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Estimation of Population Median in Two-Occasion Rotation Sampling

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Abstract: The problem of estimation of finite population median at current occasion in two occasion successive sampling has been considered using the additional auxiliary variate which is dynamic over time and is readily available at both the occasions. Properties of the proposed estimators including the optimum replacement strategies have been elaborated. The density functions appearing in the results have been estimated by the method of generalized nearest neighbour density estimator related to kernel estimator. The dominance of the proposed estimators is shown over sample median estimator when there is no matching from previous occasion as well as over the ratio type estimator proposed by Singh et al. [H.P. Singh, R. Tailor, S. Singh and J.M. Kim, *Journal of the Korean Statistical Society*, **36**(4), 543-556 (2007)] for second quantile. The behaviours of the proposed estimators are justified by empirical interpretations and validated by means of simulation study with the help of a natural population.

Keywords: Exponential type, Population median, Skewed distribution, Successive sampling, Bias, Mean square error, Optimum replacement strategy, Dynamic auxiliary information

1 Introduction

Survey often get repeated on many occasions for estimating same characteristics at different point of time technically called repetitive sampling or sampling over successive occasions. It has been given considerable attention by some survey statisticians, when a population is subject to change, a survey carried out on a single occasion will provide information about the characteristic of the surveyed population for the given occasion only, the survey estimates are therefore time specific. Generally, the main objective of successive surveys is to estimate the change with a view to study the effect of the forces acting upon the population as this scheme consists of selecting sample units on different occasions such that some units are common with sample drawn on previous occasions. This retention of a part of sample in periodic surveys provides efficient estimates as compared to other alternatives by eliminating some of the old elements from the sample and adding new elements to the sample each time.

The problem of sampling on two successive occasions was first considered by [1] and latter this idea was extended by [2,3,4,7,13,14,16,21,24] and many others. All the above efforts were devoted to the estimation of population mean or variance on two or more occasion successive sampling.

When a distribution is skewed, when end-values are not known, or when one requires reduced importance to be attached to outliers because they may be measurement errors, median can be used as a measure of central location. Median is defined on ordered one-dimensional data, and is independent of any distance metric so it can be seen as a better indication of central tendency (less susceptible to the exceptionally large value in data) than the arithmetic mean.

Most of the studies related to median have been developed by assuming simple random sampling or its ramification in stratified random sampling considering only the variable of interest without making explicit use of auxiliary variables (see e.g. [5,6,8]). Some of the researchers namely [10,11,12,15,18] etc. have utilized the auxiliary information for the estimation of population median.

Very few researchers namely [19,20] and [23] have proposed estimators for population median in successive sampling.

The work done in [22] has proposed estimator to estimate population median in two-occasion successive sampling assuming that a guess value of the population median is known. In all the above quoted papers, related to the study of

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median, they have assumed that the density functions appearing in the results are known. But, in general being a population parameter they are not known. Hence, using the additional stable auxiliary variable available on both the occasions, [25] and [26] have proposed estimators for population median in successive sampling. In these papers they have also estimated the unknown density functions by using the method of generalized nearest neighbour density estimator related to kernel estimator.

But in practice, one may find that if the gap between two successive occasions is large, the stability character of the auxiliary variate may not sustain. In addition to this, we may also find several other situations where auxiliary variate may not be stable over time, whatever is the duration between two surveys. In such situations the use of dynamic auxiliary variate (changing over time) which are readily available on different occasions, may be efficiently utilized for estimating the population median at current occasions.

Hence, focusing on the above problems in this work we have proposed more effective and relevant estimators of population median at current occasion in two occasion successive sampling using additional auxiliary information which is dynamic over time and is readily available at both the occasions. Properties of the proposed estimators are discussed. The density functions appearing in the results have been estimated by the method of generalized nearest neighbour density estimator related to kernel estimator.

Optimum replacement strategies are elaborated for the proposed estimators. Proposed estimators at optimum conditions are compared with the sample median estimator when there is no matching from the previous occasion as well as with the ratio type estimator proposed by Singh et al. [20] for second quantile, when no additional auxiliary information was used at any occasion. The behaviours of the proposed estimators are justified by empirical interpretations and validated by the means of simulation study with the help of some natural populations.

2 Sample Structure and Notations

Let $U=(U_1,U_2,\dots,U_N)$ be the finite population of N units, which has been sampled over two occasions. It is assumed that size of the population remains unchanged but values of units change over two occasions. The character under study be denoted by $x(y)$ on the first (second) occasions respectively. It is assumed that information on an auxiliary variable whose population medians are known and dynamic over occasions are readily available on both the occasions and positively correlated to x and y respectively. Let z_1 be the auxiliary variable on the first occasion which changes to z_2 on second (current) occasions. Simple random sample (without replacement) of n units is taken on the first occasion. A random subsample of $m=n\lambda$ units is retained for use on the second occasion. Now at the current occasion a simple random sample (without replacement) of $u=(n-m)=n\mu$ units is drawn afresh from the remaining $(N-n)$ units of the population so that the sample size on the second occasion is also n. μ and λ, $(\mu+\lambda=1)$ are the fractions of fresh and matched samples respectively at the second (current) occasion. The following notations are considered for the further use:

$M_x, M_y, M_{z_1}, M_{z_2}$	:	Population median of the variables x,y,z_1 and z_2 respectively.
$\hat{M}_y(u), \hat{M}_{z_1}(u), \hat{M}_{z_2}(u)$	:	Sample median of variables y, z_1 and z_2 based on the sample of size u.
$\hat{M}_x(m), \hat{M}_y(m), \hat{M}_{z_1}(m), \hat{M}_{z_2}(m)$	:	Sample median of variables x,y,z_1 and z_2 based on the sample of size m.
$\hat{M}_x(n), \hat{M}_{z_1}(n), \hat{M}_{z_2}(n)$	:	Sample medians of variables x,z_1 and z_2 based on the sample of size n.
$f_x(M_x), f_y(M_y), f_{z_1}(M_{z_1}), f_{z_2}(M_{z_2})$	:	The marginal densities of variables x,y,z_1 and z_2 respectively.

3 Proposed Estimator T_{ij} $(i,j=1,2)$

To estimate the population median M_y on the current (second) occasion, two sets of estimators have been proposed utilizing the concept of exponential ratio type estimators. First set of estimators $\{T_{1u},T_{2u}\}$ is based on sample of the size $u=n\mu$ drawn afresh on the current (second) occasion and the second set of estimators $\{T_{1m},T_{2m}\}$ is based on sample size $m=n\lambda$ common to the both occasions. The two sets of the proposed estimators are given as

$$T_{1u}=M_{z_2}\left(\frac{\hat{M}_y(u)}{\hat{M}_{z_2}(u)}\right) \tag{1}$$

$$T_{2u}=\hat{M}_y(u)\exp\left(\frac{M_{z_2}-\hat{M}_{z_2}(u)}{M_{z_2}+\hat{M}_{z_2}(u)}\right) \tag{2}$$

$$T_{1m} = \hat{M}_x(n)\left(\frac{\hat{M}_y(m)}{\hat{M}_x(m)}\right)\exp\left(\frac{M_{z_2}-\hat{M}_{z_2}(m)}{M_{z_2}+\hat{M}_{z_2}(m)}\right) \tag{3}$$

$$T_{2m} = \hat{M}_x^*(n)\left(\frac{\hat{M}_y^*(m)}{\hat{M}_x^*(m)}\right) \tag{4}$$

where $\hat{M}_y^*(m) = \hat{M}_y(m)\exp\left(\frac{M_{z_2}-\hat{M}_{z_2}(m)}{M_{z_2}+\hat{M}_{z_2}(m)}\right)$, $\hat{M}_x^*(m) = \hat{M}_x(m)\exp\left(\frac{M_{z_1}-\hat{M}_{z_1}(m)}{M_{z_1}+\hat{M}_{z_1}(m)}\right)$ and $\hat{M}_x^*(n) = \hat{M}_x(n)\exp\left(\frac{M_{z_1}-\hat{M}_{z_1}(n)}{M_{z_1}+\hat{M}_{z_1}(n)}\right)$.

Considering the convex linear combination of the two sets of estimators T_{iu} $(i=1,2)$ and T_{jm} $(j=1,2)$, we have the final estimators of population median M_y on the current occasion as

$$T_{ij} = \phi_{ij}T_{iu} + (1-\phi_{ij})T_{jm}\ ; \quad (i,j=1,2) \tag{5}$$

where ϕ_{ij} $(i,j=1,2)$ are the unknown constants to be determined so as to minimise the mean square error of the estimators T_{ij} $(i,j=1,2)$.

