Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090248

A Weighted-Path-Following Interior-Point Algorithm for Second-Order Cone Optimization

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Received: 19 Jun. 2014, Revised: 17 Sep. 2014, Accepted: 19 Sep. 2014
Published online: 1 Mar. 2015

Abstract: We present a weighted-path-following interior-point algorithm for solving second-order cone optimization. This algorithm starts from an initial point which is not on the central path. It generates iterates that simultaneously get closer to optimality and closer to centrality. At each iteration, we use only full Nesterov-Todd step; no line searches are required. We derive the complexity bound of the algorithm with small-update method, namely, $O\left(\sqrt{N}\log\frac{N}{\varepsilon}\right)$, where N denotes the number of second order cones in the problem formulation and ε the desired accuracy. This bound is the currently best known iteration bound for second-order cone optimization.

Keywords: Second-order cone optimization, interior-point method, small-update method, polynomially complexity

1 Introduction

The second-order cone (SOC) in R^n, also called the Lorentz cone or the ice-cream cone, is defined as

$$L := \left\{(x_1,x_2,...,x_n) \in R^n : x_1^2 \geq \sum_{j=2}^{n} x_j^2,\ x_1 \geq 0\right\}, \quad (1)$$

where $n \geq 2$ is some natural number. Second-order cone optimization (SOCO) is convex optimization problem in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of second-order cones. In this paper we consider SOCO in standard format

$$(P)\ \ \min\left\{c^{\mathrm{T}}x : Ax = b,\ \ x \in K\right\},$$

where $A \in R^{m\times n}, c \in R^n$ and $b \in R^m$, and $K \subset R^n$ is the Cartesian product of second-order cones, i.e., $K = L^1 \times \cdots \times L^N$, with $L^i \subset R^{n_i}$ for each i, $i = 1,2,...,N$.

The dual problem of (P) is given by

$$(D)\ \ \max\left\{b^{\mathrm{T}}y : A^{\mathrm{T}}y + s = c,\ \ s \in K\right\}.$$

Without loss of generality, throughout the paper, we assume that rows of the matrix A are linearly independent. Hence, if the pair (y,s) is dual feasible then y is uniquely determined by s. Therefore, we will feel free to say that s is dual feasible, without mentioning y.

The study of SOCO is vast important because it covers linear optimization, convex quadratic optimization, quadratically constraint convex quadratic optimization as well as other problems [1]. In the last few years, the SOCO problem has received considerable attention from researchers for its wide range of applications in many fields, such as engineering, optimal control and design, machine learning, robust optimization and combinatorial optimization and so on. We refer the interested readers to the survey paper [9] and the references therein.

Many researchers have worked on interior-point methods (IPMs) for solving SOCO (e.g., [2,11,13,15]). Notice that the IPMs in those papers required the initial point and iteration points to be on, or close to, the central path. However, practical implementations often don't use perfectly centered starting points. Therefore, it is worth analyzing the case when the starting point is not on the central path. Recently, Jansen et al. [7] presented a class of primal-dual target-following interior-point methods for linear optimization. This class starts from an initial non-central point and generates iterates that

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simultaneously get closer to optimality and closer to centrality. Darvay [4,5] investigated the weighted-path-following interior-point method, a particular case of target-following methods, for linear optimization. These methods were also studied by Ding et al. [3] for linear complementarity problems, by Jin et al. [8] for convex quadratic optimization and by Roos et al. [14] for linear optimization.

Motivated by their work, in this paper we propose a weighted-path-following interior-point algorithm for SOCO. We adopt the basic analysis used in [4,5] to the SOCO case. The currently best known iteration bound for the algorithm with small-update method, namely, $O\left(\sqrt{N}\log\frac{N}{\varepsilon}\right)$, is obtained. To compare with the path-following (primal-dual) interior-point methods, our method has the following advantages:

(i) it can start from any strictly feasible initial point, and works for optimality and centrality at the same time.

(ii) it uses only full Nesterov-Todd step and no any line searches are required at each iteration. While some path-following (primal-dual) interior point methods (e.g., [2,11, 13,15]) need do some line searches to find a suitable step size which keeps the iterations be feasible and be on, or close to, the central path.

The paper is organized as follows. In Section 2, we recall some useful results on second-order cones and their associated Jordan algebra. In Sections 3, we present the new search directions for SOCO. In Section 4, we propose the weighted-path-following interior-point algorithm. In Section 5, we analyze the algorithm and derive the currently best known iteration bound with small-update method. The conclusions are given in Section 6.

Some notations used throughout the paper are as follows. R^n, R^n_+ and R^n_{++} denote the set of vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. We use ";" for adjoining vectors in a column. For instance, for column vectors x,y and z we have $(x;\ y;\ z) := (x^{\mathrm{T}},\ y^{\mathrm{T}},\ z^{\mathrm{T}})^{\mathrm{T}}$. As usual, I denotes the identity matrix with suitable dimension. $\|\cdot\|$ denotes the 2-norm of the vector x and the index set J is $J = \{1,2,...,N\}$. Finally, for any $x,y \in R^n$, we write $x \succeq_K y$ (respectively, $x \succ_K y$) if $x-y \in K$ (respectively, $x-y \in \mathrm{int}K$, where int$K$ denotes the interior of K).

