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Relations for Generalized Order Statistics from Doubly Truncated Generalized Exponential Distribution and its Characterization

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Abstract: In this paper we give some recurrence relations satisfied by single and product moments of generalized order statistics from doubly truncated generalized exponential distribution. These relations are deduced for moments of record values and order statistics. Further, using a recurrence relation for single moments we obtain a characterization of generalized exponential distribution.

Keywords: Characterization; generalized exponential distribution; generalized order statistics; order statistics; record values; recurrence relations; truncation; single and product moment.

1 Introduction

The concept of generalized order statistics (gos) was introduced by Kamps [12]. A variety of order models of random variables is contained in this concept.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables (rv) with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Assuming that $k > 0$, $n \in N$, $m \in \Re$ and $\gamma_r = k + (n-r)(m+1) > 0$. If the random If the random variables $X(r,n,m,k)$, $r = 1, 2, \ldots, n$, possess a joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)(1-F(x_n))^{k-1} f(x_n), \tag{1.1}$$

on the cone $F^{-1}(0) < x_1 \le \ldots \le x_n < F^{-1}(1)$,

then they are called generalized order statistics of a sample from a distribution with df $F(x)$. Note that in the case $m = 0$, $k = 1$, this model reduces to the joint pdf of the ordinary order statistics, and when $m = -1$ we get the joint pdf of the $k-$th upper record values. We shall also take $X(0,n,m,k) = 0$. On using (1.1), the pdf of the $r-$th gos is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s - 1} f(y), \qquad x < y, \tag{1.3}$$

where

$$\bar{F}(x) = 1 - F(x), \qquad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad r = 1, 2, \ldots, n,$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ \ln\left(\dfrac{1}{1-x}\right), & m = -1 \end{cases}$$

$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1).$$

Many recurrence relations between moments of generalized order statistics are available in the literature. Reference may be made to Cramer and Kamps [7], Pawlas and Szynal [21], Ahmad and Fawzy [2], Al-Hussaini, et al. [4], Ahmad [1], Khan *et al.* [16, 17], Mahmoud and Al-Nagar [20], Khan *et al.* [18, 19], Athar and Nayabuddin [5] and references therein.

Kamps and Gather [14] characterized the exponential distribution based on generalized order statistics. Keseling [15] characterized some continuous distributions based on conditional distributions of generalized order statistics. Ahsanullah [3] characterized the exponential distribution based on independence of functions of generalized order statistics and presented the estimators of its parameters. Bieniek and Szynal [6] characterized some distributions via linearity of regression of generalized order statistics. Cramer *et al*. [8] gave a unifying approach of characterization via linear regression of ordered random variables. Kamps [13] investigated the importance of recurrence relations of order statistics in characterization.

The doubly truncated case of a distribution is the most general case since it includes the right, left and non-truncated distribution as special cases.

In this paper, we have obtained some recurrence relations for single and product moments of generalized order statistics from doubly truncated generalized exponential distribution. Further, various deductions and particular cases are discussed and a characterization of generalized exponential distribution has been obtained on using a recurrence relation for single moments.

A random variable X is said to have generalized exponential distribution if its pdf is of the form

$$f_1(x) = (1-\alpha x)^{(1/\alpha)-1}, \qquad 0 \le x \le 1/\alpha, \ 0 \le \alpha < 1 \tag{1.4}$$

and the corresponding df is

$$F_1(x) = 1 - (1-\alpha x)^{(1/\alpha)}, \qquad 0 \le x \le 1/\alpha, \ 0 \le \alpha < 1. \tag{1.5}$$

Now if for given P_1 and Q_1

$$\int_0^{Q_1} f_1(x)dx = Q, \qquad \int_0^{P_1} f_1(x)dx = P,$$

then the truncated pdf $f(x)$ is given by

$$f(x) = \frac{(1-\alpha x)^{(1/\alpha)-1}}{P-Q}, \ x \in (Q_1, P_1) \tag{1.6}$$

and the corresponding df $F(x)$ is

$$\bar{F}(x) = -P_2 + (1-\alpha x) f(x), \tag{1.7}$$

where

$$P_2 = \frac{1-P}{P-Q} \quad \text{and} \quad Q_2 = \frac{1-Q}{P-Q}.$$

More details on this distribution and its applications see, Saran and Pandey [22].

