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Numerical Study of Heavy Meson's Spectra Using the Matrix Numerov's Method

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Abstract: In this paper, we calculate the spectra of bottomonium by applying the non relativistic potential model of heavy mesons. Numerov's method for solving Schrödinger's equation is reintroduced by transforming it into a matrix form. The method gives high accuracy results which are in good agreement with other methods and with recently published experimental data.

Keywords: Heavy mesons; Numerov's method; Schrödinger's equation; bottomonium spectra.

1 Introduction

The properties of mesons are one of the most exciting topics in particle and nuclear physics. Theoretical predictions of the spectra of heavy mesons have been based on potential models which describe the physical environment of mesons. By using a suitable potential model, one can predict good agreement with the observed spectra obtained for the meason's energy states. Selecting an efficient method to solve Schrdinger's equations is the most important task to obtain a correct solution and good agreement with experiments. Schrdinger's equation still a subject for various studies, aims to extend its field of applications and to develop more efficient analytical and approximation methods for obtaining its solutions.

Approximation methods for solutions, such as variation [1], Wentzel-Kramers-Brillouin(WKB) [2] and perturbation [1], have been used extensively but their application range is almost restricted for practical problems. To overcome these restrictions, numerical methods of solutions by shooting [3] or matching wave functions obtained by Numerov's method [4,5] have been developed for atomic structure calculations early on. An additional method of solution is the discretized matrix eigenvalue problem.

The main aim of the paper is to introduce a highly accurate method to solve Schrödinger's equation by using the simplest possible method. Here, we will first discuss the solution of the time-independent 1-D Schrödinger equation which is a problem almost identical to solve the radial wave in three dimensions. We will derive and use Numerov's method, which is a specialized integration formula for numerically integrating differential equations to transform it into a new representation of a matrix form on a discrete lattice depending only on the displacement of grid d and the number of grid N by studying the stability of N and r_{max} , where $d = \frac{r_{max}}{N}$, r_{max} is the maximum value of the distance between the two particles.

2 Theoretical Basis

2.1 Solving Schrödinger's Equation with the Matrix Numerov's Method

Numerov's Method is a specialized integration formula for numerical integration of the differential equation:

$$\psi''(x) = f(x)\psi(x) \tag{1}$$

For the time-independent 1D Schrdinger equation, we have

$$f(x) = \frac{-2m(E-V(x))}{\hbar^2} \tag{2}$$

By using a lattice of x_i points spaced by a distance d, the integration formula is:

$$\psi_{i+1} = \frac{\psi_{i-1}(12-d^2 f_{i-1} - 2\psi_i(5d^2 f_i + 12))}{d^2 f_{i+1} - 12} \tag{3}$$

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Substitute from Eqs. (2 & 3) in Eq. (1) we have,

$$\frac{-2md^2}{\hbar^2}[(E\psi_{i-1}+V_{i-1}\psi_{i-1})+(10E\psi_i+10V_i\psi_i)+ (E\psi_{i+1}+10V_{i+1}\psi_{i+1})] = 12\psi_{i-1}-2\psi_i+\psi_{i+1} \quad (4)$$

where $\psi_i = \psi(x_i)$. By re-arranging the above equation, then:

$$\frac{-\hbar^2}{2m}\frac{\psi_{i-1}-2\psi_i+\psi_{i+1}}{d^2}+\frac{(V_{i-1}\psi_{i-1}+10V_i\psi_i+V_{i+1}\psi_{i+1})}{12} = E\frac{(\psi_{i+1}+10\psi_i+\psi_{i+1})}{12} \quad (5)$$

Now, we transform the well-known Numerov's method into a representation of the matrix form on a discrete lattice depending only on the grid number d and the matrix size N. To do thatψwill be represented by a column vector$(\ldots., \psi_{i-1}, \psi_i, \psi_{i+1}, \ldots.)$. Where i run from 1 to N and defines the matrices

$A_{N,N} = \frac{I_{-1}-2I_0+I_1}{d^2}, B_{N,N} = \frac{I_{-1}+10I_0+I_1}{12}, V_N = diag(\ldots, V_{i-1}, V_i, V_{i+1})$ Where $I_{-1}, I_0 and I_1$ represent sub-, main-, and up- diagonal unit matrices respectively.
Hence, Eq. (5) could be transformed into a matrix form as follows

$$\frac{-\hbar^2}{2m}A_{N,N}\psi_i + B_{N,N}V_N\psi_i = E_iB_{N,N}\psi_i \quad (6)$$

Multiplying by $B_{N,N}^{-1}$, we get:

$$\frac{-\hbar^2}{2m}A_{N,N}B_{N,N}^{-1}\psi_i + V_N\psi_i = E_i\psi_i \quad (7)$$

