Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090319

A Hybrid LQP Alternating Direction Method for Solving Variational Inequality Problems with Separable Structure

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Received: 20 Jul. 2014, Revised: 21 Oct. 2014, Accepted: 22 Oct. 2014
Published online: 1 May 2015

Abstract: In this paper, we presented a logarithmic-quadratic proximal alternating direction method for structured variational inequalities. The method generates the new iterate by searching the optimal step size along the descent direction. Global convergence of the new method is proved under certain assumptions.

Keywords: Variational inequalities, monotone operator, logarithmic-quadratic proximal method, projection method, alternating direction method.
2010 AMS Subject Classification: 49J40, 65N30

1 Introduction

The problem concerned in this paper is the following variational inequalities, find $u \in \Omega$ such that:

$$(u' - u)^T F(u) \geq 0, \qquad \forall\, u' \in \Omega, \tag{1}$$

with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \tag{2}$$

and

$$\Omega = \{(x,y) | x \in \mathscr{R}^n_{++}, y \in \mathscr{R}^m_{++}, Ax + By = b\} \tag{3}$$

where $A \in \mathscr{R}^{l \times n}, B \in \mathscr{R}^{l \times m}$ are given matrices, $b \in \mathscr{R}^l$ is a given vector, and $f : \mathscr{R}^n_{++} \to \mathscr{R}^n$, $g : \mathscr{R}^m_{++} \to \mathscr{R}^m$ are given monotone operators. Studies and applications of such problems can be found in [7,9,10,11,12,13,14]. By attaching a Lagrange multiplier vector $\lambda \in \mathscr{R}^l$ to the linear constraints $Ax + By = b$, the problem (1)-(3) can be explained as find $w \in \mathscr{W}$ such that:

$$(w' - w)^T Q(w) \geq 0, \qquad \forall w' \in \mathscr{W}, \tag{4}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \qquad Q(w) = \begin{pmatrix} f(x) - A^T \lambda \\ g(y) - B^T \lambda \\ Ax + By - b \end{pmatrix}, \tag{5}$$

$$\mathscr{W} = \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l. \tag{6}$$

Problem (4)-(6) is referred to as SVI (*structured variational inequalities*).

The alternating direction method (ADM) is a powerful method for solving the structured problem (4)-(6), since it decomposes the original problems into a series subproblems with lower scale, which was originally proposed by Gabay and Mercier [11] and Gabay [10]. The classical proximal alternating direction method (PADM) [6,8,15] is an effective numerical approach for solving variational inequalities with separable structure. To make the PADM more efficient and practical, He et al. [15] proposed a modified PADM as following. For given $(x^k, y^k, \lambda^k) \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$, the new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is obtained via the following steps:

Step 1.Solve the following inequality to obtain x^{k+1}:

$$\begin{aligned}&(x' - x^{k+1})^T \{ f(x^{k+1}) - A^T [\lambda^k - H_k(Ax^{k+1} + By^k - b)] \\ &+ R_k(x^{k+1} - x^k)\} \geq 0, \qquad \forall x' \in \mathscr{R}^n_{++}\end{aligned} \tag{7}$$

Step 2.Solve the following inequality to obtain y^{k+1}:

$$\begin{aligned}&(y' - y^{k+1})^T \{ g(y^{k+1}) - B^T [\lambda^k - H_k(Ax^{k+1} + By^{k+1} - b)] \\ &+ S_k(y^{k+1} - y^k)\} \geq 0, \qquad \forall y' \in \mathscr{R}^m_{++}\end{aligned} \tag{8}$$