Remark 3.1. For estimating the median on each occasion, the estimators T_{iu} $(i=1,2)$ are suitable, which implies that more belief on T_{iu} could be shown by choosing ϕ_{ij} $(i,j=1,2)$ as 1 (or close to 1), while for estimating the change from occasion to occasion, the estimators T_{jm} $(j=1,2)$ could be more useful so ϕ_{ij} might be chosen as 0 (or close to 0). For asserting both problems simultaneously, the suitable (optimum) choices of ϕ_{ij} are desired.

4 Properties of the Proposed Estimators T_{ij} $(i,j=1,2)$

4.1 Assumptions

The properties of the proposed estimators T_{ij} $(i,j=1,2)$ are derived under the following assumptions:

(i) Population size is sufficiently large (i.e. $N\to\infty$), therefore finite population corrections are ignored.
(ii) As $N\to\infty$, the distribution of the bivariate variable (a,b) where a and $b\in\{x,y,z_1,z_2\}$ and $a\neq b$ approaches a continuous distribution with marginal densities $f_a(\cdot)$ and $f_b(\cdot)$ respectively, (see [11]).
(iii) The marginal densities $f_x(\cdot)$, $f_y(\cdot)$, $f_{z_1}(\cdot)$ and $f_{z_2}(\cdot)$ are positive.
(iv) The sample medians $\hat{M}_y(u)$, $\hat{M}_y(m)$, $\hat{M}_x(m)$, $\hat{M}_x(n)$, $\hat{M}_{z_1}(u)$, $\hat{M}_{z_2}(u)$, $\hat{M}_{z_1}(m)$, $\hat{M}_{z_2}(m)$, $\hat{M}_{z_1}(n)$ and $\hat{M}_{z_2}(n)$ are consistent and asymptotically normal (see [6]).
(v) Following [11], let P_{ab} be the proportion of elements in the population such that $a\le\hat{M}_a$ and $b\le\hat{M}_b$ where a and $b\in\{x,y,z_1,z_2\}$ and $a\neq b$.
(vi) Following large sample approximations are assumed:

$$\hat{M}_y(u)=M_y(1+e_0),\ \hat{M}_y(m)=M_y(1+e_1),\ \hat{M}_x(m)=M_x(1+e_2), \hat{M}_x(n)=M_x(1+e_3),\ \hat{M}_{z_2}(u)=M_{z_2}(1+e_4),$$
$$\hat{M}_{z_2}(m)=M_{z_2}(1+e_5),\ \hat{M}_{z_1}(m)=M_{z_1}(1+e_6)\ \text{ and }\ \hat{M}_{z_1}(n)=M_{z_1}(1+e_7),$$

such that $|e_i|<1\ \forall\ i=0,1,2,3,4,5,6$ and 7.

The values of various related expectations can be seen in [17] and [18].

4.2 Bias and Mean Square Errors of the Estimators T_{ij} $(i,j=1,2)$

The estimators T_{iu} and T_{jm} $(i,j=1,2)$ are ratio, exponential ratio, ratio to exponential ratio and chain type ratio to exponential ratio type in nature respectively. Hence they are biased for population median M_y. Therefore, the final estimators T_{ij} $(i,j=1,2)$ defined in equation (5) are also biased estimators of M_y. Bias $B(\cdot)$ and mean square errors $M(\cdot)$ of the proposed estimators T_{ij} $(i,j=1,2)$ are obtained up to first order of approximations and thus we have following theorems:

Theorem 4.2.1. Bias of the estimators T_{ij} $(i,j=1,2)$ to the first order of approximations are obtained as

$$B(T_{ij}) = \phi_{ij}B(T_{iu}) + (1-\phi_{ij})B(T_{jm})\ ; \quad (i,j=1,2) \tag{6}$$

where

$$B(T_{1u}) = \frac{1}{u}\left\{\frac{[f_{z_2}(M_{z_2})]^{-2}M_y}{4M_{z_2}^2} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}}{4M_{z_2}}\right\} \tag{7}$$

$$B(T_{2u}) = \frac{1}{u}\left\{\frac{3[f_{z_2}(M_{z_2})]^{-2}M_y}{32M_{z_2}^2} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}}{8M_{z_2}}\right\} \tag{8}$$

$$B(T_{1m}) = \frac{1}{m}\left\{\frac{[f_x(M_x)]^{-2}M_y}{4M_x^2} - \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}}{4M_x} + \frac{3[f_{z_2}(M_{z_2})]^{-2}M_y}{32M_{z_2}^2} + \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{8M_xM_{z_2}} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}}{8M_{z_2}}\right\} + \frac{1}{n}\left\{\frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}}{4M_x} - \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{8M_xM_{z_2}} - \frac{[f_x(M_x)]^{-2}M_y}{4M_x^2}\right\} \tag{9}$$

$$B(T_{2m}) = \frac{1}{m}\left\{\frac{[f_x(M_x)]^{-2}M_y}{4M_x^2} + \frac{3[f_{z_2}(M_{z_2})]^{-2}M_y}{32M_{z_2}^2} - \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}}{4M_x} - \frac{(4P_{xz_1}-1)[f_x(M_x)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y}{8M_xM_{z_1}} + \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{8M_xM_{z_2}} + \frac{(4P_{yz_1}-1)[f_y(M_y)]^{-1}[f_{z_1}(M_{z_1})]^{-1}}{8M_{z_1}} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}}{8M_{z_2}} - \frac{(4P_{z_1z_2}-1)[f_{z_1}(M_{z_1})]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{16M_{z_1}M_{z_2}} - \frac{[f_{z_1}(M_{z_1})]^{-2}M_y}{32M_{z_1}^2}\right\} + \frac{1}{n}\left\{\frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}}{4M_x} + \frac{(4P_{xz_1}-1)[f_x(M_x)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y}{8M_xM_{z_1}} - \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{8M_xM_{z_2}} - \frac{(4P_{yz_1}-1)[f_y(M_y)]^{-1}[f_{z_1}(M_{z_1})]^{-1}}{8M_{z_1}} + \frac{(4P_{z_1z_2}-1)[f_{z_1}(M_{z_1})]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{16M_{z_1}M_{z_2.}} - \frac{[f_x(M_x)]^{-2}M_y}{4M_x^2} + \frac{[f_{z_1}(M_{z_1})]^{-2}M_y}{32M_{z_1}^2}\right\} \tag{10}$$

Proof The bias of the estimators T_{ij} $(i,j=1,2)$ are given by

$$B(T_{ij}) = E[T_{ij}-M_y] = \phi_{ij}B(T_{iu}) + (1-\phi_{ij})B(T_{jm})$$

where $B(T_{iu}) = E[T_{iu}-M_y]$ and $B(T_{jm}) = E[T_{jm}-M_y]$.

Using large sample approximations assumed in Section 4.1 and retaining terms upto the first order of approximations, the expression for $B(T_{iu})$ and $B(T_{jm})$ are obtained as in equations (7)-(10) and hence the expression for bias of the estimators T_{ij} $(i,j=1,2)$ are obtained as in equation (6).

Theorem 4.2.2. Mean square errors of the estimators T_{ij} $(i,j=1,2)$ to the first order of approximations are obtained as

$$M(T_{ij}) = \phi_{ij}^2 M(T_{iu}) + (1-\phi_{ij})^2 M(T_{jm}) + 2\phi_{ij}(1-\phi_{ij})\mathrm{cov}(T_{iu},T_{jm})\,; \quad (i,j=1,2) \tag{11}$$

where

$$M(T_{1u}) = \frac{1}{u}A_1 \tag{12}$$

$$M(T_{2u}) = \frac{1}{u}A_4 \tag{13}$$

$$M(T_{1m}) = \frac{1}{m}A_2 + \frac{1}{n}A_3 \tag{14}$$

$$M(T_{2m}) = \frac{1}{m}A_5 + \frac{1}{n}A_6 \tag{15}$$

$$A_1 = \left\{\frac{[f_y(M_y)]^{-2}}{4} + \frac{[f_{z_2}(M_{z_2})]^{-2}M_y^2}{4M_{z_2}^2} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{2M_{z_2}}\right\},$$

$$A_2 = \left\{ \frac{[f_y(M_y)]^{-2}}{4} + \frac{[f_x(M_x)]^{-2}M_y^2}{4M_x^2} + \frac{[f_{z_2}(M_{z_2})]^{-2}M_y^2}{16M_{z_2}^2} - \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}M_y}{2M_x} \right.$$
$$\left. - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{4M_{z_2}} + \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y^2}{4M_xM_{z_2}} \right\},$$