2 Relations for Single Moments

Note that for the doubly truncated generalized exponential distribution defined in (1.6)

$$f(x)(1-\alpha x) = P_2 + \bar{F}(x). \tag{2.1}$$

Recurrence relations for single moments of gos from (1.6) can be obtained in the following theorem.

Theorem 2.1 For the given generalized exponential distribution and $n \in N$, $m \in \Re$, $r = 1$

$$E[X^j(1,n,m,k)] = \frac{j}{(\gamma_1 + \alpha j)} E[X^{j-1}(1,n,m,k)]$$

$$-\frac{\gamma_1}{(\gamma_1 + \alpha j)} \{P_2 E[X^j(1,n,m,k)] - Q_1^j(1+P_2)\} \tag{2.2}$$

and for $2 \le r \le n$

$$E[X^j(r,n,m,k)] = \frac{1}{(\gamma_r + \alpha j)} \{jE[X^{j-1}(r,n,m,k)] + \gamma_r E[X^j(r-1,n,m,k)]\}$$

$$+\frac{P_2 \gamma_r K}{(\gamma_r + \alpha j)} \{E[X^j(r-1,n-1,m,k+m)] - E[X^j(r,n-1,m,k+m)]\} \tag{2.3}$$

where

$$K = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \left(\frac{\gamma_i}{\gamma_i - 1} \right), \quad C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1} \gamma_i^{(n-1,k+m)},$$

$$\gamma_i^{(n-1,k+m)} = (k+m) + (n-1-i)(m+1).$$

Proof From equation (1.2), we have

$$E[X^{j-1}(r,n,m,k)] - \alpha E[X^j(r,n,m,k)]$$

$$= \frac{C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} x^{j-1}(1-\alpha x)(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

using (2.1), we have

$$E[X^{j-1}(r,n,m,k)] - \alpha E[X^j(r,n,m,k)]$$

$$= \frac{P_2 C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r - 1} g_m^{r-1}(F(x)) dx + \frac{C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x)) dx$$

$$= \frac{C_{r-1}}{(r-1)!} \left\{ P_2 I_{j-1}^{(n-1,k+m)} + I_{j-1}^{(n,k)} \right\} \tag{2.4}$$

where

$$I_{t-1}^{(n-1,k+m)}(x) = \int_{Q_1}^{P_1} x^{t-1}(\bar{F}(x))^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}(F(x)) dx$$

and

$$I_{t-1}^{(n,k)}(x) = \int_{Q_1}^{P_1} x^{t-1} (\overline{F}(x))^{\gamma_r} g_m^{r-1}(F(x)) dx$$

Integrating by parts treating x^{t-1} for integration and the rest of the integrand for differentiation, we get

$$I_{t-1}^{(n-1,k+m)}(x) = \begin{cases} \frac{\gamma_1}{t}\left(E[X^t(1,n-1,m,k+m)] - Q_1^t\right), & r=1 \\ \frac{\gamma_r (r-1)!}{t\, C_{r-2}^{(n-1,k+m)}}\left(E[X^t(r,n-1,m,k+m)]\right. \\ \left. -E[X^t(r-1,n-1,m,k+m)]\right), & 2 \le r \le n \end{cases}$$

and

$$I_t^{(n,k)}(x) = \begin{cases} \frac{\gamma_1}{t}\left(E[X^t(1,n,m,k)] - Q_1^t\right), & r=1 \\ \frac{\gamma_r (r-1)!}{t\, C_{r-2}}\left(E[X^t(r,n,m,k)] - E[X^t(r-1,n,m,k)]\right), & 2 \le r \le n \end{cases}.$$

Now substituting for $I_{j-1}^{(n-1,k+m)}(x)$ and $I_{j-1}^{(n,k)}(x)$ in equation (2.4) and simplifying the resulting expression, we derive the relations in (2.2) and (2.3).