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Step 3.Update λ^k via

$$\lambda^{k+1} = \lambda^k - H_k(Ax^{k+1} + By^{k+1} - b). \tag{9}$$

Very recently, Yuan and Li [24] have proposed a new type of ADM by substituting in the alternating directions method (7)-(9) the term $R(x-x^k)$ and $S(y-y^k)$ by $R[(x-x^k)+\mu(x^k - X_k^2 x^{-1})]$ and $S[(y-y^k)+\mu(y^k - Y_k^2 y^{-1})]$, respectively. The new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ in [24] is obtained via the following steps: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$, and $\mu \in (0,1)$, the new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ in [24] is obtained via solving the following system:

$$f(x) - A^T[\lambda^k - H(Ax + By^k - b)] \\ + R[(x-x^k) + \mu(x^k - X_k^2 x^{-1})] = 0,$$

$$g(y) - B^T[\lambda^k - H(Ax^{k+1} + By - b)] \\ + S[(y-y^k) + \mu(y^k - Y_k^2 y^{-1})] = 0,$$

$$\lambda^{k+1} = \lambda^k - H(Ax^k + By^k - b).$$

Motivated and inspired by the works of [24], we proposed a new inexact alternating direction method for SVI. Each iteration of the above method contains a prediction and a correction, the predictor is obtained via solving the LQP system approximately under significantly relaxed accuracy criterion and new iterate is obtained by a convex combination of the previous point and the one generated by a projection type method along the descent direction. Our results can be viewed as significant extensions of the previously known results.

2 The proposed method

In this section, we suggest and consider the new LQP alternating direction method (LQP-ADM) for solving SVI. In course we always make the following standard assumptions:

Assumption A. $f(x)$ is monotone with respect to $\mathscr{R}^n_{++}$ and $g(y)$ is monotone with respect to $\mathscr{R}^m_{++}$,

Assumption B. The solution set of SVI, denoted by $\mathscr{W}^*$, is nonempty.

Then the iterative scheme of the proposed method is given as follows.

Prediction step: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$, and $\mu \in (0,1)$, the predictor $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$ is obtained via solving the following system:

$$f(x) - A^T[\lambda^k - H(Ax + By - b)] + R[(x-x^k) \\ + \mu(x^k - X_k^2 x^{-1})] =: \xi_x^k \approx 0, \tag{10}$$

$$g(y) - B^T[\lambda^k - H(Ax + By - b)] \\ + S[(y-y^k) + \mu(y^k - Y_k^2 y^{-1})] =: \xi_y^k \approx 0, \tag{11}$$

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b) \tag{12}$$

where

$$\|G^{-1}\xi^k\|_G^2 \le \frac{1-\mu}{1+\mu}\eta^2 \|w^k - \tilde{w}^k\|_G^2. \quad \eta \in (0,1), \tag{13}$$

$$\xi^k = \begin{pmatrix} \xi_x^k \\ \xi_y^k \\ 0 \end{pmatrix} \tag{14}$$

and

$$G = \begin{pmatrix} (1+\mu)R & & \\ & (1+\mu)S & \\ & & H^{-1} \end{pmatrix} \tag{15}$$

is a positive definite (block diagonal) matrix.

Correction step: The new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by:

$$w^{k+1} = \rho w^k + (1-\rho) P_{\mathscr{W}}[w^k - \alpha_k d_k], \qquad \rho \in (0,1) \tag{16}$$

where

$$\alpha_k = \frac{\varphi_k}{\|d_k\|_G^2}, \tag{17}$$

$$\varphi_k := \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ + (w^k - \tilde{w}^k)^T \xi^k \tag{18}$$

and

$$d_k := w^k - \tilde{w}^k + G^{-1}\xi^k. \tag{19}$$

Remark 2.1. Note that if $\xi_x^k = A^T HB(y - y^k)$ and $\xi_y^k = 0$ in (10) and (11), respectively, the new iterate in [24] is produced via solving (10)-(12).

