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Applications of First Integral Method to Some Complex Nonlinear Evolution Systems

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Abstract: The aim of the present paper is to study nonlinear system of partial differential equations (PDEs) involving both complex- and real-valued unknown functions. We shall extend the use of the first integral method ”based on the theory of commutative algebra” to construct new solutions to the coupled Higgs field equations, the Davey-Sterwatson (DS) equations and the coupled Klein-Gordon-Zakharov equations. All the algebraic computations in this work are performed using Mathematica software.

Keywords: First integral method, Coupled Higgs field equations, Davey-Sterwatson equations, Coupled Klein-Gordon-Zakharov equations.

1 Introduction

Nonlinear partial differential equations (PDEs) appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibres, biology, solid state physics, chemical physics and Stochastic control with relevance to information sciences. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as bilinear transformation, the tanh-sech method extended tanh method, sine-cosine method, exp-function method and others were used to study the solutions of these PDEs [13]-[30].

Our interest in the present work is in implementing the first integral method. The first integral method was first proposed by Feng [1] in solving Burgers- KdV equation. It is a direct algebraic method based on the commutative algebra. Recently, it was successfully used for constructing exact solutions to a variety of nonlinear problems see [2,3,4,5,6,7,8,9]. In this work, we consider the following mathematical models:

First, we study the Coupled Higgs field equations

$$u_{tt}-u_{xx}-au+b|u|^2u-2uv=0$$
$$v_{tt}+v_{xx}-b(|u|^2)_{xx}=0 \tag{1}$$

describing a system of conserved scalar nucleons interacting with a neutral scalar meson. Where $a>0$, $b>0$, $u=u(x,t)$ is a complex-valued function and $v=v(x,t)$ is a real-valued function. Authors in [10] apply the functional variable method and obtained analytical solution to this system.

Second, we study The Davey-Sterwatson (DS) equations

$$iu_t+\frac{1}{2}b^2(u_{xx}+b^2u_{yy})+a|u|^2u-uv=0,$$
$$v_{xx}-b^2v_{yy}-2a(|u|^2)_{xx}=0, \tag{2}$$

where a is a real constant, $b^2=\pm1$, $u(x,y,t)$ is a complex valued function and $v(x,y,t)$ is a real valued function. These equations were introduced in order to discuss the instability of uniform trains of weakly nonlinear water waves in two dimensional space. Yomba use the extended F-expansion method and general projective Riccati equations method to construct exact solutions to Davey-Sterwatson (DS) equations see [11,12]

Finally, we study the Coupled Klein-Gordon-Zakharov equations

$$u_{tt}-c_0\nabla^2u+f_0^2u+\delta uv=0$$
$$v_{tt}-c_0^2\nabla^2v-b\nabla^2(|u|^2)=0, \tag{3}$$

where c_0, f_0, and b are constants, $u(x,y,z,t)$ is a complex valued function and $v(x,y,z,t)$ is a real valued function. General projective Riccati equations method in [11] is applied to construct exact solution for (3). Also the

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extended F-expansion method is used to solve the same equation [12].

Our goal in this work is implementing the first integral method with help of the symbolic computational Mathematica software to show its applicability in handling nonlinear equations, so that one can apply it to models of various types of nonlinear equations.

2 Analysis of the first integral method

In this section we go briefly over the procedure of the first integral method [1,4,9]. Consider the nonlinear PDE for a function u of two variables, x and t :

$$F(u, u_t, u_x, u_{tt}, u_{tx}, ...) = 0. \tag{4}$$

introduce the wave variable $\xi = x - ct$ so that $u(x,t) = u(\xi)$. Based on this we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \\ &\frac{\partial^2}{\partial t^2}(.) = c^2\frac{d^2}{d\xi^2}(.), \quad \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.). \end{aligned} \tag{5}$$

Using (5) changes the PDE in (4) to an ODE

$$G(u, u', u'', u''', ...) = 0, \tag{6}$$

where the prime denotes the derivatives with respect to ξ. Next, we introduce a new independent variables

$$X(\xi) = u(\xi), \quad Y(\xi) = u'(\xi), \tag{7}$$

This yields to a system of ODEs

$$X'(\xi) = Y(\xi), \quad Y'(\xi) = H(X(\xi), Y(\xi)), \tag{8}$$

According to the qualitative theory of ordinary differential equations , if we can find integrals to (8), we can reduce (8) to a first-order ODE to be solved directly. But in general (it is really difficult for us to realize this because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are) see [1].

