Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/030204

Fractional S-Transform for Boehmians

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Received: 7 Feb. 2015, Revised: 30 Mar. 2015, Accepted: 31 Mar. 2015
Published online: 1 Jul. 2015

Abstract: In this paper we extend the results of ultradistributions for the fractional S-transform given by Singh [18] to the Boehmian spaces namely, tempered Boehmian and ultraBoehmian spaces.

Keywords: Fractional Fourier transform, fractional S-transform, S-transform, wavelet transform, ultradistributions, tempered distribution, tempered Boehmian, ultraBoehmian.
Subject Classifications: 46E15, 46E40, 46F12

1 Introduction

In this section we give the definition and some basic properties of the S-transform. The S-transform was first introduced by Stockwell *et al.* [19] in 1996 as invertible time-frequency spectral localization technique. The S-transform is an extension to the ideas of continuous wavelet transform, and is based on a moving and scalable localizing Gaussian window and has characteristics superior to both of the Fourier transform and the wavelet transform [4,20,21].

The one-dimensional continuous S-transform of $u(t)$ is defined as [20]

$$(Su)(\tau,f)=S(u(t))(\tau,f)=\int_{\mathbb{R}}u(t)\omega(\tau-t,f)e^{-i2\pi ft}dt, \tag{1}$$

where the window ω is assumed to satisfy the following:

$$\int_{\mathbb{R}}\omega(t,f)dt=1 \text{ for all } f\in\mathbb{R}\setminus\{0\}. \tag{2}$$

The most usual window ω is the Gaussian one

$$\omega(t,f)=\frac{|f|}{k\sqrt{2\pi}}e^{-\frac{f^2t^2}{2k^2}},\ k>0, \tag{3}$$

where f is the frequency, t is the time variable, and k is a scaling factor that controls the number of oscillations in the window.

Then, Equation (1) can be rewritten as a convolution

$$(Su)(\tau,f)=\left(u(\cdot)e^{-i2\pi f\cdot}*\omega(\cdot,f)\right)(\tau). \tag{4}$$

Applying the convolution property for the Fourier transform, we obtain

$$(Su)(\tau,f)=\mathscr{F}^{-1}\{\hat{u}(\cdot+f)\hat{\omega}(\cdot,f)\}(\tau), \tag{5}$$

where $\mathscr{F}^{-1}$ is the inverse Fourier transform. For the Gaussian window case (3),

$$\mathscr{F}\{\omega(t,f)\}(\alpha,f)=\hat{\omega}(\alpha,f)=e^{-2(\pi k\alpha/f)^2}. \tag{6}$$

Thus we can write the S-transform in the following form:

$$(Su)(\tau,f)=\int_{\mathbb{R}}\hat{u}(\alpha+f)e^{-2(\pi k\alpha/f)^2}e^{i2\pi\alpha\tau}d\alpha. \tag{7}$$

Also, if $\hat{u}(f)$ and $(Su)(\tau,f)$ are the Fourier transform and S-transform of u respectively, then

$$\hat{u}(f)=\int_{\mathbb{R}}(Su)(\tau,f)d\tau, \tag{8}$$

so that

$$u(t)=\mathscr{F}^{-1}\left(\int_{\mathbb{R}}(Su)(\tau,\cdot)d\tau\right)(t). \tag{9}$$

Some basic properties of S-transform can be found in [16, 17].

2 Fractional Fourier transform

The fractional Fourier transform(FRFT) has played an important role in signal processing. The a^{th} order FRFT

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of a signal $u(t)$ is defined as [1]:

$$F_u^a(f) = \int_{\mathbb{R}} u(t) K_a(t,f)dt, \tag{10}$$

where the transform kernel $K_a(t,f)$ is defined as

$$K_a(t,f) = \begin{cases} A_\theta e^{i\pi(f^2\cot\theta - 2ft\csc\theta + t^2\cot\theta)}, & \text{if } \theta \neq n\pi \\ \delta(t-f), & \text{if } \theta = 2n\pi \\ \delta(t+f), & \text{if } \theta + \pi = 2n\pi, \end{cases} \tag{11}$$