The main task of the prediction is to find an approximate solution of the following equations

$$f(x) - A^T[\lambda^k - H(Ax + By - b)] \\ + R[(x-x^k) + \mu(x^k - X_k^2 x^{-1})] = 0, \tag{20}$$

$$g(y) - B^T[\lambda^k - H(Ax + By - b)] \\ + S[(y-y^k) + \mu(y^k - Y_k^2 y^{-1})] = 0. \tag{21}$$

The exact solution of

$$f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)] \\ + R[(x-x^k) + \mu(x^k - X_k^2 x^{-1})] = 0. \tag{22}$$

denoted by $\tilde{x}^k$, as the approximate solution of (20). Then the exact solution of

$$g(y^k) - B^T[\lambda^k - H(A\tilde{x}^k + By^k - b)] \\ + S[(y-y^k) + \mu(y^k - Y_k^2 y^{-1})] = 0, \tag{23}$$

denoted by $\tilde{y}^k$, as the approximate solution of (21).

It following from (10)-(12) and (22)- (23) that

$$\xi^k = \begin{pmatrix} \xi_x^k \\ \xi_y^k \\ 0 \end{pmatrix} = \begin{pmatrix} f(\tilde{x}^k) - f(x^k) + A^T HA(\tilde{x}^k - x^k) + A^T HB(\tilde{y}^k - y^k) \\ g(\tilde{y}^k) - g(y^k) + B^T HB(\tilde{y}^k - y^k) \\ 0 \end{pmatrix}$$

Note that if $R = rI$ and $S = sI$, the positive solution of (22)- (23) can be obtained explicitly by

$$\tilde{x}_i^k = \left(s_i^k + \sqrt{(s_i^k)^2 + 4\mu(x_i^k)^2}\right)/2r \tag{24}$$

$$\tilde{y}_i^k = \left(p_i^k + \sqrt{(p_i^k)^2 + 4\mu(y_i^k)^2}\right)/2s \tag{25}$$

with

$$s^k = r(1-\mu)x^k - (f(x^k) - A^T[\lambda^k - H(Ax^k + By^k - b)]), \tag{26}$$

$$p^k = s(1-\mu)y^k - (g(y^k) - B^T[\lambda^k - H(A\tilde{x}^k + By^k - b)]). \tag{27}$$

It is easy to verify that $\tilde{y}^k > 0, \tilde{x}^k > 0$ whenever $y^k > 0, x^k > 0$.

We need the following result in the convergence analysis of the proposed method.

Lemma 2.1[24] Let $q(u) \in \mathscr{R}^n$ be a monotone mapping of u with respect to $\mathscr{R}^n_+$ and $R \in \mathscr{R}^{n\times n}$ be positive definite diagonal matrix. For given $u^k > 0$, if we let $U_k := \text{diag}(u_1^k, u_2^k, \cdots, u_n^k)$ and u^{-1} be an n-vector whose j-th element is $1/u_j$, then the equation

$$q(u) + R[(u - u^k) + \mu(u^k - U_k^2 u^{-1})] = 0 \tag{28}$$

has a unique positive solution u. Moreover, for any $v \geq 0$, we have

$$(v-u)^T q(u) \geq \tfrac{1+\mu}{2}\left(\|u-v\|_R^2 - \|u^k - v\|_R^2\right) + \tfrac{1-\mu}{2}\|u^k - u\|_R^2. \tag{29}$$

In the next theorem we show that α_k is lower bounded away from zero and it is one of the keys to prove the global convergence results.

Theorem 2.1 For given $w^k \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$, let $\tilde{w}^k$ be generated by (10)-(12), then we have the following

$$2\varphi_k - \|d_k\|_G^2 \geq \tfrac{1-\mu}{1+\mu}(1-\eta^2)\|w^k - \tilde{w}^k\|_G^2 \tag{30}$$