Remark 2.1: Setting $m=0$, $k=1$ in Theorem 2.1, we obtain recurrence relations for single moments of order statistics for the doubly truncated generalized exponential distribution in the form

$$E(X_{1:n}^{j}) = \frac{j}{(n+\alpha\, j)} E(X_{1:n}^{j-1}) - \frac{n}{(n+\alpha\, j)}\{P_2 E(X_{1:n-1}^{j}) - Q_1(1+P_2)\}$$

and

$$E(X_{r:n}^{j}) = \frac{1}{[(n-r+1)+\alpha\, j]}\{jE(X_{r:n}^{j-1}) + (n-r+1)E(X_{r-1:n}^{j-1})\}$$

$$+\frac{nP_2}{[(n-r+1)+\alpha\, j]}\{E(X_{r-1:n-1}^{j-1}) + (n-r+1)E(X_{r:n-1}^{j-1})\}.$$

when $Q=0$, $P=1$ and $\alpha \to 0$ the above results agree with Joshi [10].

Remark 2.2: Substituting $m=-1$ in Theorem 2.1, relations for single moments of records can be obtained.

Remark 2.3: Putting $Q=0$, $P=1$ (for non-truncation case) in (2.3) we can deduce the following recurrence relations for the single moments of generalized order statistics from the generalized exponential distribution as the form

$$(\gamma_1 + \alpha\, j)E[X^j(1,n,m,k)] = jE[X^{j-1}(1,n,m,k)]$$

and

$$(\gamma_r + \alpha\, j)E[X^j(r,n,m,k)] = jE[X^{j-1}(r,n,m,k)] + \gamma_r E[X^j(r-1,n,m,k)]\},$$

for $j = i+1$ the above result agrees with Saran and Pandey [22].

3 Relations for Product Moments

Theorem 3.1 For the given generalized exponential distribution and $n \geq 2$, $m \in \Re$, $1 \leq r < r+1 \leq n$

$$E[X^i(r,n,m,k)X^j(r+1,n,m,k)]$$
$$= \frac{1}{(\alpha j + \gamma_{r+1})}\Big(jE[X^i(r,n,m,k)X^{j-1}(r+1,n,m,k)] + \gamma_{r+1}E[X^{i+j}(r,n,m,k)]\Big)$$
$$+ \frac{\gamma_{r+1}P_2K^*}{(\alpha j + \gamma_{r+1})}\Big(E[X^{i+j}(r,n-1,m,k+m)]$$
$$- E[X^i(r,n-1,m,k+m)X^j(r+1,n-1,m,k+m)]\Big) \tag{3.1}$$

and for $1 \leq r < s \leq n$, $s - r \geq 2$ and $i, j \geq 0$

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = \frac{1}{(\alpha j + \gamma_s)}\Big(jE[X^i(r,n,m,k)X^{j-1}(s,n,m,k)]$$
$$+ \gamma_s E[X^i(r,n,m,k)X^j(s-1,n,m,k)]\Big)$$
$$+ \frac{\gamma_s K^{**}P_2}{(\alpha j + \gamma_s)}\Big(E[X^i(r,n-1,m,k+m)X^j(s-1,n-1,m,k+m)]$$
$$- E[X^i(r,n-1,m,k+m)X^j(s,n-1,m,k+m)]\Big), \tag{3.2}$$

where

$$K^* = \frac{C_{r-1}}{C_{r-1}^{(n-1,k+m)}} = \prod_{i=1}^{r}\left(\frac{\gamma_i}{\gamma_i - 1}\right), \quad K^{**} = \frac{C_{s-2}}{C_{s-2}^{(n-1,k+m)}} = \prod_{i=1}^{s-1}\left(\frac{\gamma_i}{\gamma_i - 1}\right).$$

