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Common Fixed Point Theorems for Infinite Families of Contractive Maps in Metric Spaces

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Abstract: In this paper, we prove some fixed and common fixed point theorems for infinite families of self mappings of a complete metric space satisfying some new conditions of common contractivity. These results generalize several well known comparable results in the literature. An example is presented to show the effectiveness of our results.

Keywords: common fixed point, coincidence point, fixed point, family of contractive maps

1 Introduction and Preliminaries

Fixed point theory constitutes an important and the core part of the subject of nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. The Banach contraction principle is the simplest and one of the most versatile elementary results in fixed point theory, which is a very popular tool for solving existence problems in many branches of mathematical analysis. Several authors have extended the Banach's fixed point theorem in various ways. The family of contraction mappings was introduced and studied by Ćirić [9] and Tasković [17]. Also in the process, the study of existence of common fixed point for finite and infinite family of self-mapping has been carried out by many authors. For example, one may refer[2,4,7,11,10,18,19,20].
Lakshmikantham and Ćirić [12] introduced the concept of commuting maps which discuss the relation from the reverse and proved fixed point theorems for single valued maps in metric spaces. Recently, existence of common fixed point and coincidence point problems has considered, and first results were obtain by Lakshmikantham and Ćirić [12]. We refer for more detail to [3,7,13,14,15,16].
In [5], Babu et al. introduced the concept of condition (B) as follows.

Definition 1. *Let (X,d) be a metric space. A map $T: X \to X$ is said to satisfy condition (B) if there exist a constant $\delta \in [0,1)$ and some $L \geq 0$ such that for all $x,y \in X$,*

$$d(Tx,Ty) \leq \delta d(x,y) + L\, min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

In 2008, Berinde [6] proved the following theorem which is a generalization of many known results.

Theorem 1. *Let (X,d) be a complete metric space and $T: X \to X$ a mapping for which there exist a $\alpha \in (0,1)$ and some $L \geq 0$ such that for all $x,y \in X$*

$$d(Tx,Ty) \leq \alpha M(x,y) + L\, min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}, \quad (1)$$

where

$$M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

Then

1. *T has a unique fixed point, i.e., $F(T) = x*$;*
2. *for any $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $Tx_n = x_{n+1}$ covnerges to some $x* \in F(T)$;*
3. *the priori estimate $d(x_n,x*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x_0,x_1)$ holds, for $n = 1,2,\ldots$;*
4. *the rate of convergence of Picard iteration is given by $d(x_n,x*) \leq \theta d(x_{n-1},x*)$ for $n = 0,1,2,\cdots$.*

Abbas and Ilić [1] introduced a new concept of generalized condition (B), called generalized almost f-contraction.

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Definition 2. *Let T and f be two self maps of a metric space (X,d). A map T is called generalized almost f-contraction if there exists $\delta\in[0,1)$ and $L\geq 0$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq\delta M(x,y)+L\,min\{d(fx,Tx),d(fy,Ty),\\ d(fx,Ty),d(fy,Tx)\},$$

where

$$M(x,y)=\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),\\ \frac{d(fx,Ty)+d(fy,Tx)}{2}\}.$$

In 2011, Ćirić et al.[8] introduced the concept almost generalized contractive condition as follows:

Definition 3. *Let f and g be two self maps of a metric space (X,d). They are said to satisfy almost generalized contractive condition if there exists $\delta\in[0,1)$ and $L\geq 0$ such that for all $x,y\in X$,*

$$d(fx,gy)\leq\delta\max\{d(x,y),d(x,fx),d(y,gy),\\ \frac{d(x,gy)+d(y,fx)}{2}\}\\ +L\,min\{d(x,fx),d(y,gy),d(x,gy),d(y,fx)\}.$$

The following interesting theorem was given by Ćirić [9] for a family of generalized contractions.

Theorem 2. *Let (X,d) be a complete metric space and let $\{T_\alpha\}_{\alpha\in J}$ be a family of self mappings of X. If there exists fixed $\beta\in J$ such that for each $\alpha\in J$:*

$$d(T_\alpha x,T_\beta y)\leq\lambda\ \max\{d(x,y),d(x,T_\alpha x),d(y,T_\beta y),\\ \frac{1}{2}[d(x,T_\beta y)+d(y,T_\alpha x)]\},\qquad (2)$$

for some $\lambda=\lambda(\alpha)\in(0,1)$ and all $x,\ y\in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha\in J$.