$$A_3 = \left\{ \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}M_y}{2M_x} - \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_z(M_{z_2})]^{-1}M_y^2}{4M_xM_{z_2}} - \frac{[f_x(M_x)]^{-2}M_y^2}{4M_x^2} \right\},$$

$$A_4 = \left\{ \frac{[f_y(M_y)]^{-2}}{4} + \frac{[f_{z_2}(M_{z_2})]^{-2}M_y^2}{16M_{z_2}^2} - \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{4M_{z_2}} \right\},$$

$$A_5 = \left\{ \frac{[f_y(M_y)]^{-2}}{4} + \frac{[f_x(M_x)]^{-2}M_y^2}{4M_x^2} + \frac{[f_{z_2}(M_{z_2})]^{-2}M_y^2}{16M_{z_2}^2} + \frac{[f_{z_1}(M_{z_1})]^{-2}M_y^2}{16M_{z_1}^2} \right.$$
$$- \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}M_y}{2M_x} + \frac{(4P_{yz_1}-1)[f_y(M_y)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y}{4M_{z_1}}$$
$$- \frac{(4P_{yz_2}-1)[f_y(M_y)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y}{4M_{z_2}} - \frac{(4P_{xz_1}-1)[f_x(M_x)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y^2}{4M_xM_{z_1}}$$
$$\left. + \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y^2}{4M_xM_{z_2}} - \frac{(4P_{z_1z_2}-1)[f_{z_1}(M_{z_1})]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y^2}{8M_{z_1}M_{z_2}} \right\}$$

and

$$A_6 = \left\{ \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1}M_y}{2M_x} - \frac{[f_x(M_x)]^{-2}M_y^2}{4M_x^2} - \frac{[f_{z_1}(M_{z_1})]^{-2}M_y^2}{16M_{z_1}^2} \right.$$
$$- \frac{(4P_{yz_1}-1)[f_y(M_y)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y}{4M_{z_1}} + \frac{(4P_{xz_1}-1)[f_x(M_x)]^{-1}[f_{z_1}(M_{z_1})]^{-1}M_y^2}{4M_xM_{z_1}}$$
$$\left. - \frac{(4P_{xz_2}-1)[f_x(M_x)]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y^2}{4M_xM_{z_2}} + \frac{(4P_{z_1z_2}-1)[f_{z_1}(M_{z_1})]^{-1}[f_{z_2}(M_{z_2})]^{-1}M_y^2}{8M_{z_1}M_{z_2}} \right\}$$

Proof The mean square errors of the estimators $T_{ij}$are given by

$$M(T_{ij}) = E[T_{ij}-M_y]^2 = E[\phi_{ij}(T_{iu}-M_y)+(1-\phi_{ij})\{T_{jm}-M_y\}]^2$$
$$= \phi_{ij}^2 M(T_{iu}) + (1-\phi_{ij})^2 M[T_{jm}] + 2\phi_{ij}(1-\phi_{ij})\mathrm{cov}(T_{iu},T_{jm})$$

where $M(T_{iu}) = E[T_{iu}-M_y]^2$ and $M[T_{jm}] = E[T_{jm}-M_y]^2$; $(i,j=1,2)$.

The estimators T_{iu} and T_{jm} are based on two independent samples of sizes u and m respectively, hence $\mathrm{cov}(T_{iu},T_{jm}) = 0$; $(i,j=1,2)$. Using large sample approximations assumed in Section 4.1 and retaining terms upto the first order of approximations, the expression for $M(T_{iu})$ and $M(T_{jm})$ are obtained as given in equations (12)-(15) and hence the expressions for mean square error of estimators T_{ij} $(i,j=1,2)$ are obtained as in equation (11).

Remark 4.2.1. The mean square errors of the estimators T_{ij} $(i,j=1,2)$ in equation (11) depend on the population parameters P_{xy}, P_{yz_1}, P_{yz_2}, P_{xz_1}, P_{xz_2}, $P_{z_1z_2}$, $f_x(M_x)$, $f_y(M_y)$, $f_{z_1}(M_{z_1})$ and $f_{z_2}(M_{z_2})$. If these parameters are known, the properties of proposed estimators can be easily studied. Otherwise, which is the most often situation in practice, the unknown population parameters are replaced by their sample estimates. The population proportions P_{xy}, P_{yz_1}, P_{yz_2}, P_{xz_1}, P_{xz_2} and $P_{z_1z_2}$ can be replaced by the sample estimate $\hat{P}_{xy}$, $\hat{P}_{xz_1}$, $\hat{P}_{xz_2}$, $\hat{P}_{yz_1}$, $\hat{P}_{yz_2}$ and $\hat{P}_{z_1z_2}$ and the marginal densities f_y, (M_y), $f_x(M_x)$, $f_{z_1}(M_{z_1})$ and $f_{z_2}(M_{z_2})$ can be substituted by their kernel estimator or nearest neighbour density estimator or generalized nearest neighbour density estimator related to the kernel estimator [Silverman (1986)]. Here, the marginal densities $f_y(M_y)$, $f_x(M_x)$, $f_{z_1}(M_{z_1})$ and $f_{z_2}(M_{z_2})$ are replaced by $\hat{f}_y(\hat{M}_y(m))$, $\hat{f}_x(\hat{M}_x(n))$, $\hat{f}_{z_1}(\hat{M}_{z_1}(n))$ and $\hat{f}_{z_2}(\hat{M}_{z_2}(n))$ respectively, which are obtained by method of generalized nearest neighbour density estimator related to kernel estimator.

To estimate $f_y(M_y)$, $f_x(M_x)$, $f_{z_1}(M_{z_1})$ and $f_{z_2}(M_{z_2})$, by generalized nearest neighbour density estimator related to the kernel estimator, following procedure has been adopted:

Choose an integer $h \approx n^{\frac{1}{2}}$ and define the distance $\delta(x_1,x_2)$ between two points on the line to be $|x_1-x_2|$.

For $\hat{M}_x(n)$, define $\delta_1(\hat{M}_x(n)) \leq \delta_2(\hat{M}_x(n)) \leq \cdots \leq \delta_n(\hat{M}_x(n))$ to be the distances, arranged in ascending order, from $\hat{M}_x(n)$ to the points of the sample.

The generalized nearest neighbour density estimate is defined by

$$\hat{f}(\hat{M}_x(n)) = \frac{1}{n\delta_h(\hat{M}_x(n))} \sum_{i=1}^{n} K\left[\frac{\hat{M}_x(n) - x_i}{\delta_h(\hat{M}_x(n))}\right]$$

where the kernel function K, satisfies the condition $\int_{-\infty}^{\infty} K(x)dx = 1$.

Here, the kernel function is chosen as Gaussian Kernel given by $K(x) = \frac{1}{2\pi}e^{-(\frac{1}{2}x^2)}$.

The estimate of $f_y(M_y)$, $f_{z_1}(M_{z_1})$ and $f_{z_2}(M_{z_2})$ can be obtained by the above explained procedure in similar manner.

5 Minimum Mean Square Errors of the Proposed Estimators T_{ij} $(i, j = 1, 2)$

Since the mean square errors of the estimators T_{ij} $(i, j = 1, 2)$ given in equation (11) are the functions of unknown constants ϕ_{ij} $(i, j = 1, 2)$, therefore, they are minimized with respect to ϕ_{ij} and subsequently the optimum values of ϕ_{ij} are obtained as

$$\phi_{ij_{opt.}} = \frac{M(T_{jm})}{M(T_{iu}) + M(T_{jm})}\ ; \quad (i, j = 1, 2) \tag{16}$$

Now substituting the values of $\phi_{ij_{opt.}}$ in equation (11), we obtain the optimum mean square errors of the estimators T_{ij} $(i, j = 1, 2)$ as

$$M(T_{ij})_{opt.} = \frac{M(T_{iu}) \cdot M(T_{jm})}{M(T_{iu}) + M(T_{jm})}\ ; \quad (i, j = 1, 2) \tag{17}$$