Suggesting that this particle is compound and it consists of two smaller particles (meson consist of two quarks), then the reduced mass in the non-relativistic model can be identified as:

$$\mu = \frac{m_q m_{\bar{q}}}{m_q+m_{\bar{q}}}$$

where $m_q = m_{\bar{q}}$ is the mass of quark and anti-quark for Quarkonium system, the last equation reads:

$$\frac{-\hbar^2}{2\mu}A_{N,N}B_{N,N}^{-1}\psi_i + V_N\psi_i = E_i\psi_i \quad (8)$$

For the 3D radial Schrdinger equation, Eq. (2) reads

$$f(r) = \frac{-2\mu(E-V_N(r))}{\hbar^2}+\frac{l(l+1)}{r^2} \quad (9)$$

Then, Eq.(8) could be written as:

$$\frac{-\hbar^2}{2\mu}A_{N,N}B_{N,N}^{-1}\psi_i + [V_N(r)+\frac{l(l+1)}{r^2}]\psi_i = E_i\psi_i \quad (10)$$

By considering the natural units $\hbar = c = 1$ then,

$$\frac{-1}{2\mu}A_{N,N}B_{N,N}^{-1}\psi_i + [V_N(r)+\frac{l(l+1)}{r^2}]\psi_i = E_i\psi_i \quad (11)$$

The first term is Matrix Numerov's representation of the kinetic energy operator and the second is Matrix Numerov's representation of the potential energy operator.

2.2 The Potential Model of Bottomonium Mesons

The potential model used in solving Eq. (10) can be written as in [6,7]:

$$V_N(r) = \frac{l(l+1)}{2\mu r^2} - \frac{4\alpha_s}{3} + br + \frac{32\pi\alpha_s\delta(r)S_bS_{\bar{b}}}{9m_b} + \frac{1}{m_b^2}[(\frac{2\alpha_s}{r^3}-\frac{b}{2r})\mathbf{L}.\mathbf{S}+\frac{4\alpha_s}{r^3}T]. \quad (12)$$

where $S_bS_{\bar{b}} = \frac{s(s-1)}{2}-\frac{3}{4}$ and μ are the reduced mass of the quark and anti-quark,m_b is the mass of the bottom quark, and S is the total spin quantum number of the meson. For the $b\bar{b}$ mesons, the parameters α_s , b, σ, and m_b are taken to be 0.4036, 0.1624 GeV2, 2.4948GeV and 4.8097 GeV, respectively [8]. T is the tensor operator and the spin-orbit operator is [9] diagonal in a $|J,L,S>$ basis with the matrix elements.

$$|\mathbf{L},\mathbf{S}> = \frac{[J(J+1)-l(l+1)-S(S+1)]}{2}$$

3 Numerical Results and Discussion

A non relativistic potential model is used to study the heavy meson spectra by using Numerov's Method. To check out Numerov's method, we started firstly, by checking the stability by changing the value of N. Then, the theoretical spectra of the 1S, 2S, 3S, 4S bottomonium states were extracted and we set the order of the matrix N at $r_{max} = 20fm$. It was obvious that the results were stable when $N \geq 98$ as shown in Fig. (1). Moreover, the theoretical spectra of the 1P, 2P, 3P bottomonium states were calculated and at $r_{max} = 20fm$, it was obvious that the results were stable when $N \geq 71$. These results are shown in Fig. (2). Secondly, the stability of the method was checked out by using different values of r_{max}. The calculation of the theoretical spectra of the 1S, 2S, 3S and 4S bottomonium states and the distance between the quark-anti quark at $N = 100$ made it, is obvious that the results were stable when $r_{max} \geq 9fm$ as shown in Fig. (3). The same procedure was used to calculate the theoretical spectra of the 1P, 2P, 3P bottomonium states. Fig. (4) shows that the results were stable when $r_{max} \geq 9fm$.

According to the figures, the value of $N \cong 200$ and the value of $r_{max} \cong 20 fm$ could be used to give the spectra of the bottominium that consist with the experimental data. This is obvious from the agreement between the theoretical results obtained by using Numerov's Method and the experimental data [10] at these values. The comparison between the experiments and theoretical spectra of other groups [8] to the matrix Numerov's calculations for some of the spectra of botommonium is given in Table ((1)).

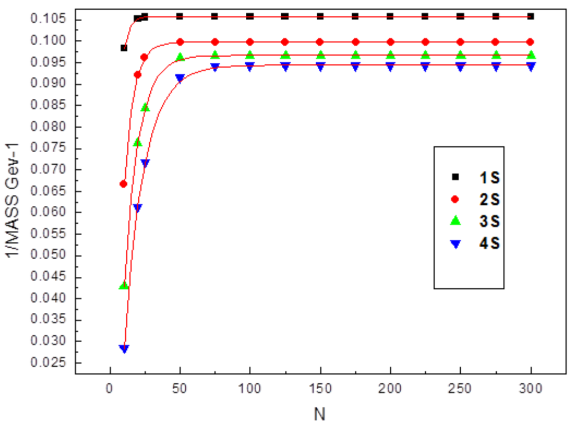

Fig. 1: The relation between the inverse of the theoretical spectra of the 1S, 2S, 3S and 4S botommonium states and the order of matrix N at $r_{max} = 20 fm$.