2 Algebraic properties of second-order cones

In this section we briefly recall some algebraic properties of the SOC L as defined by (1) and its associated Euclidean Jordan algebra. For any vectors $x,s \in R^n$, their Jordan product associated with the SOC L is defined by

$$x \circ s := (x^{\mathrm{T}}s;\ x_1 s_{2:n} + s_1 x_{2:n}),$$

where $x_{2:n} := (x_2;...;x_n)$ and $s_{2:n} := (s_2;...;s_n)$. $(R^n,\circ)$ is an Euclidean Jordan algebra with the vector $\mathbf{e} := (1;0;...;0)$ as the identity element. Obviously, $\mathbf{e}^{\mathrm{T}}(x \circ s) = x^{\mathrm{T}}s$. In the sequel, we denote the vector $(x_2;...;x_n)$ shortly as $x_{2:n}$. So $x = (x_1;x_{2:n})$. Given an element $x \in R^n$, define the symmetric matrix

$$L(x) := \begin{bmatrix} x_1 & x_{2:n}^{\mathrm{T}} \\ x_{2:n} & x_1 I \end{bmatrix} \in R^{n\times n},$$

where I represents the $(n-1)\times(n-1)$ identity matrix. Notice that $x \circ s = L(x)s$ for any $x,s \in R^n$.

The so-called *spectral decomposition* of a vector $x \in R^n$ is given by

$$x = \lambda_{\min}(x)u^{(1)} + \lambda_{\max}(x)u^{(2)}, \tag{2}$$

where $\lambda_{\min}(x), \lambda_{\max}(x)$ and $u^{(1)}, u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_{\min}(x) := x_1 - \|x_{2:n}\|, \quad \lambda_{\max}(x) := x_1 + \|x_{2:n}\|,$$

$$u^{(i)} = \begin{cases} \frac{1}{2}\left(1;(-1)^i \frac{x_{2:n}}{\|x_{2:n}\|}\right), & x_{2:n} \neq 0, \\ \frac{1}{2}\left(1;(-1)^i \upsilon\right), & x_{2:n} = 0, \end{cases} \quad i = 1,2,$$

with $\upsilon \in R^{n-1}$ being any vector satisfying $\|\upsilon\| = 1$.

Using (2), for each $x \in R^n$ we define the following [1]:

• square root: $x^{\frac{1}{2}} = (\lambda_{\min}(x))^{\frac{1}{2}}u^{(1)} + (\lambda_{\max}(x))^{\frac{1}{2}}u^{(2)}$, for any $x \in L$;

• inverse: $x^{-1} = (\lambda_{\min}(x))^{-1}u^{(1)} + (\lambda_{\max}(x))^{-1}u^{(2)}$, for any $x \in \mathrm{int}L$;

• square: $x^2 = (\lambda_{\min}(x))^2u^{(1)} + (\lambda_{\max}(x))^2u^{(2)}$, for any $x \in R^n$.

Indeed, one has $x^2 = x \circ x = (\|x\|^2; 2x_1x_{2:n})$, and $(x^{\frac{1}{2}})^2 = x$. If x^{-1} is defined, then $x \circ x^{-1} = \mathbf{e}$, and we call x be invertible.

Lemma 2.1. *Let $x,s \in R^n$, one has*

(i) $\lambda_{\min}(x) + \lambda_{\min}(s) \leq \lambda_{\min}(x+s) \leq \lambda_{\max}(x+s) \leq \lambda_{\max}(x) + \lambda_{\max}(s)$;

(ii) $\lambda_{\min}(x^2) = \lambda_{\min}(x)^2$, $\lambda_{\max}(x^2) = \lambda_{\max}(x)^2$;

(iii) *If x is invertible, then $\lambda_{\min}(x^{-1}) = \lambda_{\max}(x)^{-1}$, and $\lambda_{\max}(x^{-1}) = \lambda_{\min}(x)^{-1}$.*

The trace and the determinant of $x \in R^n$ are

$$\mathbf{Tr}(x) := \lambda_{\min}(x) + \lambda_{\max}(x) = 2x_1,$$

$$\mathbf{det}(x) := \lambda_{\min}(x)\lambda_{\max}(x) = x_1^2 - \|x_{2:n}\|^2.$$

Lemma 2.2.[2] *For all $x,s,t \in L$, one has*

$$\mathbf{Tr}((x \circ s) \circ t) = \mathbf{Tr}(x \circ (s \circ t)).$$

The natural inner product is given by

$$\langle x,s\rangle := \mathbf{Tr}(x \circ s) = 2x^{\mathrm{T}}s, \quad x,s \in R^n.$$

Hence, the norm induced by this inner product, which is denoted as $\|\cdot\|_F$, satisfies

$$\|x\|_F = \sqrt{\langle x \circ x\rangle} = \sqrt{\mathbf{Tr}(x \circ x)}$$

$$= \sqrt{\lambda_{\min}(x)^2 + \lambda_{\max}(x)^2} = \sqrt{2}\|x\|. \quad (3)$$

It is obvious that

$$\lambda_{\min}(x) \le \|x\|_F, \text{ and } \lambda_{\max}(x) \le \|x\|_F. \quad (4)$$