Further, substituting the values of the mean square error of the estimators defined in equation (12) to equation (15) in equation (16) and (17), the simplified values $\phi_{ij_{opt.}}$ and $M(T_{ij})_{opt.}$ are obtained as

$$\phi_{11_{opt.}} = \frac{\mu_{11}[\mu_{11}A_3 - (A_2 + A_3)]}{[\mu_{11}^2 A_3 - \mu_{11}(A_2 + A_3 - A_1) - A_1]} \tag{18}$$

$$\phi_{12_{opt.}} = \frac{\mu_{12}[\mu_{12}A_6 - (A_5 + A_6)]}{[\mu_{12}^2 A_6 - \mu_{12}(A_5 + A_6 - A_1) - A_1]} \tag{19}$$

$$\phi_{21_{opt.}} = \frac{\mu_{21}[\mu_{21}A_3 - (A_2 + A_3)]}{[\mu_{21}^2 A_3 - \mu_{21}(A_2 + A_3 - A_4) - A_4]} \tag{20}$$

$$\phi_{22_{opt.}} = \frac{\mu_{22}[\mu_{22}A_6 - (A_5 + A_6)]}{[\mu_{22}^2 A_6 - \mu_{22}(A_5 + A_6 - A_4) - A_4]} \tag{21}$$

$$M(T_{11})_{opt.} = \frac{1}{n}\frac{[\mu_{11}C_1 - C_2]}{[\mu_{11}^2 A_3 - \mu_{11}C_3 - A_1]} \tag{22}$$

$$M(T_{12})_{opt.} = \frac{1}{n}\frac{[\mu_{12}C_4 - C_5]}{[\mu_{12}^2 A_6 - \mu_{12}C_6 - A_1]} \tag{23}$$

$$M(T_{21})_{opt.} = \frac{1}{n}\frac{[\mu_{21}C_7 - C_8]}{[\mu_{21}^2 A_3 - \mu_{21}C_9 - A_4]} \tag{24}$$

$$M(T_{22})_{opt.} = \frac{1}{n}\frac{[\mu_{22}C_{10} - C_{11}]}{[\mu_{22}^2 A_6 - \mu_{22}C_{12} - A_4]} \tag{25}$$

where

$C_1 = A_1A_3$, $C_2 = A_1A_2 + A_1A_3$, $C_3 = A_2 + A_3 - A_1$, $C_4 = A_1A_6$,
$C_5 = A_1A_5 + A_1A_6$, $C_6 = A_5 + A_6 - A_1$, $C_7 = A_3A_4$, $C_8 = A_2A_4 + A_3A_4$,
$C_9 = A_2 + A_3 - A_4$, $C_{10} = A_4A_6$, $C_{11} = A_4A_5 + A_4A_6$, $C_{12} = A_5 + A_6 - A_4$ and μ_{ij} $(i, j = 1, 2)$

are the fractions of the sample drawn afresh at the current(second) occasion.

Remark 5.1. $M(T_{ij})_{opt.}$ derived in equation (22)-(25) are the functions of μ_{ij} $(i, j = 1, 2)$. To estimate the population median on each occasion the better choices of μ_{ij} $(i, j = 1, 2)$ are 1 (case of no matching); however, to estimate the change in median from one occasion to other, μ_{ij} $(i, j = 1, 2)$ should be 0 (case of complete matching). But intuition suggests that an optimum choices of μ_{ij} $(i, j = 1, 2)$ are desired to devise the amicable strategy for both the problems simultaneously.

6 Optimum Replacement Strategies for the Estimators $T_{ij}\ (i,j=1,2)$

The key design parameter affecting the estimates of change is the overlap between successive samples. Maintaining high overlap between repeats of a survey is operationally convenient, since many sampled units have been located and have some experience in the survey. Hence to decide about the optimum value of $\mu_{ij}\ (i,j=1,2)$ (fractions of samples to be drawn afresh on current occasion) so that M_y may be estimated with maximum precision and minimum cost, we minimize the mean square errors $M(T_{ij})_{opt.}\ (i,j=1,2)$ in equation (22) to (25) with respect to $\mu_{ij}\ (i,j=1,2)$ respectively.

The optimum value of $\mu_{ij}\ (i,j=1,2)$ so obtained is one of the two roots given by

$$\hat{\mu}_{11}=\frac{D_2\pm\sqrt{D_2^2-D_1D_3}}{D_1} \tag{26}$$

$$\hat{\mu}_{12}=\frac{D_5\pm\sqrt{D_5^2-D_4D_6}}{D_4} \tag{27}$$

$$\hat{\mu}_{21}=\frac{D_8\pm\sqrt{D_8^2-D_7D_9}}{D_7} \tag{28}$$

$$\hat{\mu}_{22}=\frac{D_{11}\pm\sqrt{D_{11}^2-D_{10}D_{12}}}{D_{10}} \tag{29}$$

where

$$D_1=A_3C_1,\ D_2=A_3C_2,\ D_3=A_1C_1+C_2C_3,\ D_4=A_6C_4,\ D_5=A_6C_5,\ D_6=A_1C_4+C_5C_6,$$
$$D_7=A_3C_7,\ D_8=A_3C_8,\ D_9=A_4C_7+C_8C_9,\ D_{10}=A_6C_{10},\ D_{11}=A_6C_{11}\ \text{and}\ D_{12}=A_4C_{10}+C_{11}C_{12}.$$

The real values of $\hat{\mu}_{ij}\ (i,j=1,2)$ exist, iff $D_2^2-D_1D_3\geq 0$, $D_5^2-D_4D_6\geq 0$, $D_8^2-D_7D_9\geq 0$, and $D_{11}^2-D_{10}D_{12}\geq 0$. For any situation, which satisfies these conditions, two real values of $\hat{\mu}_{ij}\ (i,j=1,2)$ may be possible, hence to choose a value of $\hat{\mu}_{ij}\ (i,j=1,2)$, it should be taken care of that $\hat{\mu}_{ij}\in(0,1)$, all other values of $\hat{\mu}_{ij}\ (i,j=1,2)$ are inadmissible. If both the real values of $\hat{\mu}_{ij}\ (i,j=1,2)$ are admissible, the lowest one will be the best choice as it reduces the total cost of the survey. Substituting the admissible value of $\hat{\mu}_{ij}$ say $\mu_{ij}^{(0)}\ (i,j=1,2)$ from equation (26) to (29) in equation (22) to (25) respectively, we get the optimum values of the mean square errors of the estimators $T_{ij}\ (i,j=1,2)$ with respect to ϕ_{ij} as well as $\hat{\mu}_{ij}\ (i,j=1,2)$ which are given as

$$M(T_{11})^*_{opt.}=\frac{[\mu_{11}^{(0)}C_1-C_2]}{n[\mu_{11}^{(0)2}A_3-\mu_{11}^{(0)}C_3-A_1]} \tag{30}$$

$$M(T_{12})^*_{opt.}=\frac{[\mu_{12}^{(0)}C_4-C_5]}{n[\mu_{12}^{(0)2}A_6-\mu_{12}^{(0)}C_6-A_1]} \tag{31}$$

$$M(T_{21})^*_{opt.}=\frac{[\mu_{21}^{(0)}C_7-C_8]}{n[\mu_{21}^{(0)2}A_3-\mu_{21}^{(0)}C_9-A_4]} \tag{32}$$

$$M(T_{22})^*_{opt.}=\frac{[\mu_{22}^{(0)}C_{10}-C_{11}]}{n[\mu_{22}^{(0)2}A_6-\mu_{22}^{(0)}C_{12}-A_4]} \tag{33}$$

7 Efficiency Comparison

To evaluate the performance of the proposed estimators, the estimators $T_{ij}\ (i,j=1,2)$ at optimum conditions are compared with respect to (i) the sample median estimator $\hat{M}_y(n)$, when there is no matching from previous occasion and (ii) the ratio type estimator Δ proposed by Singh et al. [20] for second quantile, where no additional auxiliary information was used at any occasion and is given by

$$\Delta=\psi\hat{M}_y(u)+(1-\psi)\hat{M}_x(n)\left(\frac{\hat{M}_y(m)}{\hat{M}_x(m)}\right) \tag{34}$$

where ψ is an unknown constant to be determined so as to minimise the mean square error of the estimator Δ. Since, $\hat{M}_y(n)$ is unbiased and Δ is biased for population median, so variance of $\hat{M}_y(n)$ and mean square error of the estimator Δ at optimum conditions are given as

$$V(\hat{M}_y(n)) = \frac{1}{n}\frac{[f_y(M_y)]^{-2}}{4} \tag{35}$$

and

$$M(\Delta)^*_{opt.} = \frac{[\mu_\Delta J_1 - J_2]}{n[\mu_\Delta^2 I_3 - \mu_\Delta J_3 - I_1]} \tag{36}$$

where

$\mu_\Delta = \frac{H_2 \pm \sqrt{H_2^2 - H_1 H_3}}{H_1}$, $H_1 = J_1 I_3$, $H_2 = J_2 I_3$, $H_3 = I_1 J_1 + J_2 J_3$, $J_1 = I_1 I_3$, $J_2 = I_1(I_2 + I_3)$,
$J_3 = I_2 + I_3 - I_1$, $I_1 = \frac{[f_y(M_y)]^{-2}}{4}$, $I_2 = \frac{[f_y(M_y)]^{-2}}{4} + \frac{[f_x(M_x)]^{-2} M_y^2}{4M_x^2} - \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1} M_y}{2M_x}$
and $I_3 = \frac{(4P_{xy}-1)[f_x(M_x)]^{-1}[f_y(M_y)]^{-1} M_y}{2M_x} - \frac{[f_x(M_x)]^{-2} M_y^2}{4M_x^2}$.