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of equations in (8) and $q(X,Y) = \sum_{i=0}^m a_i(X)Y^i$ is an irreducible polynomial in complex domain $C[X,Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{9}$$

where $a_i(X), (i = 0, 1, 2, ..., m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (9) is called the first integral to (8). According to the division theorem, there exists a polynomial $g(X) + h(X)Y$ in a complex domain $C[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = [g(X) + h(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \tag{10}$$

3 Coupled Higgs field equations

In this section we study the Coupled Higgs field equations

$$\begin{aligned} &u_{tt} - u_{xx} - au + b|u|^2u - 2uv = 0 \\ &\qquad v_{tt} + v_{xx} - b(|u|^2)_{xx} = 0 \end{aligned} \tag{11}$$

First, we make the following transformation:

$$\begin{aligned} &u(x,t) = u(\xi)\exp(i\eta), \quad v(x,t) = v(\xi) \\ &\qquad \xi = k(x + \lambda t), \quad \eta = \alpha x + \beta t. \end{aligned} \tag{12}$$

Substituting (12) into (11) we obtain

$$\begin{aligned} &0 = bu^3 + (a - \alpha^2 + \beta^2 + 2v)u + k^2(\lambda^2 - 1)u'' \\ &\quad + i(-2k(\alpha - \beta\lambda)u'), \\ &0 = -2bu'^2 - 2buu' + (1 + \lambda^2)v'', \end{aligned} \tag{13}$$

where the prime denotes the derivation with respect to ξ. We divide the first equation of (13) into two parts imaginary part and real part as follow

$$Im : (-2k(\alpha - \beta\lambda)u') = 0. \tag{14}$$

$$Re : bu^3 + (a - \alpha^2 + \beta^2 + 2v)u + k^2(\lambda^2 - 1)u'' = 0. \tag{15}$$

We only solve (15), instead of both (14) and (15), provided that

$$\alpha = \beta\lambda. \tag{16}$$

integrating the second equation of (13) twice and setting the constant of integration to be zero. We find

$$v = \frac{bu^2}{1 + \lambda^2}. \tag{17}$$

Substituting (17) into (15) we have

$$-(a - \alpha^2 + \beta^2)u + (b - \frac{2b}{1 + \lambda^2})u^3 + k^2(\lambda^2 - 1)u'' = 0. \tag{18}$$

Using (7) we obtain

$$X'(\xi) = Y(\xi), \tag{19}$$

$$Y'(\xi) = \frac{a - \alpha^2 + \beta^2}{k^2(\lambda^2 - 1)}X(\xi) - \frac{b}{k^2(\lambda^2 + 1)}X(\xi)^3. \tag{20}$$

Suppose that $m = 1$ in (9), then

$$q[X,Y] = a_0(X) + a_1(x)Y = 0.$$

From (10) we obtain

$$\begin{aligned}\frac{dq}{d\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi}+\frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}\\ &= [a_0'(X)+a_1'(X)Y]Y\\ &+ a_1(X)[\frac{a-\alpha^2+\beta^2}{k^2(\lambda^2-1)}X-\frac{b}{k^2(\lambda^2+1)}X^3]\\ &= a_1'(X)Y^2+a_0'(X)Y\\ &+ [\frac{a-\alpha^2+\beta^2}{k^2(\lambda^2-1)}X-\frac{b}{k^2(\lambda^2+1)}X^3]a_1(X). \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\frac{dq}{d\xi} &= [g(X)+h(X)Y](a_0(X)+a_1(x)Y)\\ &= h(X)a_1(X)Y^2+[g(X)a_1(X)+h(X)a_0(X)]Y\\ &+ g(X)a_0(X). \qquad (22)\end{aligned}$$