Using Lemma 2.2, one can easily verify that

$$\|(x \circ s) \circ t)\|_F = \|(x \circ (s \circ t))\|_F, \ \forall x,s,t \in L. \quad (5)$$

Lemma 2.3. [16] *For any $x,s \in R^n$, one has*
(i) $\|x^2\|_F \le \|x\|_F^2$; *equality holds if and only if* $x_1 = \|x_{2:n}\|$;
(ii) $\mathbf{Tr}\left[(x \circ s)^2\right] \le \mathbf{Tr}(x^2 \circ s^2)$;
(iii) $\|x \circ s\|_F \le \lambda_{\max}(x)\|s\|_F \le \|x\|_F\|s\|_F$.
Lemma 2.4. *Let $s \in R^n$ and $x \in \text{int}L$. If $\lambda_{\min}(x) > \|s\|_F$, then $x - s \succ_L 0$.*
Proof. From (4), it follows that

$$\lambda_{\min}(-s)^2 \le \|-s\|_F^2 = \|s\|_F^2,$$

then

$$-\|s\|_F \le \lambda_{\min}(-s) \le \|s\|_F.$$

Thus, using Lemma 2.1 (i), we have

$$\lambda_{\min}(x-s) \ge \lambda_{\min}(x) + \lambda_{\min}(-s) \ge \lambda_{\min}(x) - \|s\|_F > 0,$$

which implies that $x - s \succ_L 0$. This proves the lemma.
Lemma 2.5. *Let $\rho \in L$, for any $v \in L$, one has*

$$\|\rho - v\|_F \le \frac{\|\rho \circ \rho - v \circ v\|_F}{\lambda_{\min}(\rho) + \lambda_{\min}(v)}.$$

Proof. From Lemma 2.1, Lemma 2.3 (iii) and (5), we have

$$\begin{aligned}
\|\rho - v\|_F &= \|((\rho+v)^{-1} \circ (\rho+v)) \circ (\rho - v)\|_F \\
&= \|(\rho+v)^{-1} \circ ((\rho+v) \circ (\rho - v))\|_F \\
&= \|(\rho+v)^{-1} \circ (\rho \circ \rho - v \circ v)\|_F \\
&\le \lambda_{\max}((\rho+v)^{-1})\|\rho \circ \rho - v \circ v\|_F \\
&= \frac{\|\rho \circ \rho - v \circ v\|_F}{\lambda_{\min}(\rho+v)} \\
&\le \frac{\|\rho \circ \rho - v \circ v\|_F}{\lambda_{\min}(\rho) + \lambda_{\min}(v)}.
\end{aligned}$$

The proof is completed.

Now we proceed by adapting the above definitions to the general case $K = L^1 \times \cdots \times L^N$ and $N > 1$. First we partition vectors $x,s \in R^n$ according to the dimensions of the successive cones L^i , so $x := (x^1;...;x^N)$ and $s := (s^1;...;s^N)$, and we define the algebra $(R^n, \circ)$ as a direct product of Jordan algebras:

$$x \circ s := (x^1 \circ s^1;...;x^N \circ s^N).$$

If $\mathbf{e}^i \in L^i$ is the unit element in the Jordan algebra for the ith cone, then $\mathbf{e} := (\mathbf{e}^1;...;\mathbf{e}^N)$ is the unit element in $(R^n, \circ)$. Since $L(x) := \text{diag}(L(x^1),...,L(x^N))$, it is easy to verify that [1]

$$\lambda_{\min}(x) = \lambda_{\min}(L(x)) = \min\{\lambda_{\min}(x^i), i \in J\},$$

$$\lambda_{\max}(x) = \lambda_{\max}(L(x)) = \max\{\lambda_{\max}(x^i), i \in J\}.$$

Furthermore,

$$\mathbf{Tr}(x) = \sum_{i=1}^N \mathbf{Tr}(x^i) = \sum_{i=1}^N \left[\lambda_{\min}(x^i) + \lambda_{\max}(x^i)\right],$$

$$\|x\|_F^2 = \sum_{i=1}^N \|x^i\|_F^2 = \sum_{i=1}^N \left(\lambda_{\min}(x^i)^2 + \lambda_{\max}(x^i)^2\right),$$

$$\mathbf{det}(x) = \prod_{i=1}^N \mathbf{det}(x^i) = \prod_{i=1}^N \lambda_{\min}(x^i)\lambda_{\max}(x^i).$$

3 The new search directions for SOCO

Throughout the paper, we assume that (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (x^0,y^0,s^0) such that $Ax^0 = b$, $x^0 \in \text{int}K$, $A^Ty^0 + s^0 = c$, $s^0 \in \text{int}K$. In fact, by using the self-dual embedding technique (e.g., [10]), we may (and will) assume that $x^0 = s^0 = \mathbf{e}$.

