag{8}$$

and

$$\mathscr{C}_2 = \int_{\frac{1}{2}}^{1}[t^\alpha-(1-t)^\alpha]e^{-t(\ln g(b)+\ln g(a))}dt$$

$$\leq \int_{\frac{1}{2}}^{1}(2t-1)^\alpha e^{-t(\ln g(b)-\ln g(a))}dt$$

$$= \frac{1}{2}\int_{1}^{2}(s-1)^\alpha e^{-\frac{1}{2}(\ln g(b)-\ln g(a))s}ds$$

$$= \frac{1}{2}e^{-(\ln g(b)-\ln g(a))}\int_{0}^{1}(1-\tau)^\alpha e^{\frac{\ln g(b)-\ln g(a)}{2}\tau}d\tau$$

$$\leq \frac{1}{2}e^{-\frac{\ln g(b)-\ln g(a)}{2}}\int_{0}^{1}(1-\tau)^\alpha d\tau$$

$$= \frac{\sqrt{\frac{g(a)}{g(b)}}}{2(\alpha+1)}, \tag{9}$$

where we have utilized Lemma 1 and Lemma 2. Combining (7), (8) and (9) completes the proof. $\square$

Theorem 9. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$ where g is any arbitrary function. If $\alpha\in(0,1]$, $f'\in L[g(a),g(b)]$ and is nondecreasing, then*

$$\Big|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))}\left[{}_HJ^\alpha_{g(a)^+}f(g(b))+{}_HJ^\alpha_{g(b)^-}f(g(a))\right]$$
$$-f\Big(\frac{g(a)+g(b)}{2}\Big)\Big|$$
$$\leq \left[\frac{g(b)-g(a)}{2}+\frac{g(b)(\ln g(b)-\ln g(a))}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]|f'(g(b))|.$$

Proof. Using Lemma 3 and the fact that f' is nondecreasing, we have

$$\Big|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))}\left[{}_HJ^\alpha_{g(a)^+}f(g(b))+{}_HJ^\alpha_{g(b)^-}f(g(a))\right]$$
$$-f\Big(\frac{g(a)+g(b)}{2}\Big)\Big|$$
$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|$$
$$+\frac{\ln g(b)-\ln g(a)}{2}\int_0^{\frac{1}{2}}[(1-t)^\alpha-t^\alpha]|f'(g(b))|dt$$
$$+\frac{\ln g(b)-\ln g(a)}{2}\int_{\frac{1}{2}}^{1}[t^\alpha-(1-t)^\alpha]|f'(g(b))|dt$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|+\frac{g(b)(\ln g(b)-\ln g(a))}{2}|f'(g(b))|$$
$$\times\Big(\int_0^{\frac{1}{2}}[(1-t)^\alpha-t^\alpha]dt+\int_{\frac{1}{2}}^{1}[t^\alpha-(1-t)^\alpha]dt\Big)$$
$$= \left[\frac{g(b)-g(a)}{2}+\frac{g(b)(\ln g(b)-\ln g(a))}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]|f'(g(b))|.$$

This completes the proof. $\square$

Theorem 10. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$, where g ia any arbitrary function. If $\alpha\in(0,1]$, $f'\in L[g(a),g(b)]$ and is nondecreasing, then*

$$\Big|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))^\alpha}\left[{}_HJ^\alpha_{g(a)^+}f(g(b))+{}_HJ^\alpha_{g(b)^-}f(g(a))\right]$$
$$-f\left(\sqrt{g(a)g(b)}\right)\Big|$$
$$\leq\left(\frac{g(b)(\ln g(b)-\ln g(a))}{2}\right)$$
$$\times\left[1+\frac{\alpha+2}{(\alpha+1)(\ln g(b)-\ln g(a))}+\frac{\sqrt{\frac{g(a)}{g(b)}}}{2(\alpha+1)}\right]|f'(g(b))|.$$