By equating the coefficients of Y^i $(i=2,1,0)$ in (21) and (22) we obtain

$$a_1'(X)=h(X)a_1(X), \qquad (23)$$

$$a_0'(X)=h(X)a_0(X)+a_1(X)g(X), \qquad (24)$$

$$a_0(X)g(X)=[\frac{a-\alpha^2+\beta^2}{k^2(\lambda^2-1)}X-\frac{b}{k^2(\lambda^2+1)}X^3]a_1(X). \qquad (25)$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, then from(23) we deduce that $a_1(X)$ is a constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X))=1$ only. Suppose that $g(X)=A_1X+A_0$ and $A_1\neq 0$, then we find that $a_0(X)$ is expressed as

$$a_0(X)=\frac{A_2}{2}X^2+A_1X+A_0. \qquad (26)$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into (25) and setting all coefficients of powers of X to be zeros, then we obtain the following system of nonlinear algebraic equations

$$\begin{aligned}0 &= A_0A_1\\ 0 &= (\frac{A_2^2}{2}+\frac{b}{k^2(1+\lambda^2)})\\ 0 &= \frac{3}{2}A_1A_2\\ 0 &= A_1^2+A_0A_2-\frac{a^2+\beta^2-\beta^2\lambda^2}{k^2(1+\lambda^2)} \qquad (27)\end{aligned}$$

Solving (27), we obtain

$$\begin{aligned}A_0 &= \pm\frac{i(a^2+\beta^2+a\lambda^2-\beta^2\lambda^4)}{\sqrt{2}\sqrt{b}k(-1+\lambda^2)\sqrt{1+\lambda^2}},\\ A_1 &= 0,\\ A_2 &= \mp i\frac{\sqrt{2}\sqrt{b}}{k\sqrt{1+\lambda^2}}, \qquad (28)\end{aligned}$$

where β,λ and k are arbitrary. Using (26) in (9), we obtain

$$Y(\xi)=-\frac{A_2}{2}X^2-A_1X-A_0. \qquad (29)$$

Combining (29) with (19), we obtain the exact solution to (20)

$$X(\xi)=-\frac{\sqrt{2}\sqrt{A_0}\tan(\frac{\sqrt{A_0}\sqrt{A_2}(\xi-2c_0)}{\sqrt{2}})}{\sqrt{A_2}}, \qquad (30)$$

where c_0 is the integration constant. Therefore, the solutions of (11) are

$$\begin{aligned}u(x,t) &= \pm\tanh(\frac{\sqrt{-a+\beta^2(-1+\lambda^2)}(k(x+\lambda t)-2c_0)}{\sqrt{2}k\sqrt{-1+\lambda^2}}\\ &\times i\exp(i\beta(\lambda x+t))\frac{\sqrt{-a+\beta^2(-1+\lambda^2)}}{\sqrt{b}\sqrt{1-\frac{2}{1+\lambda^2}}}. \qquad (31)\end{aligned}$$

$$v(x,t)=\frac{\mu}{\lambda^2-1}\tanh^2(\frac{\sqrt{\mu}(k(x+\lambda t)-2c_0)}{\sqrt{2}k\sqrt{-1+\lambda^2}}, \qquad (32)$$

where $\mu=a-\beta^2(-1+\lambda^2)$.

4 Davey−Sterwatson (DS) equations

Consider the Davey-Sterwatson (DS) equations

$$\begin{aligned}&iu_t+\frac{1}{2}b^2(u_{xx}+b^2u_{yy})+a|u|^2u-uv=0,\\ &v_{xx}-b^2v_{yy}-2a(|u|^2)_{xx}=0. \qquad (33)\end{aligned}$$

Apply the following transformations:

$$\begin{aligned}&u(x,y,t)=u(\xi)\exp(i\eta), \quad v(x,y,t)=v(\xi),\\ &\xi=k(x+ly+\lambda t), \quad \eta=\alpha x+\beta y+\gamma t. \qquad (34)\end{aligned}$$

Substitution (34) into (33) yield the following system of ODEs

$$\begin{aligned}0 &= 2au^3-(b^2\alpha^2+b^4\beta^2+2\gamma+2v)u+b^2k^2(1+b^2l^2)u'',\\ 0 &= -4au'^2-4auu''+(1-b^2l^2)v'', \qquad (35)\end{aligned}$$

where $\lambda=-b^2\alpha-b^4l\beta$ and the prime denotes the derivation with respect to ξ.