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system:

$$\begin{aligned}
Ax &= b, \ x \in K, \\
A^Ty + s &= c, \ s \in K, \\
L(x)s &= 0.
\end{aligned} \quad (6)$$

The basic idea of IPMs is to replace the third equation in system (6), the so-called *complementarity condition* for (P) and (D), by the parameterized equation $L(x)s = \mu\mathbf{e}$, with $\mu > 0$. Thus we consider the system

$$\begin{aligned}
Ax &= b, \ x \in K, \\
A^Ty + s &= c, \ s \in K, \\
L(x)s &= \mu\mathbf{e}.
\end{aligned} \quad (7)$$

For each $\mu > 0$, the parameterized system (7) has a unique solution $(x(\mu),y(\mu),s(\mu))$. We call $x(\mu)$ the μ-center of (P) and $(y(\mu),s(\mu))$ the μ-center of (D). The set of μ-center (with μ running through all positive real numbers) gives a homotopy path, which is called *the central path* of (P) and (D). If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $L(x)s = 0$, it naturally yields optimal solution for (P) and (D).

The weighted-path-following approach starts from the observation that system (7) can be generalized by replacing the vector $\mu\mathbf{e}$ with an arbitrary vector $\kappa \in \text{int}K$. Thus we obtain the following system

$$\begin{aligned}
Ax &= b, \ x \in K, \\
A^Ty + s &= c, \ s \in K, \\
L(x)s &= \kappa.
\end{aligned} \quad (8)$$

Since the IPC holds and A has full rank, system (8) has a unique solution. Hence we can apply Newton's method for system (8) to develop a weighted-path-following algorithm.

Suppose that we have $Ax = b$ and $A^T y + s = c$ for a triple (x,y,s) such that $x \in \text{int}K$ and $s \in \text{int}K$, hence x and s are strictly feasible. By linearizing system (8) we obtain the following system for the search directions:

$$\begin{aligned} A\Delta x &= 0, \\ A^{\mathrm{T}}\Delta y + \Delta s &= 0, \\ L(x)\Delta s + L(s)\Delta x &= \kappa - L(x)s. \end{aligned} \tag{9}$$

This system has a unique solution if and only if the matrix $AL(s)^{-1}L(x)A^{\mathrm{T}}$ is nonsingular. Unfortunately, this might not be the case, even if A has full rank. This is due to the fact that x and s do not operator commute in general (i.e. $L(x)L(s) \neq L(s)L(x)$). However, the system can be solved by using some scaling schemes. In this paper we take the NT-scaling scheme which results in the well-known NT-direction. Below, we only briefly outline the NT-scaling scheme. The readers who are interested in this may find the detail description in [15].

For any $x^i, s^i \in \text{int}L^i$ and $i \in J$, we denote the NT-scaling matrix W^i as:

$$\omega_i = \left(\frac{(s_1^i)^2 - \|s_{2:n}^i\|^2}{(x_1^i)^2 - \|x_{2:n}^i\|^2} \right)^{\frac{1}{4}},$$

$$\vec{s^i} = (\vec{s_1^i}, \vec{s_{2:n}^i}) = \omega_i^{-1}(s_1^i, s_{2:n}^i),\ \vec{x^i} = (\vec{x_1^i}, \vec{x_{2:n}^i}) = \omega_i(x_1^i, x_{2:n}^i),$$

$$\zeta^i = (\zeta_1^i, \zeta_{2:n}^i) = (\vec{x_1^i} + \vec{s_1^i}, \vec{s_{2:n}^i} - \vec{x_{2:n}^i}),$$

$$\alpha_i = \frac{\zeta_1^i}{\gamma(\zeta^i)},\ \ \beta_i = \frac{\zeta_{2:n}^i}{\gamma(\zeta^i)},\ \ \gamma(\zeta^i) = \sqrt{(\zeta_1^i)^2 - (\zeta_{2:n}^i)^{\mathrm{T}}\zeta_{2:n}^i},$$

$$W^i = \omega_i \begin{bmatrix} \alpha_i & \beta_i^{\mathrm{T}} \\ \beta_i & I + \frac{\beta_i\beta_i^{\mathrm{T}}}{1+\alpha_i} \end{bmatrix},$$

where I represents the $n_i \times n_i$ identity matrix.

For the above choices one has (see, Proposition 7.6. in [15])

$$W^i x^i = (W^i)^{-1} s^i,\ i \in J.$$

We define

$$W := \text{diag}(W^1, ..., W^N).$$

Then

$$v := (v^1; ...; v^N) = Wx \quad (= W^{-1}s). \tag{10}$$

Let us further denote

$$\overline{A} := AW^{-1}, \quad d_x := W\Delta x, \quad d_s := W^{-1}\Delta s. \tag{11}$$

Applying (10) and (11), system (9) can be rewritten as follows:

$$\begin{aligned} \overline{A}d_x &= 0, \\ \overline{A}^{\mathrm{T}}\Delta y + d_s &= 0, \\ L(W^{-1}v)Wd_s + L(Wv)W^{-1}d_x &= \kappa - L(W^{-1}v)Wv. \end{aligned} \tag{12}$$

Since this system is equivalent to system (9), it may not have a unique solution. To overcome this difficulty, in the same way as Bai et al. did in [2], we replace the third equation in system (12) by

$$L(v)d_s + L(v)d_x = \kappa - L(v)v,$$

which, after multiplying of both sides from the left with $L(v)^{-1}$, becomes

$$d_s + d_x = L(v)^{-1}\kappa - v. \tag{13}$$

Thus the system defining the scaled search directions becomes

$$\begin{aligned} \overline{A}d_x &= 0, \\ \overline{A}^{\mathrm{T}}\Delta y + d_s &= 0, \\ d_s + d_x &= L(v)^{-1}\kappa - v. \end{aligned} \tag{14}$$

Since the matrix $\overline{A}^{\mathrm{T}}\overline{A}$ is positive definite, this system has a unique solution.