where $A_\theta = \sqrt{1 - i\cot\theta}$, $\theta = a\pi/2$, $a \in [0,4)$, i is the complex unit, n is an integer, and f is the fractional Fourier frequency(FRFfr). The inverse FRFT of equation (10) is :

$$u(t) = \int_{\mathbb{R}} F_u^a(f) \overline{K_a(t,f)} df. \tag{12}$$

We can write (10) as

$$\begin{aligned} F_u^a(f) &= \int_{\mathbb{R}} u(t) A_\theta e^{i\pi(f^2\cot\theta - 2ft\csc\theta + t^2\cot\theta)} dt \\ &= A_\theta e^{i\pi f^2\cot\theta} \int_{\mathbb{R}} u(t) e^{-i2\pi ft\csc\theta} e^{i\pi t^2\cot\theta} dt \\ &= A_\theta e^{i\pi f^2\cot\theta} [\widehat{e^{i\pi t^2\cot\theta} u(t)}](f\csc\theta). \end{aligned} \tag{13}$$

Replacing $u(t)$ by $e^{-i\pi t^2\cot\theta}\phi(t)$, we have

$$F_u^a(f) = A_\theta e^{i\pi f^2\cot\theta} [\widehat{\phi(t)}](f\csc\theta). \tag{14}$$

Put $f = \xi\sin\theta$, we have

$$F_u^a(\xi\sin\theta) = A_\theta e^{i\pi\xi^2\sin2\theta/2} \hat{\phi}(\xi). \tag{15}$$

$$\hat{\phi}(\xi) = \frac{1}{A_\theta} e^{-i\pi\xi^2\sin2\theta/2} F_u^a(\xi\sin\theta), \tag{16}$$

where $u(t) = e^{-i\pi t^2\cot\theta}\phi(t)$.

3 The fractional S-transform

The fractional S-transform(FRST) is a generalization of the S-transform. The a^{th} order continuous fractional S-transform of $u(t)$ is defined as [5]:

$$FRST_u^a(\tau,f) = \int_{\mathbb{R}} u(t) g(\tau - t, f) K_a(t,f) dt, \tag{17}$$

where the window g is:

$$g(t,f) = \frac{|f\csc\theta|^p}{k\sqrt{2\pi}} e^{-t^2(f\csc\theta)^{2p}/2k^2};\ k,p > 0, \tag{18}$$

and satisfy the condition:

$$\int_{\mathbb{R}} g(t,f)dt = 1 \text{ for all } f \in \mathbb{R}\setminus\{0\}. \tag{19}$$

Inverse fractional S-transform is defined by

$$u(t) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} FRST_u^a(\tau,f) d\tau\right] \overline{K_a(t,f)} df. \tag{20}$$

Note that the fractional S-transform depends on a parameter θ and can be interpreted as a rotation by an angle θ in the time-frequency plane. An FRST with $\theta = \frac{\pi}{2}$ corresponds to the S-transform, and an FRST with $\theta = 0$ corresponds to the zero operator. The parameters p and k can be used to adjust the window function space. Let

$$h(t,\tau,f) = g(\tau - t, f) K_a(t,f), \tag{21}$$

and

$$\begin{aligned} H_a(\tau,f,f_1) &= \int_{\mathbb{R}} h(t,\tau,f) \overline{K_a(t,f_1)} \\ &= \int_{\mathbb{R}} g(\tau - t, f) K_a(t,f) \overline{K_a(t,f_1)} dt. \end{aligned} \tag{22}$$

Since

$$K_a(t,f)\overline{K_a(t,f_1)} = A_\theta \overline{A}_\theta e^{i\pi[(f^2 - f_1^2)\cot\theta - 2(f-f_1)t\csc\theta]}. \tag{23}$$