Proof. Using Lemma 4 and the fact that f' is nondecreasing, we have

$$\Big|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))^\alpha}\left[{}_HJ^\alpha_{g(a)^+}f(g(b))+{}_HJ^\alpha_{g(b)^-}f(g(a))\right]$$
$$-f\sqrt{g(a)g(b)}\Big|$$
$$\leq\frac{\ln g(b)-\ln g(a)}{2}$$
$$\times\Big[\int_0^1 e^{\ln g(b)-t(\ln g(b)-\ln g(a))}|f'(e^{\ln g(b)-t(\ln g(b)-\ln g(a))})|dt$$
$$+\int_0^1|(1-t)^\alpha-t^\alpha|e^{\ln g(b)-t(\ln g(b)-\ln g(a))}|$$
$$\times f'(e^{\ln g(b)-t(\ln g(b)-\ln g(a))})|dt\Big]$$
$$\leq\frac{\ln g(b)-\ln g(a)}{2}$$
$$\times\Big[\int_0^1 g(b)|f'(g(b))|dt+\int_0^1|(1-t)^\alpha-t^\alpha|$$
$$\times e^{\ln g(b)-t(\ln g(b)-\ln g(a))}|f'(e^{\ln g(b)-t(\ln g(b)-\ln g(a))})|dt\Big]$$
$$\leq\Big(\frac{g(b)(\ln g(b)-\ln g(a))}{2}\Big)$$
$$\times\Big[1+\frac{\alpha+2}{(\alpha+1)(\ln g(b)-\ln g(a))}+\frac{\sqrt{\frac{g(a)}{g(b)}}}{2(\alpha+1)}\Big]|f'(g(b))|.$$

This completes the proof. $\square$

Theorem 11. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$, where g ia any arbitrary function. If $\alpha\in(0,1]$, $f'\in L[g(a),g(b)]$ and is nondecreasing, then*

$$\Big|\frac{\Gamma(\alpha+1)}{2(\ln(g(b))-\ln(g(a)))^{\alpha}}\left[{}_HJ^{\alpha}_{g(a)^+}f(g(b))+{}_HJ^{\alpha}_{g(b)^-}f(g(a))\right]$$
$$-f\sqrt{g(a)g(b)}\Big|$$
$$\leq\left(\frac{g(b)(\ln g(b)-\ln g(a))}{2}\right)\left[1+\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]|f'(g(b))|.$$

Proof. Using Lemma 4 and the fact that f' is nondecreasing the proof is obvious.

Theorem 12. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$ and $f'\in[g(a),g(b)]$. Let f' be GA-nonconvex function. Then for $|f'(g(x))|\leq M$, $g(x)\in[g(a),g(b)]$ we have*

$$|\Gamma(\alpha+1)[{}_HJ^{\alpha}_{g(x)^-}f(g(a))+{}_HJ^{\alpha}_{g(x)^+}f(g(b))]$$
$$-[f(g(a))(\ln g(x)-\ln g(a))^{\alpha}+f(g(b))(\ln g(b)-\ln g(x))^{\alpha}]|$$
$$\leq\frac{\alpha M(g(b))}{\alpha+1}[(\ln g(b)-\ln g(x))^{\alpha+1}+(\ln g(x)-\ln g(a))^{\alpha+1}].$$

Proof. Using Lemma 5 and the fact that f' is GA-nonconvex function, we have

$$|\Gamma(\alpha+1)[{}_HJ^{\alpha}_{g(x)^-}f(g(a))+{}_HJ^{\alpha}_{g(x)^+}f(g(b))]$$
$$-[f(g(a))(\ln g(x)-\ln g(a))^{\alpha}+f(g(b))(\ln g(b)-\ln g(x))^{\alpha}]|$$
$$\leq(\ln g(b)-\ln g(x))^{\alpha+1}\int_0^1(1-t^{\alpha})e^{t\ln g(x)+(1-t)\ln g(b)}$$
$$\times|f'(e^{t\ln g(x)+(1-t)\ln g(b)})|dt$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}\int_0^1(1-t^{\alpha})e^{t\ln g(x)+(1-t)\ln g(a)}$$
$$\times|f'(e^{t\ln g(x)+(1-t)\ln g(a)})|dt$$
$$=(\ln g(b)-\ln g(x))^{\alpha+1}\int_0^1(1-t^{\alpha})(g(x))^t(g(b))^{1-t}$$
$$\times|f'((g(x))^t(g(b))^{1-t})|dt$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}\int_0^1(1-t^{\alpha})(g(x))^t(g(a))^{1-t}$$
$$\times|f'((g(x))^t(g(a))^{1-t})|dt$$
$$\leq(\ln g(b)-\ln g(x))^{\alpha+1}\int_0^1(1-t^{\alpha})(g(x))^t(g(b))^{1-t}$$
$$\times[t|f'(g(x))|+(1-t)|f'(g(b))|]dt$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}\int_0^1(1-t^{\alpha})(g(x))^t(g(a))^{1-t}$$
$$\times[t|f'(g(x))|+(1-t)|f'(g(a))|]dt$$
$$\leq M(g(b))[(\ln g(b)-\ln g(x))^{\alpha+1}$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}]\int_0^1[t(1-t^{\alpha})+(1-t)(1-t^{\alpha})]dt$$
$$\leq\frac{\alpha M(g(b))}{\alpha+1}[(\ln g(b)-\ln g(x))^{\alpha+1}+(\ln g(x)-\ln g(a))^{\alpha+1}].$$