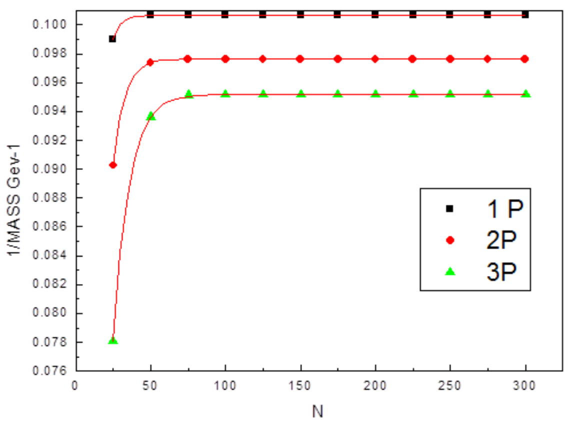

Fig. 2: The relation between the inverse of the theoretical spectra of the 1P, 2P and 3P botommonium states and the order of matrix N at $r_{max} = 20 fm$.

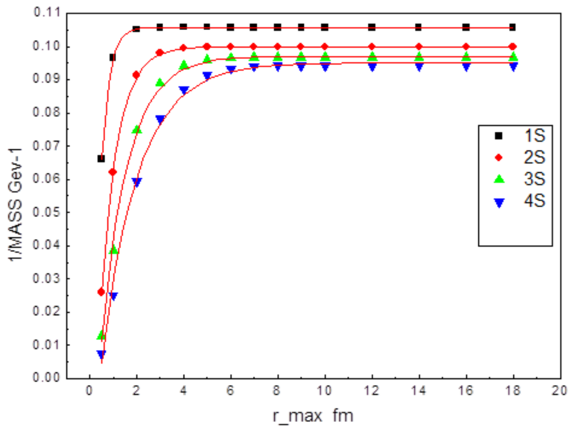

Fig. 3: The theoretical spectra of the 1S, 2S, 3S and 4S bottomonium states versus the distance between the quark- anti quark r_{max} at N = 100.

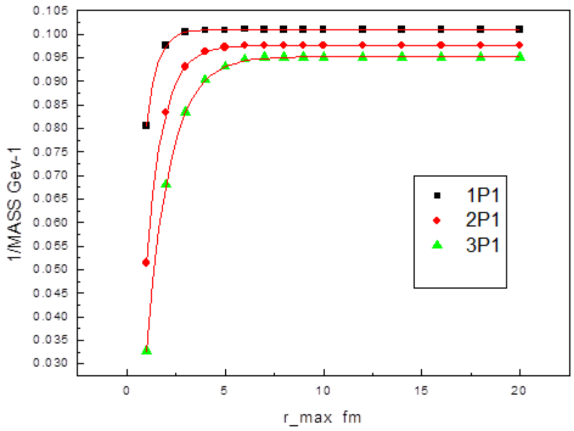

Fig. 4: he theoretical spectrum of 1P, 2P and 3P bottomonium states versus the distance between the quark- anti quark r_{max} at N = 100.

Table 1: The obtained theoretical spectra of bottomonium $b\bar{b}$ compared to other groups.

State	Name	Theo. Spectra Gev	A. Aly [8]	Exp.masses [10]
1^1S_0	$\eta_b\ (1S)$	9.393	9.389	9,390.9
2^1S_0	$\eta_b\ (2S)$	9.996	9.994	
3^1S_0		10.33	10.328	
4^1S_0		10.596	10.593	
1^3S_1	$\Upsilon\ (1S)$	9.458	9.459	9,460.30
2^3S_1	$\Upsilon\ (2S)$	10.017	10.015	10,023.26
3^3S_1	$\Upsilon\ (3S)$	10.345	10.354	10,355.2
4^3S_1	$\Upsilon\ (4S)$	10.607	10.738	10,579.4
1^3P_2	$\chi_{b2}\ (1P)$	9.936	9.935	9,912.21
2^3P_2	$\chi_{b2}\ (2P)$	10.272	10.27	10,268.65
1^3P_1	$\chi_{b1}\ (1P)$	9.904	9.912	9,892.76
2^3P_1	$\chi_b\ (2P)$	10.244	10.251	10,255.46
1^3P_0	$\chi_{b0}\ (1P)$	9.884	9.879	9,859.44
2^3P_0	$\chi_{b0}\ (2P)$	10.234	10.228	10,232.5
1^1P_1	$h_b\ (1P)$	9.92	9.92	
2^1P_1	$h_b\ (2P)$	10.258	10.258	

4 Conclusion

The spectra of bottomonium were calculated by applying the non relativistic potential model of heavy mesons. The calculated spectra show agreement with previous theoretical and experimental spectra. So, we can point out that it is recommended to use Numerov's matrix method to solve the 3D radial Schrödinger equation, for the following reasons: It is easier to use compared to other methods. Also, it saves time, both in implementation and extraction results.

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