The aim of this paper is to define some new conditions of common contractivity for an infinite family of mappings and give some new results on the existence and uniqueness of common fixed points in the setting of complete metric space.

Definition 4. *Let X be a nonempty set and let $\{T_n\}$ be a family of self mappings on X. A point $x_0\in X$ is called a common fixed point for this family iff $T_n(x_0)=x_0$, for each $n\in N$.*

Definition 5. *Let $\{T_n\}$ be a sequence of mappings and g be a self mapping on X. If $y=gx=T_nx$ for all $n\in N$ and for some $x\in X$, then x is called coincidence point of $\{T_n\}$ and g, where y is called a point of coincidence of $\{T_n\}$ and g.*

2 Common fixed point theorems for a family of mappings

In this section, we prove existence of a unique common fixed point for a family of contractive type self maps on a complete metric space.

Theorem 3. *Let (X,d) be a complete metric space and $0\leq a_{i,j}\quad(i,j=1,2,...)$, satisfy*

i) for each j, $\overline{\lim}_{i\longrightarrow\infty}a_{i,j}<1$,

ii) $\sum_{n=1}^{\infty}A_n<\infty$ where $A_n=\prod_{i=1}^{n}\frac{a_{i,i+1}}{1-a_{i,i+1}}$.

If $\{T_n\}$ is a sequence of self maps on X satisfying

$$d(T_ix,T_jy)\leq a_{i,j}M_{i,j}(x,y)+LN_{i,j}(x,y),\qquad (3)$$

where,

$$M_{i,j}(x,y)=\max\{d(x,y),d(x,T_ix),d(y,T_jy),\\ \frac{d(x,T_jy)+d(y,T_ix)}{2}\},$$

and

$$N_{i,j}(x,y)=\max\{d(x,T_ix),d(y,T_jy),d(x,T_jy),d(y,T_ix)\},$$

for all $x,y\in X,\ i,j\in\mathbb{N}$ with $x\neq y$, $i\neq j$ and $L\geq 0$, then all T_n,s have a unique common fixed point in X. Further, if $x\in X$ be unique common fixed point of $\{T_n\}$,s then x is a unique fixed point for all T_n, s.

Proof. For any $x_0\in X$, let $x_n=T_n(x_{n-1}),\ n=1,2,...,$ then using (3) we obtain

$$d(x_1,x_2)=d(T_1(x_0),T_2(x_1))\\ \leq a_{1,2}M_{1,2}(x_0,x_1)+LN_{1,2}(x_0,x_1),$$

where,

$$M_{1,2}(x_0,x_1)=\max\{d(x_0,x_1),d(x_0,T_1x_0),d(x_1,T_2x_1),\\ \frac{d(x_0,T_2(x_1))+d(x_1,T_1(x_0))}{2}\}\\ =\max\{d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),\\ \frac{d(x_0,x_2)+d(x_1,x_1)}{2}\}\\ =\max\{d(x_0,x_1),d(x_1,x_2),\frac{1}{2}d(x_0,x_2)\}\\ \leq d(x_0,x_1)+d(x_1,x_2),$$

and

$$N_{1,2}(x_0,x_1)=\min\{d(x_0,T_1x_0),d(x_1,T_2x_1),d(x_0,T_2(x_1)),\\ d(x_1,T_1(x_0))\}\\ =\min\{d(x_0,x_1),d(x_1,x_2),d(x_0,x_2),d(x_1,x_1)\}\\ =0.$$

Therefore

$$d(x_1,x_2)\leq\frac{a_{1,2}}{1-a_{1,2}}d(x_0,x_1).\qquad (4)$$

Also,

$$d(x_2,x_3) = d(T_2(x_1),T_3(x_2)) \leq a_{2,3}M_{2,3}(x_2,x_3) + LN_{2,3}(x_2,x_3),$$

where

$$\begin{aligned} M_{2,3}(x_1,x_2) &= \max\{d(x_1,x_2), d(x_1,T_2x_1), d(x_2,T_3x_2), \frac{d(x_1,T_3(x_2)) + d(x_2,T_2(x_1))}{2}\} \\ &= \max\{d(x_1,x_2), d(x_1,x_2), d(x_2,x_3), \frac{d(x_1,x_3) + d(x_2,x_2)}{2}\} \\ &= \max\{d(x_1,x_2), d(x_2,x_3), \frac{1}{2}d(x_1,x_3)\} \\ &\leq d(x_1,x_2) + d(x_2,x_3), \end{aligned}$$

and

$$\begin{aligned} N_{2,3}(x_1,x_2) &= \min\{d(x_1,T_2x_1), d(x_2,T_3x_2), d(x_1,T_3(x_2)), d(x_2,T_2(x_1))\} \\ &= \min\{d(x_1,x_2), d(x_2,x_3), d(x_1,x_3), d(x_2,x_2)\} \\ &= 0. \end{aligned}$$