By using (18) and (23) in (22) we obtain

$$\begin{aligned} H_a(\tau,f,f_1) &= \int_{\mathbb{R}} \frac{|f\csc\theta|^p}{k\sqrt{2\pi}} e^{-(\tau-t)^2(f\csc\theta)^{2p}/2k^2} A_\theta \overline{A}_\theta \\ &\quad \times e^{i\pi[(f^2-f_1^2)\cot\theta - 2(f-f_1)t\csc\theta]} dt \\ &= \frac{|f\csc\theta|^p}{k\sqrt{2\pi}} A_\theta \overline{A}_\theta e^{i\pi[(f^2-f_1^2)\cot\theta]} \\ &\quad \times \int_{\mathbb{R}} e^{-(\tau-t)^2(f\csc\theta)^{2p}/2k^2} e^{-i2\pi(f-f_1)t\csc\theta} dt. \end{aligned} \tag{24}$$

By using the technique of [5], we obtain

$$\begin{aligned} H_a(\tau,f,f_1) &= A_\theta \overline{A}_\theta e^{i\pi[(f^2-f_1^2)\cot\theta]} e^{-i2\pi(f-f_1)\tau\csc\theta} \\ &\quad \times e^{-2\pi^2k^2(f-f_1)^2(\csc\theta)^2/(f\csc\theta)^{2p}}, \end{aligned} \tag{25}$$

and by using (23) we can write

$$H_a(\tau,f,f_1) = e^{-[2\pi^2k^2(f-f_1)^2(\csc\theta)^2/(f\csc\theta)^{2p}]} K_a(t,f)\overline{K_a(t,f_1)}. \tag{26}$$

Also, the FRST can be also defined as operations on the fractional Fourier domain

$$\begin{aligned} FRST_u^a(\tau,f) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} F_u^a(f)\overline{K_a(t,f)} df\right] g(\tau - t, f) K_a(t,f) dt \\ &= \int_{\mathbb{R}} F_u^a(f_1) H_a(\tau,f,f_1) df_1. \end{aligned} \tag{27}$$

By using (13) and (25) we can write (27) as

$$FRST_u^a(\tau,f)=$$
$$\int_{\mathbb{R}} A_\theta e^{i\pi f_1^2\cot\theta}[e^{i\pi t^2\cot\theta}\widehat{u(t)}](f_1\csc\theta)A_\theta\overline{A}_\theta e^{i\pi[(f^2-f_1^2)\cot\theta]}$$
$$\times e^{-i2\pi(f-f_1)\tau\csc\theta}e^{-2\pi^2k^2(f-f_1)^2(\csc\theta)^2/(f\csc\theta)^{2p}}df_1$$
$$=A_\theta|A_\theta|^2\int_{\mathbb{R}} e^{i\pi f_1^2\cot\theta}[e^{i\pi t^2\cot\theta}\widehat{u(t)}](f_1\csc\theta)e^{i\pi[(f^2-f_1^2)\cot\theta]}$$
$$\times e^{-i2\pi(f-f_1)\tau\csc\theta}e^{-2\pi^2k^2(f-f_1)^2(\csc\theta)^2/(f\csc\theta)^{2p}}df_1$$
$$=A_\theta|A_\theta|^2e^{i\pi f^2\cot\theta}\int_{\mathbb{R}}[e^{i\pi t^2\cot\theta}\widehat{u(t)}](f_1\csc\theta)$$
$$\times e^{-i2\pi(f-f_1)\tau\csc\theta}e^{-2\pi^2k^2(f-f_1)^2(\csc\theta)^2/(f\csc\theta)^{2p}}df_1. \tag{28}$$

The S-transform has been studied on the spaces of type S and spaces of tempered ultradistributions by S. K. Singh [16]. Pathak and Singh [11] have studied the wavelet transform of Tempered Ultradistributions. The fractional S-transform on ultradistribution space is studied by Singh [18], which is, in this paper, extended to tempered and ultra Boehmians.

4 Tempered Boehmians for the fractional S-transform

The concept of Boehmian is motivated by the so called Regular operators, introduced by Boehme [3]. Regular operators form a subalgebra of the field of Mikusinski operators and hence they include only such functions whose support is bounded from the left. Boehmians contain all regular operators, all distributions and some objects which are neither operators nor distributions, they have an algebraic character of Mikusinski operators, and at the same time do not have any restrictions on the support, and they contain all Schwartz distributions, Roumieu ultradistributions, regular operators and tempered distribution. Mikusinski [9] introduced the space of tempered Boehmians, and its Fourier transform is defined as a classical distribution. In another paper [10], he enlarged the space of tempered Boehmians, by introducing larger class of delta sequence, which is identified with the space of ultradistribution.