Integrating the second equation of (35) twice and setting the constant of integration to be zero. We find

$$v=\frac{2a}{1-b^2l^2}u^2. \qquad (36)$$

Substituting (36) into the first equation of (35) we have

$$\begin{aligned}0 &= b^2k^2(1+b^2l^2)u''-(b^2\alpha^2+b^4\beta^2+2\gamma)u\\ &+ a(2+\frac{4}{b^2l^2-1})u^3. \qquad (37)\end{aligned}$$

Using (7) we obtain

$$X'(\xi)=Y(\xi), \qquad (38)$$

$$Y'(\xi) = \frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)}X(\xi) - \frac{2a}{b^2k^2(-1+b^2l^2)}X(\xi)^3. \quad (39)$$

Suppose that $m=1$ in (9), then

$$q[X,Y] = a_0(X)+a_1(x)Y = 0.$$

From (10) we obtain

$$\begin{aligned} \frac{dq}{d\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} \\ &= a_1(X)[\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)}X - \frac{2a}{b^2k^2(-1+b^2l^2)}X^3] \\ &+ [a_0'(X)+a_1'(X)Y]Y. \\ &= [\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)}X - \frac{2a}{b^2k^2(-1+b^2l^2)}X^3]a_1(X) \\ &+ a_1'(X)Y^2 + a_0'(X)Y. \end{aligned} \quad (40)$$

and

$$\begin{aligned} \frac{dq}{d\xi} &= [g(X)+h(X)Y](a_0(X)+a_1(x)Y) \\ &= h(X)a_1(X)Y^2 + [g(X)a_1(X)+h(X)a_0(X)]Y \\ &+ g(X)a_0(X). \end{aligned} \quad (41)$$

By equating the coefficients of Y^i $(i=2,1,0)$ in (40) and (41) we obtain

$$a_1'(X) = h(X)a_1(X), \quad (42)$$

$$a_0'(X) = h(X)a_0(X)+a_1(X)g(X), \quad (43)$$

$$a_0(X)g(X) = [\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)}X]a_1(x) - [\frac{2a}{b^2k^2(-1+b^2l^2)}X^3]a_1(X). \quad (44)$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, then from (42) we deduce that $a_1(X)$ is a constant and $h(X)=0$. For simplicity, take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X))=1$ only. Suppose that $g(X)=A_2X+A_1$ and $A_2\neq 0$, then we find that $a_0(X)$ is expressed as

$$a_0(X) = \frac{A_2}{2}X^2 + A_1X + A_0. \quad (45)$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into (44) and setting all coefficients of powers of X to be zeros, then we obtain the following system of nonlinear algebraic equations

$$\begin{aligned} 0 &= A_0A_1 \\ 0 &= \frac{A_2^2}{2} + \frac{2a}{b^2k^2(-1+b^2l^2)} \\ 0 &= \frac{3}{2}A_1A_2 \\ 0 &= A_1^2 + A_0A_2 - \frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)} \end{aligned} \quad (46)$$

Solving (46), we obtain

$$\begin{aligned} A_0 &= \pm\frac{(-1+b^2l^2)\sqrt{\frac{a}{b^2-b^4l^2}}(b^2\alpha^2+b^4\beta^2+2\gamma)}{2ak(1+b^2l^2)}, \\ A_1 &= 0, \\ A_2 &= \mp\frac{2\sqrt{\frac{a}{b^2-b^4l^2}}}{k}, \end{aligned} \quad (47)$$

where α,β,γ,l and k are arbitrary. Using (47) in (9), we obtain

$$Y(\xi) = -\frac{A_2}{2}X^2 - A_1X - A_0. \quad (48)$$

Combining (48) with (38), we obtain the exact solution to (39)

$$X(\xi) = -\frac{\sqrt{2}\sqrt{A_0}\tan(\frac{\sqrt{A_0}\sqrt{A_2}(\xi-2c_0)}{\sqrt{2}})}{\sqrt{A_2}}, \quad (49)$$

where c_0 is the integration constant. Therefore, the solutions of (33) are

$$\begin{aligned} u(x,t) &= \pm\tan(\sqrt{\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{2b^2k^2+2b^4k^2l^2}}(\xi-2c_0)) \\ &\times k\frac{\sqrt{\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{b^2k^2(1+b^2l^2)}}}{\sqrt{2}\sqrt{\frac{a}{b^2-b^4l^2}}}\exp(i(\alpha x+\beta y+\gamma t)) \end{aligned} \quad (50)$$

$$\begin{aligned} v_(x,t) &= \tan^2(\sqrt{\frac{b^2\alpha^2+b^4\beta^2+2\gamma}{2b^2k^2+2b^4k^2l^2}}(\xi-2c_0)) \\ &\times \frac{b^2\alpha^2+b^4\beta^2+2\gamma}{1+b^2l^2}, \end{aligned} \quad (51)$$

where $\lambda = -b^2\alpha - b^4l\beta$.