and

$$\alpha_k \geq \frac{1}{2}. \tag{31}$$

Proof. It follows from (18), (19) and under Condition (13), we have

$$\begin{aligned}
2\varphi_k &= 2\|x^k - \tilde{x}^k\|_R^2 + 2\|y^k - \tilde{y}^k\|_S^2 + 2\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\
&\quad + 2(w^k - \tilde{w}^k)^T \xi^k \\
&= \|w^k - \tilde{w}^k + G^{-1}\xi^k\|_G^2 - \|G^{-1}\xi^k\|_G^2 + (1-\mu)\|x^k - \tilde{x}^k\|_R^2 \\
&\quad + (1-\mu)\|y^k - \tilde{y}^k\|_S^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\
&= \|d_k\|_G^2 + \tfrac{1-\mu}{1+\mu}((1+\mu)\|x^k - \tilde{x}^k\|_R^2 + (1+\mu)\|y^k - \tilde{y}^k\|_S^2 \\
&\quad + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2) - \|G^{-1}\xi^k\|_G^2 + \frac{2\mu}{1+\mu}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\
&\geq \|d_k\|_G^2 + \tfrac{1-\mu}{1+\mu}\|w^k - \tilde{w}^k\|_G^2 - \|G^{-1}\xi^k\|_G^2 \\
&\geq \|d_k\|_G^2 + \tfrac{1-\mu}{1+\mu}(1-\eta^2)\|w^k - \tilde{w}^k\|_G^2.
\end{aligned}$$

Therefore, it follows from (17) and (30) that

$$\alpha_k \geq \frac{1}{2}.\square$$

3 Main Results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method. The first result is due to applying Lemma 2.1 to the LQP systems in prediction step of the proposed method.

Theorem 3.1 For given $w^k = (x^k, y^k, \lambda^k) \in \mathscr{R}^n_{++} \times \mathscr{R}^m_{++} \times \mathscr{R}^l$, let $\tilde{w}^k$ be generated by (10)-(12). Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathscr{W}^*$, we have

$$(w^k - w^*)^T G d_k \geq \varphi_k. \tag{32}$$

Proof. Applying Lemma 2.1 to (10) by setting $u^k = x^k, u = \tilde{x}^k, v = x^*$ in (29)) and

$$q(u) = f(\tilde{x}^k) - A^T[\lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b)] - \xi_x^k,$$

we get

$$\begin{aligned}
&(x^* - \tilde{x}^k)^T\left\{f(\tilde{x}^k) - A^T\left[\lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b)\right] - \xi_x^k\right\} \\
&\geq \tfrac{1+\mu}{2}\left(\|\tilde{x}^k - x^*\|_R^2 - \|x^k - x^*\|_R^2\right) + \tfrac{1-\mu}{2}\|x^k - \tilde{x}^k\|_R^2.
\end{aligned} \tag{33}$$

Recall

$$(x^* - \tilde{x}^k)^T R(x^k - \tilde{x}^k) = \frac{1}{2}\left(\|\tilde{x}^k - x^*\|_R^2 - \|x^k - x^*\|_R^2\right) + \frac{1}{2}\|x^k - \tilde{x}^k\|_R^2. \tag{34}$$

Adding (33) and (34), we obtain

$$(x^* - \tilde{x}^k)^T\left\{(1+\mu)R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^T\tilde{\lambda}^k + \xi_x^k\right\} \leq \mu\|x^k - \tilde{x}^k\|_R^2. \tag{35}$$

Similarly, applying Lemma 2.1 to (11), substituting $u^k = y^k$, $u = \tilde{y}^k$, $v = y^*$ and replacing R, n with S, m respectively in (29) and

$$q(u) = g(\tilde{y}^k) - B^T[\lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b)] - \xi_y^k,$$

we get

$$\begin{aligned}
&(y^* - \tilde{y}^k)^T\{g(\tilde{y}^k) - B^T[\lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b)] - \xi_y^k\} \\
&\geq \tfrac{1+\mu}{2}\left(\|\tilde{y}^k - y^*\|_S^2 - \|y^k - y^*\|_S^2\right) + \tfrac{1-\mu}{2}\|y^k - \tilde{y}^k\|_S^2.
\end{aligned} \tag{36}$$

Recall

$$(y^* - \tilde{y}^k)^T S(y^k - \tilde{y}^k) = \frac{1}{2}\left(\|\tilde{y}^k - y^*\|_S^2 - \|y^k - y^*\|_S^2\right) + \frac{1}{2}\|y^k - \tilde{y}^k\|_S^2. \tag{37}$$