Tempered Boehmians : The pair of sequence (f_n,φ_n) is called a quotient of sequence, denoted by f_n/φ_n, whose numerator belongs to some set $\mathscr{G}$ and the denominator is a delta sequence such that

$$f_n\sharp\varphi_m = f_m\sharp\varphi_n\ ,\ \forall n,m\in\mathbb{N}. \tag{29}$$

Two quotients of sequence f_n/φ_n and g_n/ψ_n are said to be equivalent if

$$f_n\sharp\psi_n = g_n\sharp\varphi_n,\ \forall n\in\mathbb{N}. \tag{30}$$

The equivalence classes are called the Boehmians. The space of Boehmians is denoted by $\mathscr{B}$, an element of which is written as $x = f_n/\varphi_n$. Application of construction of Boehmians to function spaces with the convolution product yields various spaces of generalized functions. The spaces, so obtained, contain the standard spaces of generalized functions defined as dual spaces. For example, if $\mathscr{G}=C(\mathbb{R}^N)$ and a delta sequence defined as sequence of functions $\varphi_n\in\mathscr{D}$ such that

(i) $\int\varphi_n dx=1$, $\forall a\in\mathbb{N}$
(ii) $\int|\varphi_n|dx\le C$, for some constant C and $\forall n\in\mathbb{N}$,
(iii) supp $\varphi_n(x)\to 0$, as $n\to\infty$,

then the space of Boehmian that is obtained, contains properly the space of Schwartz distributions. Similarly, this space of Boehmians also contains properly the space of tempered distributions S' , when $\mathscr{G}$ is the space of slowly increasing functions with delta sequence. The fractional S-transform of tempered Boehmian form a proper subspace of Schwartz distribution $\mathscr{D}'$. Since $\mathscr{D}$ is dense in S there cannot more than one element to S'(dual of S) having the same restriction to $\mathscr{D}$. Therefore, this type of correspondence between S' and $\mathscr{D}'$ is one to one and so we express it by saying that $S'\subset\mathscr{D}'$. Boehmian space have two types of convergence, namely, the δ- and Δ- convergences, which are stated as:

(i) A sequence of Boehmians (x_n) in the Boehmian space $\mathscr{B}$ is said to be δ- convergent to a Boehmian x in $\mathscr{B}$, which is denoted by $x_n\xrightarrow{\delta}x$ if there exists a delta sequence (δ_n) such that $(x_n\sharp\delta_n),(x\sharp\delta_n)\in\mathscr{G},\forall n\in\mathbb{N}$ and $(x_n\sharp\delta_k)\to(x\sharp\delta_k)$ as $n\to\infty$ in $\mathscr{G}$, $\forall k\in\mathbb{N}$.
(ii) A sequence of Boehmians (x_n) in $\mathscr{B}$ is said to be Δ- convergent to a Boehmian x in $\mathscr{B}$, denoted by $x_n\xrightarrow{\delta}x$ if there exists a delta sequence $(\delta_n)\in\Delta$ such that $(x_n-x)\sharp\delta_n\in\mathscr{G},\forall n\in\mathbb{N}$ and $(x_n-x)\sharp\delta_n\to 0$ as $n\to\infty$ in $\mathscr{G}$.

For details of the properties and convergence of Boehmians one can refer to [7]. We have employed following notations and definitions.

A complex valued infinitely differentiable function f, defined on $\mathbb{R}^N$, is called rapidly decreasing, if

$$\sup_{|\alpha|\le m}\sup_{x\in\mathbb{R}^N}\left(1+x_1^2+x_2^2+\cdots+x_N^2\right)^m|D^\alpha f(x)|<\infty,$$

for every non-negative integer m. Here $|\alpha|=|\alpha_1|+\cdots+|\alpha_N|$, and

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}.$$

The space of all rapidly decreasing functions on $\mathbb{R}^N$ is denoted by S. The delta sequence, i.e., sequence of real valued functions $\varphi_1,\varphi_2,\ldots\in S$, is such that

(i) $\int\varphi_n dx=1,\ \forall a\in\mathbb{N}$
(ii) $\int|\varphi_n|dx\le C$, for some constant C and $\forall n\in\mathbb{N}$,
(iii) $\lim_{n\to\infty}\int_{\|x\|\ge\varepsilon}\|x\|^k|\varphi_n|dx=0$, for every $k\in\mathbb{N},\varepsilon>0$.