So,

$$d(x_2,x_3) \leq \frac{a_{2,3}}{1-a_{2,3}} d(x_1,x_2). \tag{5}$$

Hence from (4) and (5), we have

$$d(x_2,x_3) \leq \frac{a_{1,2}}{1-a_{1,2}} \cdot \frac{a_{2,3}}{1-a_{2,3}} d(x_0,x_1).$$

In general

$$d(x_n,x_{n+1}) \leq \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}} d(x_0,x_1). \tag{6}$$

Therefore, for $m,n \in N,\ m \geq n$, and using (6), we have

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \prod_{i=1}^{k} \frac{a_{i,i+1}}{1-a_{i,i+1}} d(x_0,x_1) \\ &= \sum_{k=n}^{m-1} A_k d(x_0,x_1). \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence and by completeness of X, $\{x_n\}$ converges to x (say) in X. So for any positive integer m,

$$\begin{aligned} d(x,T_mx) &\leq d(x,x_n) + d(x_n,T_mx) = d(x,x_n) + d(T_nx_{n-1},T_mx) \\ &\leq a_{n,m}M_{n,m}(x_{n-1},x) + LN_{n,m}(x_{n-1},x), \end{aligned}$$

Taking $\overline{\lim}$ as $n \longrightarrow \infty$, we get

$$d(x,T_mx) \leq \overline{\lim_{n\longrightarrow\infty}} a_{n,m} d(x,T_mx). \tag{7}$$

Indeed,

$$\begin{aligned} M_{n,m}(x_{n-1},x) &= \max\{d(x_{n-1},x), d(x_{n-1},T_nx_{n-1}), d(x,T_mx), \frac{d(x_{n-1},T_mx) + d(x,T_nx_{n-1})}{2}\} \\ &= \max\{d(x_{n-1},x), d(x_{n-1},x_n), d(x,T_mx), \frac{d(x_{n-1},T_mx) + d(x,x_n)}{2}\}, \end{aligned}$$

and

$$\begin{aligned} N_{n,m}(x_{n-1},x) &= \min\{d(x_{n-1},T_nx_{n-1}), d(x,T_mx), d(x_{n-1},T_mx), d(x,T_nx_{n-1})\} \\ &= \min\{d(x_{n-1},x), d(x_{n-1},x_n), d(x,T_mx), d(x_{n-1},T_mx), d(x,x_n)\}, \end{aligned}$$

which,

$$\overline{\lim_{n\longrightarrow\infty}} M_{n,m} = d(x,T_mx)$$

$$\overline{\lim_{n\longrightarrow\infty}} N_{n,m} = 0.$$

From condition (i) and (7), it follows that $d(x,T_mx) = 0$ gives x as a common fixed point of $\{T_m\}$. Let y be another fixed point of $\{T_n\}$, then

$$\begin{aligned} d(x,y) &= d(T_nx,T_my) \\ &\leq a_{n,m} \max\{d(x,y), d(x,T_nx), d(y,T_my), \frac{d(x,T_my) + d(y,T_nx)}{2}\} \\ &\quad + L \min\{d(x,T_nx), d(y,T_my), d(x,T_my), d(y,T_nx)\}. \end{aligned}$$

Taking $\overline{\lim}$ as $n \longrightarrow \infty$, we get

$$d(x,y) \leq \overline{\lim_{n\longrightarrow\infty}} a_{n,m} d(x,y),$$

which is possible only when $x = y$. Hence x is the unique common fixed point of $\{T_n\}$. Further, if $y \in X$ is a unique fixed point of T_k, then according to $\overline{\lim_{i\longrightarrow\infty}} a_{i,k} < 1$, there exists an $i_k \in \mathbb{N}$ such that $a_{i_k,k} < 1$. Thus, by (3), we have

$$\begin{aligned} d(x,y) &= d(T_{i_k}x,T_ky) \\ &\leq a_{i_k,k} \max\{d(x,y), d(x,T_{i_k}x), d(y,T_ky), \frac{d(x,T_ky) + d(y,T_{i_k}x)}{2}\} \\ &\quad + L \min\{d(x,T_{i_k}x), d(y,T_ky), d(x,T_ky), d(y,T_{i_k}x)\} \\ &\leq a_{i_k,k} d(x,y), \end{aligned}$$

which implies $d(x,y) = 0$ and hence $x = y$.