The percent relative efficiencies $E_{ij}^{(1)}$ and $E_{ij}^{(2)}$ of the estimators T_{ij} $(i,j=1,2)$ (under their respective optimum conditions) with respect to $\hat{M}_y(n)$ and Δ are respectively given by

$$E_{ij}^{(1)} = \frac{V(\hat{M}_y(n))}{M(T_{ij})^*_{opt.}} \times 100 \text{ and } E_{ij}^{(2)} = \frac{M(\Delta)^*_{opt.}}{M(T_{ij})^*_{opt.}} \times 100\,; \quad (i,j=1,2) \tag{37}$$

8 Empirical Illustrations and Monte Carlo Simulation

Empirical validation can be carried out by Monte Carlo Simulation. Real life situation of completely known finite population has been considered.

Population Source (Free access to the data by Statistical Abstracts of the United States). The population comprise of $N = 51$states of United States. Let x_i be the Percentage of Advanced Degree Holders or More during 1990 in the i^{th} state of U. S., y_i represent the Percentage of Advanced Degree Holders or More during 2009 in the i^{th} state of U.S., z_{1_i} denote Percentage of Bachelor Degree Holders or More during 1990 in the i^{th}state of U.S. and z_{2_i} denote the Percentage of Bachelor Degree Holders or More during 2009 in the i^{th} state of U.S. and The data are presented in Figure 1.

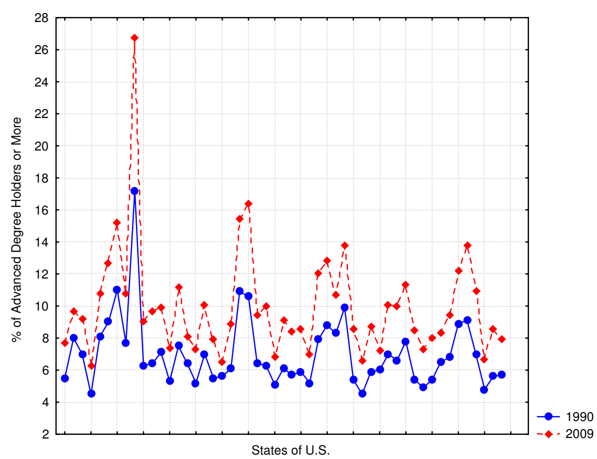

Fig. 1 Percentage Advanced Degree Holders or More during 1990 and 2009 versus different states of United States.

For the considered population, the optimum values of μ_{ij} $(i,j=1,2)$ defined in equation (26) to (29) and percent relative efficiencies $E_{ij}^{(1)}$ and $E_{ij}^{(2)}$ defined in equation (37) of T_{ij} $(i,j=1,2)$ (under their respective optimality conditions) with respect to $\hat{M}_y(n)$ and Δ have been computed and are presented in Table 2.

To validate the empirical results quoted in Table 2, Monte Carlo simulation have also been performed. 5000 samples of size $n = 20$ states are selected using simple random sampling without replacement in the year 1990. The sample medians $\hat{M}_{x|k}(n)$ and $\hat{M}_{z_1|k}(n)$, $k = 1,2,\cdots,5000$ are computed. From each one of the selected samples, $m = 17$ states are retained and new u=3 states are selected out of $N - n = 51 - 20 = 31$ states of U.S. using simple random sampling without replacement in the year 2009. From the m units retained in the sample at the current occasion, the sample medians $\hat{M}_{x|k}(m)$, $\hat{M}_{y|k}(m)$, $\hat{M}_{z_1|k}(m)$ and $\hat{M}_{z_2|k}(m)$, $k = 1,2,\cdots,5000$ are computed. From the new unmatched units selected on the current occasion the sample medians $\hat{M}_{y|k}(u)$ and $\hat{M}_{z_2|k}(u)$, $k = 1,2,\cdots,5000$ are also calculated. The parameters ϕ and ψ are selected between 0.1 and 0.9 with a step of 0.1.

The percent relative efficiencies of the proposed estimators T_{ij} with respect to $\hat{M}_y(n)$ and Δ are obtained as a result of above simulation and are respectively given as:

$$E_{ij}(1) = \frac{\sum_{k=1}^{5000}[\hat{M}_{y|k}(n) - M_y]^2}{\sum_{k=1}^{5000}[T_{ijk} - M_y]^2} \times 100 \quad \text{and} \quad E_{ij}(2) = \frac{\sum_{k=1}^{5000}[\Delta_k - M_y]^2}{\sum_{k=1}^{5000}[T_{ijk} - M_y]^2} \times 100\,; \quad (i,j = 1,2).$$

For better analysis, the above simulation experiments were repeated for different choices of μ. For convenience the different choices of μ are considered as different sets for the considered Population which is shown below:

Set I : $n = 20$, $\mu = 0.15$, $(m = 17, u = 3)$
Set II : $n = 20$, $\mu = 0.20$, $(m = 16, u = 4)$
Set III : $n = 20$, $\mu = 0.35$, $(m = 13, u = 7)$
Set IV : $n = 20$, $\mu = 0.50$, $(m = 10, u = 10)$

The simulation results obtained are presented in Table 3 to Table 7.

Table 1 Descriptive statistics for the population considered

	% of Advanced Degree Holders or More (1990) (x)	% of Advanced Degree Holders or More (2009) (y)	% of Bachelor's Degree or More (1990) (z_1)	% of Bachelor's Degree or More (2009) (z_2)
Mean	5.7	10.00	20.00	27.40
Median	6.40	7.90	19.30	26.30
Standard deviation	4.70	11.23	16.98	30.46
Kurtosis	8.43	11.04	0.79	2.70
Skewness	2.34	2.69	0.70	1.09
Minimum	5.7	6.30	12.30	17.1
Maximum	17.2	26.7	33.37	48.2
Count	51	51	51	51

9 Mutual Comparison of the Proposed Estimators T_{ij} $(i, j = 1,2)$

The performances of the proposed estimators T_{ij} $(i, j = 1,2)$ have been elaborated empirically as well as through simulation studies in above Section 8 and the results obtained are presented in Table 2 to Table 7. In this section the mutual comparison of the four proposed estimators have been elaborated though different graphs given in Figure 2 to Figure 5.

Table 2 Comparison of the proposed estimators T_{ij} (at optimum conditions) with respect to the estimators $\hat{M}_y(n)$ and Δ (at their respective optimum conditions)

$\mu_{11}^{(0)}$	$*$
$\mu_{12}^{(0)}$	0.8389
$\mu_{21}^{(0)}$	0.5278
$\mu_{22}^{(0)}$	0.5603
$E_{11}^{(1)}$	–
$E_{12}^{(1)}$	200.75
$E_{21}^{(1)}$	155.52
$E_{22}^{(1)}$	165.02
$E_{11}^{(2)}$	–
$E_{12}^{(2)}$	171.79
$E_{21}^{(2)}$	133.08
$E_{22}^{(2)}$	141.21

Note. '$*$' indicates that $\mu_{ij}^{(0)}$; $(i,j=1,2)$ do not exist.

Table 3 Monte Carlo Simulation results when the proposed estimator T_{ij} is compared to $\hat{M}_y(n)$.