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_n is a delta sequence. A complex-valued function f on $\mathbb{R}^N$ is called slowly increasing if there exists a polynomial p on $\mathbb{R}^N$ such that $f(x)/p(x)$ is bounded. The space of all slowly increasing continuous functions on $\mathbb{R}^N$ is denoted by $\mathscr{I}$. If $f_n \in \mathscr{I}$, $\{\varphi_n\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathscr{B}_{\mathscr{I}}$, elements of which will be called tempered Boehmians.
For $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$, define $D^\alpha F = [(f_n \sharp D^\alpha \varphi_n)/(\varphi_n \sharp \varphi_n)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathscr{B}_{\mathscr{I}}$.

If $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$ and $f_n \in S$, for all $n \in \mathbb{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathscr{B}_S$. If $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$ and $G = [g_n/\psi_n] \in \mathscr{B}_S$,then the convolution is

$$F \sharp G = [(f_n \sharp g_n)/(\varphi_n \sharp \psi_n)] \in \mathscr{B}_{\mathscr{I}} \quad .$$

The convolution quotient is denoted by f/φ and $\frac{f}{\varphi}$ denotes a usual quotient. Let $f \in \mathscr{I}$. Then the fractional S-transformation of f, denoted as $\tilde{f}$, is defined for distribution spaces of slowly increasing function f in the following form:

$$\langle \tilde{f}, \varphi \rangle = \langle f, \tilde{\varphi} \rangle, \quad \varphi \in S(\mathbb{R}).$$

Moreover, as we know that the Fourier transform of the convolution of two functions is the product of their Fourier transform. Whereas, in the present case, when we consider this condition for fractional S-transform, it does not seem to be as nice or as practical. Actually, the convolution operation defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t) dt \tag{31}$$

is not the right sort of convolution for the fractional S-transform. Therefore, we define fractional S convolution in terms of basic function [12, p.119]:

$$(f \sharp g)(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(x,y,z) f(x) g(y) dy, \tag{32}$$

usually, translation τ is defined by

$$(\tau_y f)(x) = f * (x,y) = \int_{-\infty}^{\infty} D(x,y,z) f(z) dz. \tag{33}$$

We assume $D(x,y,z) = 1$ for fractional S-transform and define

$$(f \sharp g)^{\sim} = \tilde{f} \cdot \tilde{g}. \tag{34}$$

Theorem 1. *If $[f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$, then the sequence $\{\tilde{f}_n\}$, $n = 1, 2, \ldots, \infty$ converges in $\mathscr{D}'$. Moreover, if $[f_n/\varphi_n] = [g_n/\psi_n]$,then $\{\tilde{f}_n\}$ and $\{\tilde{g}_n\}$ converges to the same limit for the fractional S-transform of tempered Boehmians.*

The proof is similar to that of Theorem 1; see [9].

Definition 1. *By the fractional S-transform $\tilde{F}$ of a tempered Boehmian $F = [f_n/\varphi_n] \in \beta_{\mathscr{I}}$ we mean the limit of the sequence $\{\tilde{f}_n\}$ is in $\mathscr{D}'$. The fractional S-transform is thus a mapping from $\beta_{\mathscr{I}}$ to $\mathscr{D}'$, which is a linear mapping.*

Theorem 2. *Let $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$ and $G = [g_n/\psi_n] \in \mathscr{B}_S$.*
Then (i) $(\tilde{G})$ is an infinitely differentiable function
(ii) $[F \sharp G]^{\sim} = \tilde{F}\tilde{G}$
and (iii) $(\tilde{F}) \cdot (\tilde{\varphi}_n) = (\tilde{f}_n)$, $\forall n \in \mathbb{N}$