Adding (36) and (37), we have

$$(y^* - \tilde{y}^k)^T\left\{(1+\mu)S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^T\tilde{\lambda}^k + \xi_y^k\right\} \leq \mu\|y^k - \tilde{y}^k\|_S^2, \tag{38}$$

Since (x^*, y^*, λ^*) is a solution of SVI, $\tilde{x}^k \in \mathscr{R}^n_{++}$ and $\tilde{y}^k \in \mathscr{R}^m_{++}$, we have

$$(\tilde{x}^k - x^*)^T(f(x^*) - A^T\lambda^*) \geq 0,$$
$$(\tilde{y}^k - y^*)^T(g(y^*) - B^T\lambda^*) \geq 0,$$

and

$$Ax^* + By^* - b = 0.$$

Using the monotonicity of f and g, we obtain

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix} \geq \begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^T \begin{pmatrix} f(x^*) - A^T\lambda^* \\ g(y^*) - B^T\lambda^* \\ Ax^* + By^* - b \end{pmatrix} \geq 0. \tag{39}$$

Adding (35), (38) and (39), we get

$$\begin{aligned}(w^* - \tilde{w}^k)^T G d_k &= (w^* - \tilde{w}^k)^T G(w^k - \tilde{w}^k + G^{-1}\xi^k) \\ &= (x^* - \tilde{x}^k)^T((1+\mu)R(x^k - \tilde{x}^k) + \xi_x^k) \\ &\quad + (y^* - \tilde{y}^k)^T((1+\mu)S(y^k - \tilde{y}^k) + \xi_y^k) \\ &\quad + (\lambda^* - \tilde{\lambda}^k)^T(A\tilde{x}^k + B\tilde{y}^k - b) \\ &\leq \mu\|x^k - \tilde{x}^k\|_R^2 + \mu\|y^k - \tilde{y}^k\|_S^2. \end{aligned} \tag{40}$$

It follows from (40) that

$$\begin{aligned}(w^k - w^*)^T G d_k &\geq (w^k - \tilde{w}^k)^T G d_k - \mu\|x^k - \tilde{x}^k\|_R^2 - \mu\|y^k - \tilde{y}^k\|_S^2 \\ &\geq \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ &\quad + (w^k - \tilde{w}^k)^T\xi^k. \end{aligned}$$

Using the definitions of φ_k the assertion of this theorem is proved. □

Theorem 3.2 Let $w^* \in \mathscr{W}^*$ be a solution of SVI and let w^{k+1} be defined by (16). Then w^k and $\tilde{w}^k$ are bounded, and

$$\|w^{k+1} - w^*\|_G^2 \leq \|w^k - w^*\|_G^2 - c\|w^k - \tilde{w}^k\|_G^2 \tag{41}$$

where

$$c := \frac{(1-\mu)(1-\rho)(1-\eta^2)}{4(1+\mu)} > 0.$$

Proof. It follows from (16), (32), (30) and (31) that

$$\begin{aligned}\|w^{k+1} - w^*\|_G^2 &= \|\rho(w^k - w^*) + (1-\rho)(P_W[w^k - \alpha_k d_k] - w^*)\|_G^2 \\ &\leq \rho\|w^k - w^*\|_G^2 + (1-\rho)\|P_W[w^k - \alpha_k d_k] - w^*\|_G^2 \\ &\leq \rho\|w^k - w^*\|_G^2 + (1-\rho)\|w^k - w^* - \alpha_k d_k\|_G^2 \\ &= \|w^k - w^*\|_G^2 - 2(1-\rho)\alpha_k(w^k - w^*)^T G d_k \\ &\quad + (1-\rho)\alpha_k^2\|d_k\|_G^2 \\ &\leq \|w^k - w^*\|_G^2 - \alpha_k(1-\rho)\varphi_k \\ &\leq \|w^k - w^*\|_G^2 - \frac{(1-\mu)(1-\rho)(1-\eta^2)}{4(1+\mu)}\|w^k - \tilde{w}^k\|_G^2. \end{aligned}$$