ϕ	SET	I	II	III	IV
0.1	$E_{11}(1)$	157.64	136.42	139.61	104.70
	$E_{12}(1)$	139.56	135.09	144.90	148.55
	$E_{21}(1)$	155.56	137.23	142.14	106.44
	$E_{22}(1)$	137.61	135.68	146.87	151.28
0.2	$E_{11}(1)$	161.27	145.61	148.14	119.39
	$E_{12}(1)$	147.08	144.78	153.24	167.79
	$E_{21}(1)$	161.27	144.25	148.66	121.37
	$E_{22}(1)$	142.42	143.43	152.85	170.72
0.3	$E_{11}(1)$	171.41	152.14	152.99	133.24
	$E_{12}(1)$	152.23	151.62	157.13	185.83
	$E_{21}(1)$	161.68	146.36	147.98	134.36
	$E_{22}(1)$	143.81	145.93	150.68	187.01
0.4	$E_{11}(1)$	172.75	151.39	153.37	146.51
	$E_{12}(1)$	153.29	151.85	157.52	202.15
	$E_{21}(1)$	157.17	138.96	141.53	145.79
	$E_{22}(1)$	140.00	139.43	143.68	199.08
0.5	$E_{11}(1)$	169.68	148.53	148.19	159.08
	$E_{12}(1)$	151.22	148.99	151.80	215.89
	$E_{21}(1)$	148.70	129.43	127.97	154.54
	$E_{22}(1)$	133.39	129.74	129.45	205.79
0.6	$E_{11}(1)$	162.03	140.99	138.28	171.10
	$E_{12}(1)$	145.87	141.54	141.20	227.47
	$E_{21}(1)$	136.36	115.71	112.57	160.04
	$E_{22}(1)$	123.84	116.09	113.30	206.50
0.7	$E_{11}(1)$	154.69	131.88	124.56	179.50
	$E_{12}(1)$	140.61	132.26	126.70	232.64
	$E_{21}(1)$	125.89	103.21	**	160.75
	$E_{22}(1)$	115.62	103.44	**	200.20
0.8	$E_{11}(1)$	144.46	119.52	112.15	182.42
	$E_{12}(1)$	132.79	119.77	113.77	229.90
	$E_{21}(1)$	113.79	**	**	156.07
	$E_{22}(1)$	105.73	**	**	187.95
0.9	$E_{11}(1)$	133.55	107.93	**	180.42
	$E_{12}(1)$	124.30	108.12	**	220.39
	$E_{21}(1)$	102.91	**	**	147.02
	$E_{22}(1)$	**	**	**	171.21

Note. '$**$' indicates no gain.

10 Interpretation of Results

The following interpretation can be drawn from Tables 2-7 and Figure 2-5:

(1) From Table 2, it is observed that

(a) Optimum values $\mu_{12}^{(0)}$, $\mu_{21}^{(0)}$ and $\mu_{22}^{(0)}$ for the estimators T_{12}, T_{21} and T_{22} respectively exist for the considered population which justifies the applicability of the proposed estimators T_{12}, T_{21} and T_{22} at optimum conditions. However, the optimum value $\mu_{11}^{(0)}$ for the estimators T_{11} does not exist for the considered population.

(b) Appreciable gain is observed in terms of precision indicating that the proposed estimators T_{12}, T_{21}, T_{22} (at their respective optimal conditions) are preferable over the estimator $\hat{M}_y(n)$ and (at optimal conditions). This result justifies the use of additional auxiliary information at both occasions which is dynamic over time in two occasion successive sampling.

(c) The values for $E_{11}^{(1)}$ and $E_{11}^{(2)}$ cannot be calculated as optimum value $\mu_{11}^{(0)}$ does not exist but simulation study vindicated in Tables 3-7 magnify the applicability of proposed estimator T_{11} over sample median estimator $\hat{M}_y(n)$ and the estimator Δ.

(2) From Table 3, it can be seen that, when T_{ij} $(i,j=1,2)$ is compared with sample median estimator $\hat{M}_y(n)$

(a) $E_{11}(1)$, $E_{12}(1)$, $E_{21}(1)$, $E_{22}(1)$ first increase and then decrease as ϕ increases for all sets.

(b) For fixed value of ϕ, $E_{11}(1)$ and $E_{21}(1)$ show no fixed behaviour as the value of μ is increased.

(c) $E_{12}(1)$ and $E_{22}(1)$ increase as μ increases.

Table 4 Monte Carlo Simulation results when the proposed estimator T_{11} is compared to the estimator Δ

ϕ	$\psi \rightarrow$ SET ↓	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	I	166.50	164.58	172.38	188.62	228.93	277.81	333.46	422.99	479.54
	II	140.62	135.80	149.73	184.74	244.06	347.10	391.73	556.52	731.76
	III	134.98	126.52	138.99	197.52	258.06	362.92	511.86	668.66	895.18
	IV	116.54	100.35	**	**	**	**	123.84	160.06	211.31
0.2	I	174.84	170.32	179.46	198.15	246.31	293.62	336.28	414.78	507.51
	II	147.81	141.46	157.48	196.01	256.57	350.54	433.21	581.35	756.12
	III	144.43	134.92	152.62	210.36	281.48	393.28	540.90	716.93	913.98
	IV	131.84	104.09	**	**	**	107.11	134.64	183.01	233.14
0.3	I	178.86	174.28	183.33	202.86	246.40	300.03	354.10	434.50	518.11
	II	152.95	145.17	161.03	200.44	263.20	355.79	454.44	603.50	776.33
	III	151.01	140.19	159.32	218.27	293.69	408.04	566.13	715.27	942.02
	IV	148.55	116.42	**	**	100.93	118.78	154.26	199.88	259.18
0.4	I	179.43	175.46	183.29	202.99	248.29	299.47	353.28	432.78	515.88
	II	152.22	145.54	160.58	200.91	265.35	357.71	460.92	610.86	760.58
	III	151.43	139.34	158.79	216.11	295.25	409.66	564.60	742.28	941.09
	IV	163.16	129.19	107.79	100.14	110.62	131.32	170.91	220.06	282.55
0.5	I	175.36	172.12	179.58	199.54	242.51	291.57	345.79	420.66	515.67
	II	149.34	142.89	157.15	197.21	261.35	352.07	452.17	607.09	749.93
	III	145.63	133.76	153.37	206.83	284.12	393.77	537.64	713.06	907.43
	IV	177.53	139.85	116.86	108.35	120.59	143.48	187.29	239.67	310.80
0.6	I	167.19	164.42	172.61	191.70	232.03	278.84	333.16	405.10	492.13
	II	141.98	136.24	149.74	187.96	246.67	333.07	429.17	569.91	709.42
	III	136.23	124.27	143.46	192.75	265.70	368.16	501.30	661.34	848.74
	IV	190.07	149.10	124.74	116.23	128.75	152.66	199.81	257.62	332.90
0.7	I	159.37	155.34	162.37	181.70	219.28	263.40	313.41	387.07	462.57
	II	132.92	125.95	138.83	174.34	229.98	308.49	397.17	528.32	661.19
	III	123.11	112.21	129.17	173.54	240.65	337.10	453.64	604.41	775.34
	IV	199.78	155.47	130.26	121.68	134.63	160.18	209.69	270.21	346.17
0.8	I	148.49	144.04	151.56	169.90	204.19	245.76	292.35	357.20	431.24
	II	120.56	114.86	126.15	160.31	210.25	284.17	360.20	477.84	601.11
	III	110.85	100.36	115.34	154.33	214.13	300.74	403.23	540.34	688.10
	IV	203.38	157.98	132.34	124.54	137.06	162.68	212.57	275.04	352.95
0.9	I	137.22	132.63	139.82	155.98	188.41	224.81	268.21	327.37	397.74
	II	108.66	104.07	114.83	145.37	189.19	255.82	325.72	431.0	544.49
	III	**	**	101.74	13525	187.25	265.51	353.08	470.35	600.77
	IV	201.06	157.02	131.81	123.53	135.94	161.63	211.07	272.07	346.66

Note. '**' indicates no gain.