Proof. (i) Let $G = [g_n/\gamma_n] \in \mathscr{B}_S$ and let U be the bounded open subset of $\mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $\{\tilde{\gamma}_n\} > 0$ on U . We have, thus

$$[\tilde{G}] = \lim_{n\to\infty} \{\tilde{g}_n\} = \lim_{n\to\infty} \frac{\{\tilde{g}_n\}\{\tilde{\gamma}_p\}}{\{\tilde{\gamma}_p\}}$$

$$= \lim_{n\to\infty} \frac{\{\tilde{g}_p\}\{\tilde{\gamma}_n\}}{\{\tilde{\gamma}_p\}} = \frac{\{\tilde{g}_p\}}{\{\tilde{\gamma}_p\}} \lim_{n\to\infty} \{\tilde{\gamma}_n\}$$

$$= \frac{\{\tilde{g}_p\}}{\{\tilde{\gamma}_p\}} \quad \text{on } U\ .$$

Since $\{\tilde{g}_p\}, \{\tilde{\gamma}_p\} \in S$ and $\{\tilde{\gamma}_p\} > 0$ on U, thus $\{\tilde{G}\}$ is an infinitely differentiable function on U.

(ii) Let $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$ and $G = [g_n/\gamma_n] \in \mathscr{B}_S$. If $\varphi \in \mathscr{D}$, then there exists $p \in \mathbb{N}$ such that $\{\tilde{\gamma}_p\} > 0$ on the support of φ. We have

$$(F \sharp G)^{\sim}\{\varphi\} = \lim_{n\to\infty} (f_n \sharp g_n)^{\sim}(\varphi)$$

$$= \lim_{n\to\infty} \{(\tilde{f}_n \cdot \tilde{g}_n)\}(\varphi) = \lim_{n\to\infty} (\tilde{f}_n)(\tilde{g}_n, \varphi)$$

$$= \lim_{n\to\infty} (\tilde{f}_n) \left\{ \frac{\tilde{g}_n \cdot \tilde{\gamma}_p}{\tilde{\gamma}_p} \cdot \varphi \right\}, \quad n, p \in \mathbb{N}$$

$$= \lim_{n\to\infty} (\tilde{f}_n) \left\{ \frac{\tilde{g}_p \cdot \tilde{\gamma}_n}{\tilde{\gamma}_p} \cdot \varphi \right\}$$

$$= \lim_{n\to\infty} (\tilde{f}_n) \left\{ \frac{\tilde{g}_p}{\tilde{\gamma}_p} \cdot \varphi(\tilde{\gamma}_n) \right\}$$

$$= \lim_{n\to\infty} (\tilde{f}_n)\{(\tilde{G})\varphi(\tilde{\gamma}_n)\} \quad ; \quad \text{from (i)}$$

$$= (\tilde{G}) \lim_{n\to\infty} \{(\tilde{f}_n)(\tilde{\gamma}_n)\}(\varphi)$$

$$= (\tilde{G}) \lim_{n\to\infty} (f_n \sharp \gamma_n)^{\sim}(\varphi)$$

$$= (\tilde{F} \cdot \tilde{G})(\varphi).$$

The last equality follows from the fact that $[f_n/\varphi_n] = [(f_n \sharp \varphi_n)/(\varphi_n \sharp \gamma_n)]$.
(iii) Let $\varphi \in \mathscr{D}$. Then

$$\langle \tilde{F} \cdot \tilde{\varphi}_p, \varphi \rangle = \langle \tilde{F}, \tilde{\varphi}_p \cdot \varphi \rangle, \quad p \in \mathbb{N}$$

$$= \lim_{n\to\infty} \langle \tilde{f}_n, \tilde{\varphi}_p \cdot \varphi \rangle = \lim_{n\to\infty} \langle \tilde{f}_n \cdot \tilde{\varphi}_p, \varphi \rangle$$

$$= \lim_{n\to\infty} \langle \tilde{f}_p \cdot \tilde{\varphi}_n, \varphi \rangle = \lim_{n\to\infty} \langle \tilde{f}_p, \tilde{\varphi}_n \cdot \varphi \rangle$$

$$\text{i.e.} \qquad = \langle \tilde{f}_p, \varphi \rangle$$

$$\text{Hence} \qquad \tilde{F} \cdot \tilde{\varphi}_p = \tilde{f}_p \, .$$

Thus, the fractional S-transform of an arbitrary distribution can be defined as a tempered Boehmian. The theorem is, therefore, completely proved.