Since $\gamma \in [1,2)$ we have

$$\|w^{k+1} - w^*\| \leq \|w^k - w^*\| \leq \ldots \leq \|w^0 - w^*\|$$

and thus $\{w^k\}$ is a bounded sequence.
It follows from (41) that

$$\sum_{k=0}^{\infty} c\|w^k - \tilde{w}^k\|_G^2 < +\infty.$$

which means that

$$\lim_{k\to\infty}\|w^k - \tilde{w}^k\|_G = 0. \tag{42}$$

Since $\{w^k\}$ is a bounded sequence, we conclude that $\{\tilde{w}^k\}$ is also bounded. □

4 Convergence of the proposed method

In this section, we prove the global convergence of the proposed method. The following results can be proved by using the technique of Lemma 5.1 and Theorem 5.1 in [2].

Lemma 4.1 For given $w^k = (x^k, y^k, \lambda^k) \in \mathscr{R}_{++}^n \times \mathscr{R}_{++}^m \times \mathscr{R}^l$, let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (10)-(12). Then for any $w = (x, y, \lambda) \in \mathscr{W}$, we have

$$(x - \tilde{x}^k)^T(f(\tilde{x}^k) - A^T\tilde{\lambda}^k - \xi_x^k) \geq (x^k - \tilde{x}^k)^T R\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\} \tag{43}$$

and

$$(y - \tilde{y}^k)^T(g(\tilde{y}^k) - B^T\tilde{\lambda}^k - \xi_y^k) \geq (y^k - \tilde{y}^k)^T S\{(1+\mu)y - (\mu y^k + \tilde{y}^k)\} \tag{44}$$

Proof. Applying Lemma 2.1 to Step 1 of LQP-ADM (by setting $u^k = x^k, u = \tilde{x}^k, q(u) = f(\tilde{x}^k) - A^T\tilde{\lambda}^k - \xi_x^k$ and $v = x$ in (29)), it follows that

$$\begin{aligned}(x - \tilde{x}^k)^T(f(\tilde{x}^k) - A^T\tilde{\lambda}^k - \xi_x^k) &\geq \tfrac{1+\mu}{2}(\|\tilde{x}^k - x\|_R^2 - \|x^k - x\|_R^2) \\ &\quad + \tfrac{1-\mu}{2}\|x^k - \tilde{x}^k\|_R^2.\end{aligned}$$

By a simple manipulation, we have

$$\begin{aligned}&\tfrac{1+\mu}{2}(\|\tilde{x}^k - x\|_R^2 - \|x^k - x\|_R^2) + \tfrac{1-\mu}{2}\|x^k - \tilde{x}^k\|_R^2 \\ &= (1+\mu)x^T R x^k - (1+\mu)x^T R\tilde{x}^k - (1-\mu)(\tilde{x}^k)^T R x^k - \mu\|x^k\|_R^2 + \|\tilde{x}^k\|_R^2 \\ &= (1+\mu)x^T R(x^k - \tilde{x}^k) - (x^k - \tilde{x}^k)^T R(\mu x^k + \tilde{x}^k) \\ &= (x^k - \tilde{x}^k)^T R\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\},\end{aligned}$$

and the assertion (43) is proved. Similarly we can prove the assertion (44). □

Now, we are ready to prove the convergence of the proposed method.

Theorem 4.1 The sequence $\{w^k\}$ generated by the proposed method converges to some w^∞ which is a solution of SVI.