This completes the proof. □

Theorem 13. *Let $f:[g(a),g(b)]\to\mathbb{R}$ be a differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$ and $f'\in[g(a),g(b)]$. Let $|f'|^q$ be GA-nonconvex function. Then for $|f'(g(x))|\leq M$, $g(x)\in[g(a),g(b)]$, we have*

$$|\Gamma(\alpha+1)[{}_HJ^{\alpha}_{g(x)^-}f(g(a))+{}_HJ^{\alpha}_{g(x)^+}f(g(b))]$$
$$-[f(g(a))(\ln g(x)-\ln g(a))^{\alpha}+f(g(b))(\ln g(b)-\ln g(x))^{\alpha}]|$$
$$\leq\frac{\alpha M(g(b))}{\alpha+1}[(\ln g(b)-\ln g(x))^{\alpha+1}+(\ln g(x)-\ln g(a))^{\alpha+1}].$$

Proof. Using Lemma 5, the fact that $|f'|^q$ is GA-nonconvex function and well known Power mean inequality, we have

$$|\Gamma(\alpha+1)[{}_HJ^{\alpha}_{g(x)^-}f(g(a))+{}_HJ^{\alpha}_{g(x)^+}f(g(b))]$$
$$-[f(g(a))(\ln g(x)-\ln g(a))^{\alpha}+f(g(b))(\ln g(b)-\ln g(x))^{\alpha}]|$$
$$\leq(\ln g(b)-\ln g(x))^{\alpha+1}\int_0^1|1-t^{\alpha})e^{t\ln g(x)+(1-t)\ln g(b)}$$
$$\times|f'(e^{t\ln g(x)+(1-t)\ln g(b)})|dt$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}\int_0^1(1-t^{\alpha})e^{t\ln g(x)+(1-t)\ln g(a)}$$
$$\times|f'(e^{t\ln g(x)+(1-t)\ln g(a)})|dt$$
$$\leq(\ln g(b)-\ln g(x))^{\alpha+1}g(b)\int_0^1(1-t^{\alpha})$$
$$\times|f'((g(x))^t(g(b))^{1-t})|dt$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}g(b)\int_0^1(1-t^{\alpha})$$
$$\times|f'((g(x))^t(g(a))^{1-t})|dt$$
$$\leq(\ln g(b)-\ln g(x))^{\alpha+1}g(b)$$
$$\times\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\right.$$
$$\left.\times|f'((g(x))^t(g(b))^{1-t})|^qdt\right)^{\frac{1}{q}}$$
$$+(\ln g(x)-\ln g(a))^{\alpha+1}g(b)$$
$$\times\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})\right.$$
$$\left.\times|f'((g(x))^t(g(a))^{1-t})|^qdt\right)^{\frac{1}{q}}$$
$$\leq M(g(b))\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times[(\ln g(b)-\ln g(x))^{\alpha+1}+(\ln g(x)-\ln g(a))^{\alpha+1}]$$
$$\times\left(\int_0^1[t(1-t^{\alpha})+(1-t)(1-t^{\alpha})]dt\right)^{\frac{1}{q}}$$
$$\leq\frac{\alpha M(g(b))}{\alpha+1}[(\ln g(b)-\ln g(x))^{\alpha+1}+(\ln g(x)-\ln g(a))^{\alpha+1}].$$

This completes the proof. □

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