5 Coupled Klein−Gordon−Zakharov equations

Consider the Coupled Klein-Gordon-Zakharov equations

$$\begin{aligned} &u_{tt} - c_0\nabla^2u + f_0^2u + \delta uv = 0 \\ &v_{tt} - c_0^2\nabla^2v - b\nabla^2(|u|^2) = 0 \end{aligned} \quad (52)$$

where c_0, f_0, and b are constants, $u(x,y,z,t)$ is a complex valued function and $v(x,y,z,t)$ is a real valued function. To solve (52), we apply the transformations

$$\begin{aligned} u(x,y,z,t) &= u(\xi)\exp(i\eta), \\ v(x,y,z,t) &= v(\xi), \\ \xi &= k(x+ly+nz+\lambda t), \\ \eta &= \alpha x+\beta y+\omega z+\gamma t. \end{aligned} \quad (53)$$

Substituting (53) into (52) yield the following system of ODEs

$$\begin{aligned} 0 &= (f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)+\delta v)u \\ &- k^2(-\lambda^2+c_0^2(1+n^2+l^2))u'', \\ 0 &= 2b(1+n^2+l^2)(u'^2+uu') - (\lambda^2-c_0^2(1+n^2+l^2))v'' \end{aligned} \quad (54)$$

where

$$\alpha = -l\beta + (-1+\frac{1}{c_0^2})\gamma\lambda - n\omega.$$

and the prime denotes the derivation with respect to ξ.

Integrating the second equation of (54) twice and setting the constant of integration to be zero. We find

$$v = -\frac{b(1+n^2+l^2)}{-\lambda^2+c_0^2(1+n^2+l^2)}u^2. \quad (55)$$

Substituting (55) into the first equation of (54) we have

$$0 = (f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)u + \frac{b\delta(1+n^2+l^2)u^3}{\lambda^2-c_0^2(1+n^2+l^2)} - k^2(-\lambda^2+c_0^2(1+n^2+l^2))u''. \quad (56)$$

Using (7) we obtain

$$X'(\xi) = Y(\xi), \quad (57)$$

$$Y'(\xi) = \frac{f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)}{k^2(-\lambda^2+c_0^2(1+n^2+l^2))}X(\xi) - \frac{b\delta(1+n^2+l^2)}{k^2(\lambda^2-c_0^2(1+n^2+l^2))^2}X(\xi)^3. \quad (58)$$

Suppose that $m = 1$ in (9), then

$$q[X,Y] = a_0(X) + a_1(x)Y = 0.$$

From (10) we obtain

$$\begin{aligned}\frac{dq}{d\xi} &= \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi}\\ &= [\frac{f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)}{k^2(-\lambda^2+c_0^2(1+n^2+l^2))}]Xa_1(X)\\ &- [\frac{b\delta(1+n^2+l^2)}{k^2(\lambda^2-c_0^2(1+n^2+l^2))^2}X^3]a_1(X)\\ &+ a_0'(X)Y + a_1'(X)Y^2. \end{aligned} \quad (59)$$

and

$$\begin{aligned}\frac{dq}{d\xi} &= [g(X)+h(X)Y](a_0(X)+a_1(x)Y)\\ &= h(X)a_1(X)Y^2 + [g(X)a_1(X)+h(X)a_0(X)]Y\\ &+ g(X)a_0(X). \end{aligned} \quad (60)$$

By equating the coefficients of Y^i $(i = 2,1,0)$ in (59) and (60) we obtain

$$a_1'(X) = h(X)a_1(X), \quad (61)$$

$$a_0'(X) = h(X)a_0(X) + a_1(X)g(X), \quad (62)$$

$$a_0(X)g(X) = [\frac{f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)}{k^2(-\lambda^2+c_0^2(1+n^2+l^2))}X]a_1(X) - [\frac{b\delta(1+n^2+l^2)}{k^2(\lambda^2-c_0^2(1+n^2+l^2))^2}X^3]a_1(X). \quad (63)$$