Theorem 3. *A distribution f is the fractional S-transform of tempered Boehmian if and only if there exists a delta sequence $\{\delta_n\}$ such that $\{f(\tilde{\delta}_n)\}^{\vee}$ is a tempered distribution for every $n \in \mathbb{N}$.*

Proof. Let $F = [f_n/\varphi_n] \in \mathscr{B}_{\mathscr{I}}$ and $f = \{\tilde{F}\}$. Then $f\{\tilde{\varphi}_n\} = \{\tilde{F}\}\{\tilde{\varphi}_n\}$. Thus, $f(\tilde{\varphi}_n)$ is a tempered distribution. Now let $f \in \mathscr{D}'$, and (δ_n) be a delta sequence such that $f(\tilde{\delta}_n)$ is tempered distribution for every $n \in \mathbb{N}$. We define

$$F = \left[\frac{\{f(\tilde{\delta}_n)\}^{\vee} \sharp \delta_n}{(\delta_n \sharp \delta_n)} \right] \tag{35}$$

where $\{f(\tilde{\delta}_n)\}^{\vee}$ is the inverse fractional S-transform of $\{f(\tilde{\delta}_n)\}$. Since $\{f(\tilde{\delta}_n)\}$ is a tempered distribution, therefore, $\{f(\tilde{\delta}_n)\}^{\vee}$ is also a tempered distribution.

5 UltraBoehmians for the fractional S-transform

Consider a complex valued infinitely differentiable function f, defined on $\mathbb{R}^n$, which satisfies

$$|z|^k |f(z)| \leq C_k e^{a|y|}, \ \forall z, \ \text{Im}(z) = y, \tag{36}$$

where k is a non-negative integer. Such a space of all entire function over the complex z-plane is denoted by Z and the delta sequence is such that

$$(i) \int \varphi_n = 1$$

and

$$(ii) \int |z|^k |\varphi_n(z)| \leq C_k e^{a|y|}, \ \ \forall k \in \mathbb{N}.$$

Considering G as Z', which satisfies (29) and (30) and the properties of the delta sequence, is called the ultraBoehmian. It is denoted by $\mathscr{B}_Z'$, where $f_n \in Z'$.

For $[f_n/\varphi_n] \in \mathscr{B}_{Z'}$, and $f_n \in Z'$, $\forall n \in \mathbb{N}$, F is called the space of entire function of Boehmians, and is denoted by $\mathscr{B}_Z$. If $[f_n/\varphi_n] \in \mathscr{B}_{Z'}$ and $G = [g_n/\psi_n] \in \mathscr{B}_Z$, then the convolution, given by

$$F \sharp G = \left[\frac{(f_n \sharp g_n)}{(\varphi_n \sharp \psi_n)} \right] \in \mathscr{B}_{Z'}, \tag{37}$$

is well defined.

Theorem 4. *If $[f_n/\varphi_n] \in \mathscr{B}_{Z'}$ then the fractional S-transform converges in $\mathscr{D}'$. Moreover, if $[f_n/\varphi_n] = [g_n/\psi_n] \in \mathscr{B}_Z$, then the fractional S-transform converges to the same limit of ultraBoehmians.*

Proof. Let $\varphi \in \mathscr{D}$ (testing function space) and $k \in \mathbb{N}$ be such that $\tilde{\varphi}_k > 0$ on the support of φ. Since $f_n \sharp \varphi_m = f_m \sharp \varphi_n$, $\forall m, n \in \mathbb{N}$, we have $\tilde{f}_n \cdot \tilde{\varphi}_m = \tilde{f}_m \cdot \tilde{\varphi}_n$. Thus,

$$\langle \tilde{f}_n, \varphi \rangle = \left\langle \tilde{f}_n, \frac{\varphi \tilde{\varphi}_k}{\tilde{\varphi}_k} \right\rangle$$

$$= \left\langle \tilde{f}_n \cdot \tilde{\varphi}_k, \frac{\varphi}{\tilde{\varphi}_k} \right\rangle$$

$$= \left\langle \tilde{f}_k \cdot \tilde{\varphi}_n, \frac{\varphi}{\tilde{\varphi}_k} \right\rangle$$

i.e.