In this paper, in order to avoid calculating the inverse matrix of $L(v)$ in system (14), motivated by [4,5], we replace the right hand side in the last equation in system (14) by $2(\kappa - v)$. Thus we will use the following system to define our new search direction:

$$\begin{aligned} \overline{A}d_x &= 0, \\ \overline{A}^{\mathrm{T}}\Delta y + d_s &= 0, \\ d_s + d_x &= 2(\kappa - v). \end{aligned} \tag{15}$$

Since system (15) has the same matrix of coefficients as system (14), also system (15) has a unique solution

By transforming back to the $x-$ and $s-$space, respectively, using (11), we obtain search directions Δx and Δs in the original spaces, with

$$\Delta x = W^{-1}d_x,\ \ \Delta s = Wd_s. \tag{16}$$

By taking a full NT-step, we construct a new triple (x_+, y_+, s_+) according to

$$x_+ = x + \Delta x,\ \ y_+ = y + \Delta y,\ \ s_+ = s + \Delta s.$$

4 The weighted-path-following interior-point algorithm

We define

$$p_v := 2(\kappa - v) = d_x + d_s. \tag{17}$$

Since $\kappa \in \text{int}K$, we have $\lambda_{\min}(\kappa) > 0$. Now for any vector $v \in \text{int}K$, we define the following proximity measure

$$\sigma(v;\kappa) := \sigma(x,s;\kappa) := \frac{\|p_v\|_F}{2\lambda_{\min}(\kappa)} = \frac{\|\kappa - v\|_F}{\lambda_{\min}(\kappa)}. \tag{18}$$

We introduce another measure

$$\sigma_c(\kappa) := \frac{\lambda_{\max}(\kappa)}{\lambda_{\min}(\kappa)}, \tag{19}$$

where

$$\lambda_{\max}(\kappa) = \max\{\lambda_{\max}(\kappa^i),\ \ i \in J\},$$
$$\lambda_{\min}(\kappa) = \min\{\lambda_{\min}(\kappa^i),\ \ i \in J\}.$$

It is obvious that $\sigma_c(\kappa) \geq 1$. Equality holds if and only if $\lambda_{\max}(\kappa) = \lambda_{\min}(\kappa)$, i.e, $\kappa = \eta\mathbf{e}$, where $\eta > 0$ is a positive constant, which implies that κ is centered. Thus $\sigma_c(\kappa)$ can be used to measure the distance of κ to the central path.

Furthermore, let us introduce the notation

$$q_v := d_x - d_s. \tag{20}$$

From the first two equations of the system (15), we obtain that d_x and d_s are orthogonal, that is $d_x^{\mathrm{T}} d_s = d_s^{\mathrm{T}} d_x = 0$. Thus, we get $\|p_v\|_F = \|q_v\|_F$. Consequently, the proximity measure can be written in the following form

$$\sigma(v;\kappa) := \frac{\|q_v\|_F}{2\lambda_{\min}(\kappa)}. \tag{21}$$

Hence, we have

$$d_x = \frac{p_v + q_v}{2},\ \ d_s = \frac{p_v - q_v}{2}, \tag{22}$$

$$d_x \circ d_s = \frac{p_v \circ p_v - q_v \circ q_v}{4}. \tag{23}$$

We now give the weighted-path-following interior-point algorithm for SOCO as follows.

Algorithm 4.1.(The weighted-path-following interior-point algorithm for SOCO)

Step 1: Let (x^0, y^0, s^0) be the strictly feasible interior point, and let $\kappa^0 = W^0 x^0 = (W^0)^{-1} s^0$. Choosing an accuracy parameter $\varepsilon > 0$, and a fixed update parameter $\theta \in (0,1)$. Set $x := x^0, y := y^0, s := s^0, \kappa := \kappa^0$.

Step 2: If $x^{\mathrm{T}} s < \varepsilon$, then stop. Otherwise, perform the following steps.

Step 3: Calculate $\kappa := (1-\theta)\kappa$.

Step 4: Solve the system (15) and via (16) to obtain $(\Delta x, \Delta y, \Delta s)$.

Step 5: Update $(x,y,s) := (x,y,s) + (\Delta x, \Delta y, \Delta s)$ and go to Step 2.

In the next section we will prove that this algorithm is well defined, and we will also give an upper bound for the number of iterations performed by the algorithm.

5 Analysis of the algorithm

5.1 Feasibility of the full NT-step

In this subsection, we find a condition that guarantees feasibility of the iterates after a full NT-step. As before, let $x, s \in \mathrm{int}K$, and let W be the NT-scaling matrix. Using (10) and (11), we obtain that

$$x_+ = x + \Delta x = W^{-1}(v + d_x), \tag{24}$$

$$s_+ = s + \Delta s = W(v + d_s). \tag{25}$$

Since W and its inverse W^{-1} are automorphisms of K, x_+ and s_+ belong to int$K$ if and only if $v + d_x$ and $v + d_s$ belong to int$K$.