Table 5 Monte Carlo Simulation results when the proposed estimator T_{12} is compared to the estimator Δ

ϕ	$\psi \rightarrow$ SET ↓	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	I	147.41	145.42	152.84	167.96	205.15	245.85	287.97	379.09	429.76
	II	139.26	130.52	147.81	182.62	247.25	348.99	396.14	552.83	737.42
	III	140.10	125.66	142.88	200.32	266.60	371.37	520.99	675.04	907.49
	IV	165.36	131.09	111.73	105.05	110.30	135.83	168.97	217.91	293.82
0.2	I	154.34	150.08	158.57	176.72	218.11	257.30	298.15	368.28	453.74
	II	146.96	138.61	155.49	196.26	260.63	353.46	438.36	579.23	761.42
	III	149.40	134.87	155.77	213.36	289.19	404.90	556.13	728.97	923.68
	IV	185.30	146.82	124.64	116.52	124.85	149.48	189.38	248.68	324.44
0.3	I	158.85	153.55	162.09	180.36	217.48	263.04	313.32	385.31	458.56
	II	152.43	143.57	159.98	201.01	266.61	357.11	456.77	602.15	780.56
	III	155.10	141.28	162.21	220.98	301.04	420.15	579.05	770.08	952.36
	IV	207.19	163.58	137.49	128.45	139.17	164.48	214.91	273.60	359.43
0.4	I	159.22	154.82	162.16	180.33	219.78	262.40	314.13	348.73	456.35
	II	152.68	144.84	159.77	201.72	267.74	359.65	462.40	608.93	764.93
	III	155.53	140.76	161.10	218.47	302.35	419.97	576.67	759.17	956.37
	IV	225.13	180.10	149.75	139.74	151.07	179.92	235.54	298.40	388.24
0.5	I	156.28	153.07	160.21	178.21	215.58	257.46	308.89	374.79	460.02
	II	149.80	142.50	156.87	198.03	262.93	354.07	452.77	604.91	755.09
	III	149.17	135.50	155.26	209.19	289.74	401.92	548.37	726.64	921.76
	IV	240.93	191.53	159.71	149.10	162.46	193.60	253.76	320.98	420.27
0.6	I	150.51	147.50	155.47	172.77	208.13	248.55	299.85	363.74	442.23
	II	142.53	135.85	149.68	188.86	247.72	335.34	429.69	568.05	713.58
	III	139.11	125.74	144.98	194.42	269.92	374.53	511.26	672.23	861.67
	IV	252.69	200.10	166.61	156.52	170.26	201.90	264.86	338.34	441.53
0.7	I	144.87	140.89	147.95	165.22	198.88	263.40	284.91	350.90	419.95
	II	133.32	125.80	138.93	175.14	230.76	310.13	398.21	527.50	665.34
	III	125.23	113.28	130.46	174.93	244.27	342.04	460.79	612.47	785.09
	IV	258.92	202.87	169.77	159.50	173.88	206.66	271.39	346.66	448.02
0.8	I	136.50	132.28	139.52	156.15	187.55	245.76	268.76	327.92	396.09
	II	120.82	114.81	127.12	160.96	210.85	285.41	360.90	477.31	604.09
	III	112.45	101.14	116.26	155.42	216.76	304.15	408.39	546.84	695.73
	IV	256.31	200.14	167.41	158.42	172.14	204.04	267.17	343.12	444.21
0.9	I	127.70	123.33	130.28	145.20	175.21	208.21	249.65	304.26	369.99
	II	108.85	104.07	114.96	145.06	189.64	255.82	326.32	430.68	546.88
	III	**	**	102.40	136.05	189.25	267.93	357.08	474.84	605.96
	IV	245.56	192.90	161.89	152.22	165.80	196.86	258.07	329.86	423.49

Note. '**' indicates no gain.

Table 6 Monte Carlo Simulation results when the proposed estimator T_{21} is compared to the estimator Δ

ϕ	$\psi \rightarrow$ SET ↓	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	I	164.30	162.04	170.17	186.41	225.19	273.63	318.27	416.48	473.41
	II	141.30	137.16	151.29	185.73	247.77	351.46	396.87	563.10	741.51
	III	137.43	128.55	141.92	200.86	263.88	369.83	522.15	679.07	910.72
	IV	118.48	105.56	**	**	**	**	125.42	162.57	214.0
0.2	I	169.22	164.43	173.74	191.94	237.26	283.01	325.54	401.66	492.17
	II	146.42	140.67	157.02	194.23	256.77	349.22	431.44	580.83	757.19
	III	144.94	134.99	153.20	210.47	282.25	393.85	543.97	717.48	920.49
	IV	134.04	117.32	**	**	**	108.93	136.67	185.99	236.66
0.3	I	168.71	163.74	172.99	191.54	232.10	281.87	333.45	407.94	489.82
	II	147.14	139.76	155.67	192.73	254.65	343.14	437.95	581.08	749.47
	III	146.06	134.74	154.03	209.81	233.18	396.35	547.58	718.77	917.21
	IV	149.80	128.47	100.11	**	101.37	120.47	155.25	201.64	261.26
0.4	I	163.25	159.18	167.59	183.77	224.98	271.79	321.96	393.64	473.0
	II	139.73	133.36	148.84	185.01	244.59	328.50	424.73	561.31	707.64
	III	139.74	127.01	145.41	197.85	270.95	376.30	518.17	675.94	871.26
	IV	162.36	135.92	107.72	100.04	109.68	131.46	169.16	218.55	281.58
0.5	I	153.68	151.0	158.66	175.83	211.75	255.99	304.54	370.36	452.13
	II	130.14	125.18	137.85	171.75	227.73	304.54	395.12	526.53	658.63
	III	125.76	114.80	131.95	178.34	246.39	341.66	465.45	612.26	789.09
	IV	172.46	140.33	114.30	105.17	116.65	140.05	180.77	232.54	301.70
0.6	I	140.70	138.80	146.75	162.06	195.09	236.45	282.72	343.0	417.54
	II	116.53	113.59	123.84	154.90	204.93	273.57	356.37	472.16	591.84
	III	110.90	100.53	115.38	157.30	215.80	300.35	408.67	537.16	688.69
	IV	177.79	139.94	117.65	109.17	120.37	144.22	186.37	240.64	311.57
0.7	I	129.70	126.10	132.55	149.58	177.75	215.08	257.43	314.25	380.18
	II	104.03	100.14	109.0	137.33	180.63	240.87	312.68	415.49	523.70
	III	**	**	**	133.60	185.24	258.53	351.47	464.06	595.42
	IV	178.91	135.46	117.81	109.46	120.76	144.66	186.96	241.53	311.63
0.8	I	116.88	113.24	119.85	135.13	159.69	193.70	232.06	282.62	342.43
	II	**	**	**	119.77	158.24	212.80	273.82	361.75	454.69
	III	**	**	**	113.23	156.67	219.80	298.48	397.24	506.82
	IV	174.0	140.86	113.71	106.67	117.35	140.40	181.37	234.29	302.86
0.9	I	105.73	101.54	107.40	120.34	143.32	173.41	207.59	253.34	306.09
	II	**	**	**	103.73	137.32	182.59	237.85	312.02	391.95
	III	**	**	**	**	132.98	187.26	250.32	336.43	430.48
	IV	163.85	128.33	107.93	100.65	110.64	132.25	171.50	221.13	285.08

Note. '**' indicates no gain.

Table 7 Monte Carlo Simulation results when the proposed estimator T_{22} is compared to the estimator Δ

ϕ	$\psi \rightarrow$ SET ↓	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	I	145.34	142.91	150.34	165.76	201.25	241.72	283.71	372.27	422.76
	II	139.86	131.72	149.28	183.75	251.11	353.73	401.21	560.08	747.12
	III	142.01	126.68	145.84	202.37	271.64	376.86	528.24	684.40	918.32
	IV	168.40	133.02	113.61	106.71	112.35	138.09	171.15	222.02	279.44
0.2	I	149.44	144.84	152.99	170.90	209.80	247.94	288.51	355.91	438.58
	II	145.59	137.80	155.01	194.71	260.76	352.53	436.69	579.70	761.35
	III	149.03	133.54	155.58	211.23	288.36	402.60	553.84	723.93	922.12
	IV	188.53	148.72	126.73	118.13	126.62	152.41	192.27	253.35	328.86
0.3	I	150.07	144.53	152.80	170.30	204.83	247.52	295.52	361.88	432.91
	II	146.71	138.09	154.92	193.58	257.84	344.89	441.02	580.43	754.38
	III	148.73	134.21	155.45	209.84	287.19	404.09	555.0	727.42	918.78
	IV	208.51	164.05	138.74	128.89	139.03	166.56	25.66	275.54	359.87
0.4	I	145.42	141.02	148.49	165.50	199.88	239.20	287.11	350.73	418.86
	II	140.20	132.59	148.35	185.95	246.69	330.71	426.58	559.47	711.77
	III	141.86	126.81	146.01	197.66	273.69	381.30	523.37	683.22	875.16
	IV	221.71	177.43	148.03	137.60	148.13	178.21	231.03	293.93	381.61
0.5	I	137.85	135.10	142.12	157.92	189.45	227.59	273.41	331.58	405.16
	II	130.45	124.88	137.89	172.41	229.23	307.19	396.44	524.90	662.82
	III	127.21	114.85	132.21	177.97	248.29	344.66	468.77	615.55	792.22
	IV	229.67	182.27	153.0	142.04	154.08	185.10	240.28	305.89	398.49
0.6	I	127.78	125.60	133.10	147.23	176.64	212.63	256.40	310.51	377.76
	II	116.90	113.38	124.03	155.66	205.91	275.77	357.22	470.98	594.70
	III	111.62	100.0	115.44	156.69	216.72	302.02	411.56	539.55	691.10
	IV	229.39	182.55	152.29	142.69	154.53	185.09	240.15	307.21	399.19
0.7	I	119.13	115.58	121.87	137.22	163.02	196.13	236.15	287.92	348.08
	II	104.27	**	109.23	137.97	181.22	242.29	313.73	414.96	526.11
	III	**	**	**	133.30	185.96	259.46	353.0	465.10	596.38
	IV	222.81	175.30	147.60	137.83	150.17	179.11	232.76	298.37	386.84
0.8	I	108.68	105.16	111.41	125.46	148.49	179.11	215.52	262.33	317.59
	II	**	**	**	120.23	158.65	212.99	274.54	361.38	456.32
	III	**	**	**	113.03	157.10	220.20	299.43	398.12	507.52
	IV	209.54	163.69	137.32	129.39	141.03	168.18	217.93	279.63	363.49
0.9	I	100.05	101.54	101.09	113.19	134.89	162.45	195.18	237.99	287.66
	II	**	**	**	104.06	137.61	183.21	238.41	311.82	393.15
	III	**	**	**	109.51	133.25	187.47	250.79	337.01	430.48
	IV	190.81	150.13	126.23	100.65	128.73	153.38	199.50	255.99	331.80

Note. '**' indicates no gain.