*Example 1.*Let $X = [0,1]$ be a complete metric space with the distance $d(x,y) = |x-y|, x,y \in X$, and $T_n : X \longrightarrow X$ be defined by

$$T_n(x) = \begin{cases} 1, & 0 < x \leq 1, \\ \frac{3}{4} + \frac{1}{n+3}, & x = 0. \end{cases}$$

Let $a_{i,j} = \frac{1}{4} + \frac{1}{|i-j|+8}, i \neq j$, then for each j, $\overline{\lim_{i \longrightarrow \infty}} a_{i,j} < 1$ and $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}} = (\frac{13}{23})^n$, therefore $\sum_{n=1}^{\infty} (\frac{13}{23})^n < \infty$.

Now we prove that for each $x,y \in X$,

$$d(T_i x, T_j y) \leq a_{i,j} M_{i,j}(x,y) + L\, N_{i,j}(x,y).$$

There are three possible cases:

(1)$x \in (0,1], y \in (0,1]$. Then

$$\begin{aligned} d(T_i x, T_j y) &= |T_i x - T_j y| = 0 \\ &\leq a_{i,j}|x-1| = a_{i,j} d(x, T_i x) \\ &\leq a_{i,j} M_{i,j}(x,y) + L\, N_{i,j}(x,y). \end{aligned}$$

(2)$x \in (0,1], y = 0$. Then

$$\begin{aligned} d(T_i x, T_j y) &= |T_i x - T_j(0)| = |\frac{1}{4} - \frac{1}{j+3}| \leq \frac{1}{4} \\ &\leq (\frac{1}{4} + \frac{1}{|i-j|+8})|0-1| = a_{i,j} d(y, T_i x) \\ &\leq a_{i,j} M_{i,j}(x,y) + L\, N_{i,j}(x,y). \end{aligned}$$

(3)$x = y = 0$, $i < j$. Then

$$\begin{aligned} d(T_i x, T_j y) &= |T_i(0) - T_j(0)| = |\frac{3}{4} + \frac{1}{i+3} - \frac{3}{4} - \frac{1}{j+3}| \\ &= \frac{1}{i+3} - \frac{1}{j+3} \\ &\leq (\frac{1}{4} + \frac{1}{j-i+8})|0 - \frac{3}{4} - \frac{1}{j+3}| \\ &\leq a_{i,j} d(x, T_j y) \\ &\leq a_{i,j} M_{i,j}(x,y) + L\, N_{i,j}(x,y). \end{aligned}$$

So all the conditions of Theorem 3 are satisfied and note that $x = 1$ is the only fixed point for all T_n.

The following result is the immediate consequence of Theorem 3.

Corollary 1.*Let (X,d) be a complete metric space and $0 \leq a_{i,j} \quad (i,j = 1,2,\ldots)$, satisfy*

i)for each j, $\overline{\lim_{i \longrightarrow \infty}} a_{i,j} < 1$,

ii)$\sum_{n=1}^{\infty} A_n < \infty$ where $A_n = \prod_{i=1}^{n} \frac{a_{i,i+1}}{1-a_{i,i+1}}$.

If $\{T_n\}$ is a sequence of self maps on X satisfying

$$d(T_i x, T_j y) \leq a_{i,j} \max\{d(x,y), d(x,T_i x), d(y,T_j y), \frac{d(x,T_j y)+d(y,T_i x)}{2}\},$$

for all $x,y \in X$, $i,j = 1,2,\ldots$ with $x \neq y$ and $i \neq j$, then all T_n,s have a unique common fixed point in X. Further, if $x \in X$ be unique common fixed point of $\{T_n\}$,s then x is a unique fixed point for all T_n,s.

Proof.Taking $L = 0$ in Theorem 3, we have the required proof.

Conclusion

We saw that the results of Berinde [6] and the results of Ćirić [9] also hold in the context of metric spaces with some simple changes in the contractive conditions. we can prove many fixed point results in this new contraction for infinite families of maps. Theorem 3 improves and extends the main results of Berinde [6] and Ćirić [9]. Example 1 is furnished in support of Theorem 3.

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