Let $0 \leq \alpha \leq 1$, we define

$$v^x(\alpha) = v + \alpha d_x \quad \text{and} \quad v^s(\alpha) = v + \alpha d_s.$$

Lemma 5.1. *Let $\sigma(v;\kappa) < 1$. Then the full* NT*-step is strictly feasible, hence $x_+ \in \mathrm{int}K$ and $s_+ \in \mathrm{int}K$.*

Proof. From (22) and (23), we have

$$\begin{aligned} v^x(\alpha) \circ v^s(\alpha) &= (v + \alpha d_x) \circ (v + \alpha d_s) \\ &= v \circ v + \alpha v \circ (d_x + d_s) + \alpha^2 d_x \circ d_s \\ &= v \circ v + \alpha v \circ p_v + \alpha^2 \frac{p_v \circ p_v - q_v \circ q_v}{4} \\ &= (1-\alpha) v \circ v + \alpha (v \circ v + v \circ p_v) + \alpha^2 \frac{p_v \circ p_v - q_v \circ q_v}{4}. \end{aligned}$$

Moreover, (17) yields $v + \frac{p_v}{2} = \kappa$, and thus $v \circ v + v \circ p_v = \kappa \circ \kappa - \frac{p_v \circ p_v}{4}$. Consequently

$$v^x(\alpha) \circ v^s(\alpha) = (1-\alpha) v \circ v + \alpha \left(\kappa \circ \kappa - (1-\alpha) \frac{p_v \circ p_v}{4} - \alpha \frac{q_v \circ q_v}{4} \right). \tag{26}$$

Since $0 \leq \alpha \leq 1$, using $\sigma(v;\kappa) < 1$, we have

$$\begin{aligned} \|(1-\alpha)\frac{p_v \circ p_v}{4} + \alpha \frac{q_v \circ q_v}{4}\|_F &\leq (1-\alpha)\frac{\|p_v \circ p_v\|_F}{4} + \alpha \frac{\|q_v \circ q_v\|_F}{4} \\ &\leq (1-\alpha)\frac{\|p_v\|_F^2}{4} + \alpha \frac{\|q_v\|_F^2}{4} \\ &= \sigma(v;\kappa)^2 \lambda_{\min}(\kappa)^2 \\ &< \lambda_{\min}(\kappa)^2 = \lambda_{\min}(\kappa \circ \kappa). \end{aligned}$$

Thus, it follows from Lemma 2.4 that

$$\kappa \circ \kappa - (1-\alpha)\frac{p_v \circ p_v}{4} - \alpha \frac{q_v \circ q_v}{4} \succ_K 0.$$

Since the set of the second-order cones is cone, we can conclude that

$$(1-\alpha) v \circ v + \left(\kappa \circ \kappa - (1-\alpha)\frac{p_v \circ p_v}{4} - \alpha \frac{q_v \circ q_v}{4} \right) \succ_K 0.$$

Thus, we have $\mathbf{det}(v^x(\alpha) \circ v^s(\alpha)) > 0$, which, together with Lemma 2.3 in [2], implies that for each $\alpha \in [0,1]$,

$$\mathbf{det}(v^x(\alpha))\mathbf{det}(v^s(\alpha)) \geq \mathbf{det}(v^x(\alpha) \circ v^s(\alpha)) > 0.$$

Therefore, by Lemma 6.1 in [16] we get $v^x(1) = v + d_x \succ_K 0$ and $v^s(1) = v + d_s \succ_K 0$. This completes the proof.

The following lemma gives an upper bound for the duality gap obtained after a full NT-step.

Lemma 5.2. *Let $\sigma(x,s;\kappa) < 1$. Then*

$$(x_+)^{\mathrm{T}} s_+ = \|\kappa\|^2 - \frac{\|q_v\|^2}{4},$$

and hence $(x_+)^{\mathrm{T}}s_+ \le \|\kappa\|^2$.
Proof. Due to (24) and (25), we may write

$$(x_+)^{\mathrm{T}}s_+ = \big(W^{-1}(v+d_x)\big)^{\mathrm{T}}\big(W(v+d_s)\big) = (v+d_x)^{\mathrm{T}}(v+d_s).$$

From (26) with $\alpha = 1$, we get

$$(v+d_x)\circ(v+d_s) = \kappa\circ\kappa - \frac{q_v\circ q_v}{4}.$$

Thus,

$$\begin{aligned} x_+^{\mathrm{T}}s_+ &= \mathbf{e}^{\mathrm{T}}((v+d_x)\circ(v+d_s)) \\ &= \mathbf{e}^{\mathrm{T}}(\kappa\circ\kappa) - \frac{\mathbf{e}^{\mathrm{T}}(q_v\circ q_v)}{4} \\ &= \|\kappa\|^2 - \frac{\|q_v\|^2}{4}. \end{aligned}$$

This completes the proof.

5.2 Quadratic convergence

In this subsection, we prove that the same condition, namely $\sigma(x,s;\kappa) < 1$, is sufficient for the quadratic convergence of the Newton process when taking full NT-steps.