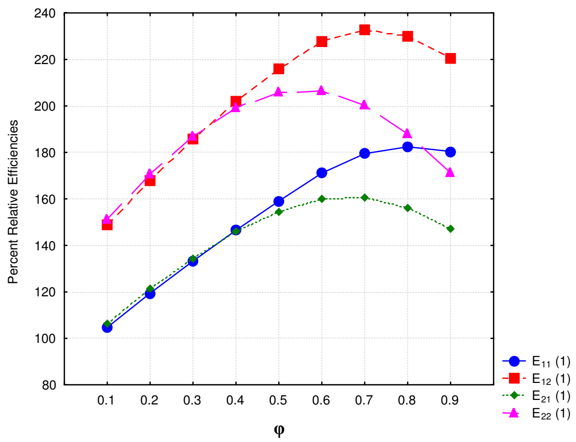

Fig. 2 Mutual Comparison of Proposed Estimator T_{ij} $(i, j = 1,2)$ when compared with the estimator $\hat{M}_y(n)$ for set IV.

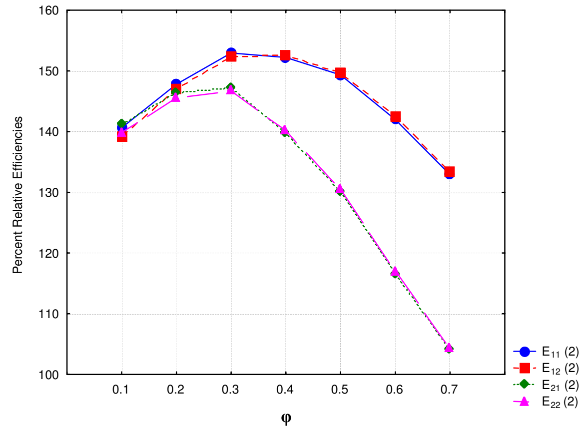

Fig. 3 Mutual Comparison of Proposed Estimators T_{ij} $(i, j = 1,2)$ when compared with the estimator Δ for $\psi = 0.1$ for set II.

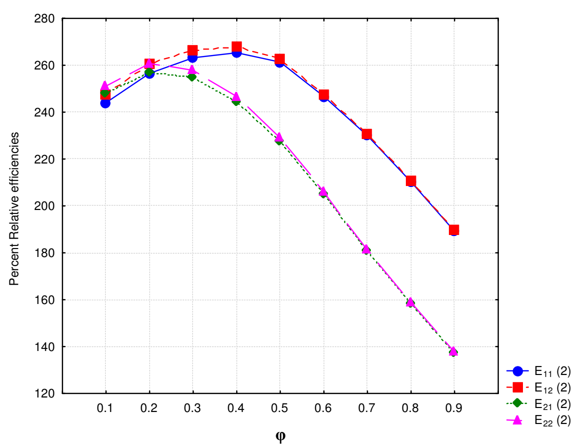

Fig. 4 Mutual Comparison of Proposed Estimators T_{ij} $(i, j = 1,2)$ when compared with the estimator Δ for $\psi = 0.5$ for set II.

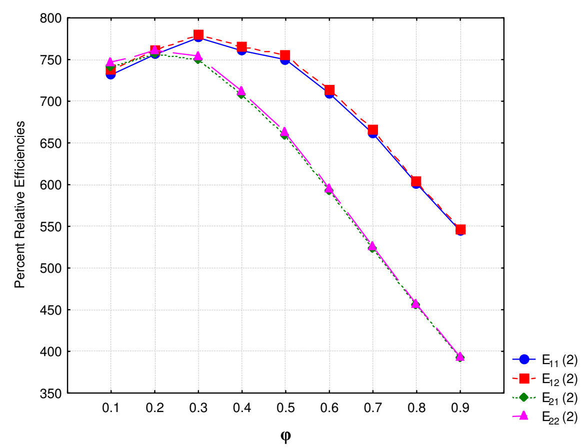

Fig. 5 Mutual Comparison of Proposed Estimators T_{ij} $(i, j = 1,2)$ when compared with the estimator Δ for $\psi = 0.9$ for set II.

(3) From Table 4, when T_{11} is compared with the estimator Δ, we see that
  (a) $E_{11}(2)$ increases as ϕ increases for all choices of ψ.
  (b) For fixed choices of ϕ as ψ increases the value of $E_{11}(2)$ increases.
  (c) As μ is increased $E_{11}(2)$ decreases.
(4) From Table 5, when T_{12} is compared with the estimator Δ, we observe that
  (a) $E_{12}(2)$ increases for all the sets as ϕ increases for all choices of ψ.
  (b) As ψ increases $E_{12}(2)$ also increases for all sets except for some of the combinations of ϕ and ψ.
  (c) No fixed pattern is observed for $E_{12}(2)$ as μ is increased.
(5) From Table 6, when $T_{21}$is compared with the estimator Δ, it can be seen that
  (a) For all choices of ψ the value of $E_{21}(2)$ first increases and then decreases as ϕ increases for all sets except for set IV.
  (b) For different choices of ϕ as ψ increases, the value of $E_{21}(2)$ also increases for set I, II and III.
  (c) For set IV, $E_{21}(2)$ first decreases as ψ increases and then increases for all choices of ϕ.
  (d) As for all choices of ϕ and ψ as μ increases, the value of $E_{21}(2)$ decreases.
(6) From Table 7, it can be concluded that

  (a) $E_{22}(2)$ first increases as ϕ increases and then decreases for different choices of ψ for all the four sets.
  (b) As ψ increases $E_{22}(2)$ also increases for all sets and for all choices of ϕ.
  (c) For set IV, $E_{22}(2)$ first decreases and then increases as ψ increases for all choices of ϕ.
  (d) No fixed behaviour is observed for $E_{22}(2)$ as portion of sample drawn afresh at current occasion increases.

(7) The mutual comparison of the four proposed estimators T_{ij}; $(i, j = 1,2)$ in Figure 2 to Figure 5, show that the estimator T_{22} comes out to be the best estimator amongst all the four proposed estimators when they are compared with sample median estimator $\hat{M}_y(n)$, since it is the most consistent and having greater precision but when T_{ij}; $(i, j = 1,2)$ are compared with estimator Δ, T_{12} comes out be the best as it possess largest gain over other proposed estimators and considerably consistent in nature for all combinations of ϕ, ψ and μ. It has also been found that the percent relative efficiency of the estimator T_{12} increases as the fraction of sample drawn at current occasion decreases and vice versa which exactly justifies the basic principles of sampling over successive occasions.

11 Conclusion

From the preceding interpretations, it may be concluded that the use of exponential ratio type estimators for the estimation of population median at current occasion in two occasion successive sampling is quite feasible as vindicated through empirical and simulation results. The use of highly correlated auxiliary information which is dynamic over time is highly rewarding in terms of precision. The mutual comparison of the proposed estimators indicates that the estimators utilizing more exponential ratio type structures perform better. It has also been observed that the estimator T_{22} in which maximum utilization of exponential ratio type structures have been considered, has turned out to be the most efficient among all the four proposed estimators when comparison is made with sample median estimator and T_{12} is most suitable amongst all when they are compared with the estimator Δ. Hence, when a highly positively correlated auxiliary information which is dynamic over time is used, the proposed estimators may be recommended for their practical use by survey practitioners.

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