Proof. It follows from (42) that

$$\lim_{k\to\infty}\|x^k - \tilde{x}^k\|_R = 0, \qquad \lim_{k\to\infty}\|y^k - \tilde{y}^k\|_S = 0 \tag{45}$$

and

$$\lim_{k\to\infty}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}} = \lim_{k\to\infty}\|A\tilde{x}^k + B\tilde{y}^k - b\|_H = 0. \tag{46}$$

Moreover, (43) and (44) imply that

$$\begin{aligned}(x - \tilde{x}^k)^T(f(\tilde{x}^k) - A^T\tilde{\lambda}^k) &\geq (x^k - \tilde{x}^k)^T R\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\} \\ &\quad + (x - \tilde{x}^k)^T\xi_x^k\end{aligned} \tag{47}$$

and

$$\begin{aligned}(y - \tilde{y}^k)^T(g(\tilde{y}^k) - B^T\tilde{\lambda}^k) &\geq (y^k - \tilde{y}^k)^T S\{(1+\mu)y - (\mu y^k + \tilde{y}^k)\} \\ &\quad + (y - \tilde{y}^k)^T\xi_y^k.\end{aligned} \tag{48}$$

We deduce from (13) and (45) that

$$\begin{cases} \lim_{k\to\infty}(x - \tilde{x}^k)^T\{f(\tilde{x}^k) - A^T\tilde{\lambda}^k\} \geq 0, & \forall x \in \mathscr{R}_{++}^n, \\ \lim_{k\to\infty}(y - \tilde{y}^k)^T\{g(\tilde{y}^k) - B^T\tilde{\lambda}^k\} \geq 0, & \forall y \in \mathscr{R}_{++}^m. \end{cases} \tag{49}$$

Since $\{w^k\}$ is bounded, so it has at least one cluster point. Let w^∞ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to w^∞. It follows from (46) and (49) that

$$\begin{cases} \lim_{j\to\infty}(x-x^{k_j})^T\{f(x^{k_j})-A^T\lambda^{k_j}\}\geq 0, & \forall x\in\mathscr{R}^n_{++}, \\ \lim_{j\to\infty}(y-y^{k_j})^T\{g(y^{k_j})-B^T\lambda^{k_j}\geq 0, & \forall y\in\mathscr{R}^m_{++}, \\ \lim_{j\to\infty}(Ax^{k_j}+By^{k_j}-b)=0. \end{cases}$$

and consequently

$$\begin{cases} (x-x^\infty)^T\{f(x^\infty)-A^T\lambda^\infty\}\geq 0, & \forall x\in\mathscr{R}^n_{++}, \\ (y-y^\infty)^T\{g(y^\infty)-B^T\lambda^\infty\}\geq 0, & \forall y\in\mathscr{R}^m_{++}, \\ Ax^\infty+By^\infty-b=0, \end{cases}$$

which means that w^∞ is a solution of SVI.

Now we prove that the sequence $\{w^k\}$ converges to w^∞. Since

$$\lim_{k\to\infty}\|w^k-\tilde{w}^k\|_G=0, \quad \text{and} \quad \{\tilde{w}^{k_j}\}\to w^\infty,$$

for any $\varepsilon>0$, there exists an $l>0$ such that

$$\|\tilde{w}^{k_l}-w^\infty\|<\frac{\varepsilon}{2} \quad \text{and} \quad \|w^{k_l}-\tilde{w}^{k_l}\|<\frac{\varepsilon}{2}. \tag{50}$$

Therefore, for any $k\geq k_l$, it follows from (41) and (50) that

$$\|w^k-w^\infty\|\leq\|w^{k_l}-w^\infty\|\leq\|w^{k_l}-\tilde{w}^{k_l}\|+\|\tilde{w}^{k_l}-w^\infty\|<\varepsilon.$$

This implies that the sequence $\{w^k\}$ converges to w^∞ which is a solution of SVI. □

5 Conclusions

In this paper, we propose a new modified logarithmic-quadratic proximal alternating direction method (LQP-ADM) for solving structured variational inequalities. Each iteration of the new LQP-ADM contains a prediction and a correction, the predictor is obtained via solving the LQP system approximately under significantly relaxed accuracy criterion and new iterate is obtained by a convex combination of the previous point and the one generated by a projection type method along the descent direction. Global convergence of the proposed method is proved under mild assumptions.

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