Lemma 5.3. *Let* $\sigma := \sigma(x,s;\kappa) < 1$. *Then*

$$\sigma(x_+,s_+;\kappa) \le \frac{\sigma^2}{1+\sqrt{1-\sigma^2}}.$$

Thus $\sigma(x_+,s_+;\kappa) < \sigma^2$, *which shows the quadratic convergence of the NT-steps.*
Proof. Taking $\alpha = 1$ in (26), we have

$$v_+\circ v_+ = \kappa\circ\kappa - \frac{q_v\circ q_v}{4}, \tag{27}$$

then it follows from (21) that

$$\begin{aligned} \lambda_{\min}(v_+)^2 &= \lambda_{\min}(v_+\circ v_+) \\ &= \lambda_{\min}(\kappa\circ\kappa - \frac{q_v\circ q_v}{4}) \\ &\ge \lambda_{\min}(\kappa)^2 - \frac{\|q_v\|_F^2}{4} \\ &= \lambda_{\min}(\kappa)^2(1-\sigma^2). \end{aligned}$$

Thus

$$\lambda_{\min}(v_+) \ge \lambda_{\min}(\kappa)\sqrt{1-\sigma^2}. \tag{28}$$

From (27) and (28), also using Lemma 2.5, one has

$$\begin{aligned} \sigma(x_+,s_+;\kappa) &= \frac{\|\kappa - v_+\|_F}{\lambda_{\min}(\kappa)} \\ &\le \frac{\|\kappa\circ\kappa - v_+\circ v_+\|_F}{\lambda_{\min}(\kappa)(\lambda_{\min}(\kappa)+\lambda_{\min}(v_+))} \\ &\le \frac{\|q_v\circ q_v\|_F}{(2\lambda_{\min}(\kappa))^2(1+\sqrt{1-\sigma^2})} \\ &\le \frac{1}{1+\sqrt{1-\sigma^2}}\left(\frac{\|q_v\|_F}{2\lambda_{\min}(\kappa)}\right)^2 \\ &= \frac{\sigma^2}{1+\sqrt{1-\sigma^2}}. \end{aligned}$$

Consequently, we have $\sigma(x_+,s_+;\kappa) < \sigma^2$. This completes the proof.

5.3 Iteration bound

Lemma 5.4. *Let* $\sigma(x,s;\kappa) < 1$, *and* $\kappa_+ = (1-\theta)\kappa$, *where* $0 < \theta < 1$. *Then*

$$\sigma(x_+,s_+;\kappa_+) \le \frac{\theta}{1-\theta}\sqrt{2N}\sigma_c(\kappa) + \frac{1}{1-\theta}\sigma(x_+,s_+;\kappa).$$

Furthermore, if $\sigma(x,s;\kappa) \le \frac{1}{2}$, $\theta = \frac{1}{5\sqrt{2N}\sigma_c(\kappa)}$ *and* $N \ge 2$, *then* $\sigma(x_+,s_+;\kappa_+) \le \frac{1}{2}$.
Proof. With triangle inequality, we have

$$\begin{aligned} \sigma(x_+,s_+;\kappa_+) &= \frac{\|\kappa_+ - v_+\|_F}{\lambda_{\min}(\kappa_+)} \\ &\le \frac{\|\kappa_+ - \kappa\|_F}{\lambda_{\min}(\kappa_+)} + \frac{\|\kappa - v_+\|_F}{\lambda_{\min}(\kappa_+)} \\ &= \frac{\theta}{1-\theta}\frac{\|\kappa\|_F}{\lambda_{\min}(\kappa)} + \frac{1}{1-\theta}\frac{\|\kappa - v_+\|_F}{\lambda_{\min}(\kappa)} \\ &= \frac{\theta}{1-\theta}\frac{\|\kappa\|_F}{\lambda_{\min}(\kappa)} + \frac{1}{1-\theta}\sigma(x_+,s_+;\kappa). \end{aligned}$$

Since

$$\begin{aligned} \|\kappa\|_F &= \sqrt{\sum_{i=1}^{N}\Big(\lambda_{\min}(\kappa^i)^2 + \lambda_{\max}(\kappa^i)^2\Big)} \\ &\le \sqrt{2\sum_{i=1}^{N}\lambda_{\max}(\kappa^i)^2} \le \sqrt{2N}\lambda_{\max}(\kappa), \end{aligned}$$

by (19) we have

$$\sigma(x_+,s_+;\kappa_+) \le \frac{\theta}{1-\theta}\sqrt{2N}\sigma_c(\kappa) + \frac{1}{1-\theta}\sigma(x_+,s_+;\kappa).$$

Thus the first part of the lemma is proved. Now, let $\sigma(x,s;\kappa) \le \frac{1}{2}$, $\theta = \frac{1}{5\sqrt{2N}\sigma_c(\kappa)}$ and $N \ge 2$. Since $\sigma_c(\kappa) \ge 1$, we have $\theta = \frac{1}{5\sqrt{2N}\sigma_c(\kappa)} \le \frac{1}{10}$. Furthermore, if $\sigma(x,s;\kappa) \le \frac{1}{2}$, then from Lemma 5.3 we deduce that $\sigma(x_+,s_+;\kappa) \le \frac{1}{4}$. Thus, the above relations yield $\sigma(x_+,s_+;\kappa_+) \le \frac{1}{2}$. The proof is completed.