$$= \left\langle \tilde{f}_k, \frac{\varphi \tilde{\varphi}_n}{\tilde{\varphi}_k} \right\rangle.$$

Since the sequence $\left\{ \frac{\varphi \tilde{\varphi}_n}{\tilde{\varphi}_k} \right\}$ converges to $\frac{\varphi}{\tilde{\varphi}_k}$ in $\mathscr{D}$, the sequence $\{\tilde{f}_n, \varphi\}$ converges in $\mathscr{D}$. This proves that the sequence $\{\tilde{f}_n\}$ converges in $\mathscr{D}'$ (dual of space $\mathscr{D}$). Now, we consider $[f_n/\varphi_n] = [g_n/\gamma_n] \in \mathscr{B}_{\mathscr{I}}$, and define

$$h_n = \begin{cases} f_{\frac{n+1}{2}} \sharp \gamma_{\frac{n+1}{2}} & \text{, if } n \text{ is odd} \\ g_{\frac{n}{2}} \sharp \varphi_{\frac{n}{2}} & \text{, if } n \text{ is even} \end{cases}.$$

and

$$\delta_n = \begin{cases} \varphi_{\frac{n+1}{2}} \sharp \gamma_{\frac{n+1}{2}} & \text{, if } n \text{ is odd} \\ \varphi_{\frac{n}{2}} \sharp \gamma_{\frac{n}{2}} & \text{, if } n \text{ is even.} \end{cases}$$

Then $[h_n/\delta_n] = [f_n/\varphi_n] = [g_n/\gamma_n]$, and the sequence $\{\tilde{h}_n\}$ converges in $\mathscr{D}'$. Moreover,

$$\lim_{n\to\infty} \langle \tilde{h}_{2n-1}, \varphi \rangle = \lim_{n\to\infty} \langle (f_n \sharp \gamma_n)\tilde{}, \varphi \rangle$$

$$= \lim_{n\to\infty} \langle \tilde{f}_n \cdot \tilde{\gamma}_n, \varphi \rangle = \lim_{n\to\infty} \langle \tilde{f}_n, \tilde{\gamma}_n \varphi \rangle$$

i.e

$$= \lim_{n\to\infty} \langle \tilde{f}_n, \varphi \rangle.$$

Thus, this proves that sequences $\{\tilde{f}_n\}$ and $\{\tilde{h}_n\}$ have the same limit. Similarly, it can be shown that the sequences $\{\tilde{h}_n\}$ and $\{\tilde{g}_n\}$ will have the same limit.

Remark. **(1)** For the classical Fourier transform, $Z \subset S \subset S^{'} \subset Z^{'}$ [21, p. 201], and owing to this fact, the elements of ultraBoehmians will be contained in the tempered Boehmians. Therefore, the fractional S-transform of ultraBoehmians and tempered Boehmians, discussed in this paper, shows that $\mathscr{B}_Z \subset \mathscr{B}_S \subset \mathscr{B}_I \subset \mathscr{B}_{Z'}$ [2].

(2) Integrable Boehmians of fractional S-transform : Let G be the space of complex-valued Lebesgue integrable functions on real line $\mathbb{R}$ and δ be the delta sequence. Then the equivalence class of quotients is called the integrable Boehmians, space of which is denoted by $\mathscr{B}_{L_1}$. One may refer to Mikusinski [8], where Fourier transform for integrable Boehmian is investigated. By using the relation between fraction Fourier transform, fractional S-transform given, one can define and find fractional S-transform for integrable Boehmians.

Acknowledgement

This work is partially supported by the University Grants Commission, Govt. of India under the Dr. D.S. Kothari Post-Doctoral Fellowship, Sanction No. F.4-2/2006(BSR)/13-663/2012(BSR). The author is grateful to the anonymous referee for their valuable comments and suggestions.

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