Since $a_i(X)$ $(i = 0,1)$ are polynomials, then from(61) we deduce that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + A_0$ and $A_1 \neq 0$, then we find that $a_0(X)$ is expressed as

$$a_0(X) = \frac{A_2}{2}X^2 + A_1X + A_0. \quad (64)$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into(63)and setting all coefficients of powers of X to be zeros, then we obtain the following system of nonlinear algebraic equations

$$\begin{aligned} 0 &= A_0A_1,\\ 0 &= A_1^2 + A_0A_2 - \frac{f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2++\omega^2)}{k^2(-\lambda^2+c_0^2(1+n^2+l^2))},\\ 0 &= \frac{3}{2}A_1A_2,\\ 0 &= \frac{A_2^2}{2} + \frac{b\delta(1+n^2+l^2)}{k^2(\lambda^2-c_0^2(1+n^2+l^2))^2}. \end{aligned} \quad (65)$$

Solving (65), we obtain

$$\begin{aligned} A_0 &= \pm\frac{f_0^2-\gamma^2+c_0^2(\alpha^2+\beta^2+\omega^2)}{\sqrt{2}\sqrt{b}k\sqrt{\delta}\sqrt{-1-l^2-n^2}},\\ A_1 &= 0,\\ A_2 &= \pm\frac{\sqrt{2}\sqrt{b}\sqrt{\delta}\sqrt{-1-l^2-n^2}}{-k\lambda^2-c_0^2k(1+n^2+l^2)}, \end{aligned} \quad (66)$$

where $\beta, \omega, \gamma, \lambda, n, l$ and k are arbitrary. Using (66) in (9), we obtain

$$Y(\xi) = -\frac{A_2}{2}X^2 - A_1X - A_0. \quad (67)$$

Combining (67) with (57), we obtain the exact solution to (58)

$$X(\xi) = -\frac{\sqrt{2}\sqrt{A_0}\tan(\frac{\sqrt{A_0}\sqrt{A_2}(\xi-2\xi_0)}{\sqrt{2}})}{\sqrt{A_2}}, \quad (68)$$

where ξ_0 is the integration constant. Therefore, the solutions of (52) are

$$u_1(x,t) = e^{i\eta}\frac{i\sqrt{f_0^2+c_0^2k_1-\gamma^2}\sqrt{k^2(c_0^2k2-\lambda^2)}}{\sqrt{b}\sqrt{k_2}\sqrt{\delta}} \times \tan(\frac{\sqrt{f_0^2+c_0^2k_1-\gamma^2\xi-2\xi_0)}}{k\sqrt{2c_0^2k_2-2\lambda^2}}) \quad (69)$$

$$v_1(x,t) = \tan^2(\frac{\sqrt{f_0^2+c_0^2k_1-\gamma^2}(\xi-2\xi_0)}{k\sqrt{2c_0^2k_2-2\lambda^2}}) \times \frac{f_0^2+c_0^2k_1-\gamma^2}{\delta} \quad (70)$$

$$u_2(x,t) = e^{i\eta} \frac{i\sqrt{-f_0^2 - c_0^2 k_1 + \gamma^2}\sqrt{k^2(c_0^2 k2 - \lambda^2)}}{\sqrt{b}\sqrt{k_2}\sqrt{\delta}} \times \tanh\left(\frac{\sqrt{-f_0^2 - c_0^2 k_1 + \gamma^2}(\xi - 2\xi_0)}{k\sqrt{2c_0^2 k_2 - 2\lambda^2}}\right) \quad (71)$$

$$v_2(x,t) = \tanh^2\left(\frac{\sqrt{f_0^2 + c_0^2 k_1 - \gamma^2}(\xi - 2\xi_0)}{k\sqrt{2c_0^2 k_2 - 2\lambda^2}}\right) \times -\frac{f_0^2 + c_0^2 k_1 - \gamma^2}{\delta} \quad (72)$$

where k_1 and k_2 are

$$k_1 = \alpha^2 + \beta^2 + \omega^2,$$
$$k_2 = 1 + l^2 + n^2, \quad (73)$$

and $\alpha = -l\beta + (-1 + \frac{1}{c_0^2})\gamma\lambda - n\omega.$

6 Conclusions

In this work, we extend the application of first integral method to solve some nonlinear evolution systems. By means of this method new exact solutions to such evolution systems are obtained. The performance of this method is found to be reliable and effective. The Mathematica software was used to solve complicated and tedious algebraic calculations. The proposed method can be extended to other nonlinear problems in mathematical sciences.

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