Remark 5.5. According to Algorithm 4.1, at the start of the algorithm we choose a strictly feasible pair (x^0,s^0) such that $\sigma(x^0,s^0;\kappa^0) < \frac{1}{2}$. Note that $\sigma_c(\kappa) = \sigma_c(\kappa^0)$ for all iterates produced by the algorithm. Thus, from Lemma 5.1 and Lemma 5.4 we know that, for $\theta = \frac{1}{5\sqrt{2N}\sigma_c(\kappa)}$, the conditions $x \in \mathrm{int}K, s \in \mathrm{int}K$ and $\sigma(x,s;\kappa) \le \frac{1}{2}$ are maintained throughout the algorithm. Hence the algorithm is well defined.

In the next lemma, we give an upper bound for the total number of iterations performed by Algorithm 4.1.

Lemma 5.6. *Assume that* x^0 *and* s^0 *are strictly feasible*

and $\kappa^0 = W^0x^0 = (W^0)^{-1}s^0$. *Moreover, let* x^k *and* s^k *be the vectors obtained after k iterations. Then the inequality* $(x^k)^{\mathrm{T}}s^k \leq \varepsilon$ *is satisfied for* $k \geq \left\lceil \frac{1}{2\theta}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon} \right\rceil$.

Proof. After k iterations, we get $\kappa = (1-\theta)^k\kappa^0$. From Lemma 5.2 we have

$$(x^k)^{\mathrm{T}}s^k \leq \|\kappa\|^2 = (1-\theta)^{2k}\|\kappa^0\|^2 = (1-\theta)^{2k}(x^0)^{\mathrm{T}}s^0.$$

Thus the inequality $(x^k)^{\mathrm{T}}s^k \leq \varepsilon$ holds if

$$(1-\theta)^{2k}(x^0)^{\mathrm{T}}s^0 \leq \varepsilon.$$

Taking logarithms, we obtain that

$$2k\log(1-\theta) + \log((x^0)^{\mathrm{T}}s^0) \leq \log\varepsilon.$$

Using the inequality $-\log(1-\theta) \geq \theta$, we deduce that the above relation holds if

$$2k\theta \geq \log((x^0)^{\mathrm{T}}s^0) - \log\varepsilon = \log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon}.$$

This proves the lemma.

Theorem 5.7. *Suppose that* x^0 *and* s^0 *are strictly feasible and* $\kappa^0 = W^0x^0 = (W^0)^{-1}s^0$. *If* $\theta = \frac{1}{5\sqrt{2N}\sigma_c(\kappa^0)}$, *then Algorithm* 4.1 *requires at most*

$$\left\lceil \frac{5\sqrt{2}}{2}\sigma_c(\kappa^0)\sqrt{N}\log\frac{(x^0)^T s^0}{\varepsilon} \right\rceil$$

iterations. The output is a primal-dual pair (x,s) *satisfying* $x^{\mathrm{T}}s \leq \varepsilon$.

Remark 5.8. Theorem 5.7 shows that the best iteration bound is obtained by following the central path. Indeed, we have $\sigma_c(\kappa^0) = 1$ in this case, and we get the well known iteration bound, namely $\left\lceil \frac{5\sqrt{2}}{2}\sqrt{N}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon} \right\rceil$. If the starting point is not perfectly centered, then $\sigma_c(\kappa^0) > 1$ and thus the iteration bound is worse.

Corollary 5.9. *If one takes* $x^0 = s^0 = \mathbf{e}$, *then the iteration bound becomes* $O\left(\sqrt{N}\log\frac{N}{\varepsilon}\right)$, *which is the currently best known iteration bound for the algorithm with small-update method.*

Conclusions

We have developed a weighted-path-following algorithm for SOCO with full NT-step and derived the currently best known iteration bound for the algorithm with small-update method. Our analysis is a relatively simple and straightforward extension of analogous results for linear optimization.

Some interesting topics remain for further research. Firstly, the search direction used in this paper is based on the NT-symmetrization scheme. It may be possible to design similar algorithms using other symmetrization schemes and to obtain polynomial-time iteration bounds. Secondly, the extension to symmetric cone optimization deserves to be investigated. In addition, it is well-known that large-update methods are much more efficient than small-update methods in practice which have a worst case iteration bound. Jansen et al. [6] proposed a long-step target-following method for linear optimization. It is an interesting question whether we can extend their method for solving SOCO by using Jordan algebra techniques.

Acknowledgement

This paper was partly supported by National Natural Science Foundation of China (11101248), Research Award Foundation for Outstanding Young Scientists of Shandong Province (BS2011SF024, BS2012SF025), Science Technology Research Projects of Education Department of Henan Province (13A110767) and Basic and Frontier Technology Research Project of Henan Province (142300410318).

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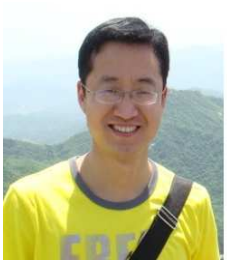

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