Proof From (1.3), we have

$$E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)] - \alpha E[X^i(r,n,m,k)X^j(s,n,m,k)]$$
$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{Q_1}^{P_1}\int_x^{P_1} x^i y^{j-1}(1-\alpha y)(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s - 1} f(y)dydx,$$
$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{Q_1}^{P_1} x^i(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))G(x)\,dx \tag{3.3}$$

where

$$G(x) = \int_x^{P_1} y^{j-1}(1-\alpha y)[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s - 1} f(y)dy.$$

Making use of the relation in (2.1) and splitting the integral according with form, we have

$$G(x) = P_2\int_x^{P_1} y^{j-1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s^{(n-1,k+m)}}dy$$
$$+ \int_x^{P_1} y^{j-1}[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s}dy$$
$$= P_2 G_{j-1}^{(n-1,k+m)}(x) + G_{j-1}^{(n,k)}(x), \tag{3.4}$$

where

$$G_{t-1}^{(n-1,k+m)}(x)=\int_x^{P_1} y^{t-1}[h_m(F(y))-h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s^{(n-1,k+m)}}\,dy$$

and

$$G_{t-1}^{(n,k)}(x)=\int_x^{P_1} y^{t-1}[h_m(F(y))-h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s}\,dy$$

Integrating by parts treating y^{t-1} for integration and the rest of the integrand for differentiation, we get for $s=r+1$

$$G_t^{(n,k)}(x)=\frac{1}{t}\left(\gamma_{r+1}\int_x^{P_1} y^t\ (\bar{F}(y))^{\gamma_{r+1}-1} f(y)dy-x^t(\bar{F}(y))^{\gamma_{r+1}}\right)$$

and

$$G_t^{(n-1,k+m)}(x)=\frac{1}{t}\left(\gamma_{r+1}^{(n-1,k+m)}\int_x^{P_1} y^t\ (\bar{F}(y))^{\gamma_{r+1}^{(n-1,k+m)}-1} f(y)dy-x^t(\bar{F}(y))^{\gamma_{r+1}^{(n-1,k+m)}}\right)$$

and for $s-r\geq 2$

$$G_t^{(n,k)}(x)=\frac{1}{t}\Bigg(\gamma_s\int_x^{P_1} y^t\ [h_m(F(y))-h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s-1} f(y)dy$$

$$-(s-r-1)\int_x^{P_1} y^t\ [h_m(F(y))-h_m(F(x))]^{s-r-2}(\bar{F}(y))^{\gamma_s+m} f(y)dy\Bigg)$$

and

$$G_t^{(n-1,k+m)}(x)=\frac{1}{t}\Bigg(\gamma_s^{(n-1,k+m)}\int_x^{P_1} y^t\ [h_m(F(y))-h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s^{(n-1,k+m)}-1} f(y)dy$$

$$-(s-r-1)\int_x^{P_1} y^t\ [h_m(F(y))-h_m(F(x))]^{s-r-2}(\bar{F}(y))^{\gamma_{s-1}^{(n-1,k+m)}-1} f(y)dy\Bigg).$$

Upon substituting for $G_{j-1}^{(n-1,k+m)}(x)$ and $G_{j-1}^{(n,k)}(x)$ in equation (3.4) and then substituting the resulting expression for $G(x)$ in equation (3.3) and simplifying, we derive the relations in (3.1) and (3.2).

Remark 3.1: Setting $m=0$, $k=1$ in (3.1) and (3.2), we obtain recurrence relations for product moments of order statistics for the doubly truncated generalized exponential distribution in the form

$$E(X_{r:n}^i X_{r+1:n}^j)=\frac{1}{(n-r+\alpha j)}\{jE(X_{r:n}^i X_{r+1:n}^{j-1})+(n-r)E(X_{r:n}^{i+j})\}$$

$$+\frac{nP_2}{(n-r+\alpha j)}\{E(X_{r:n-1}^{i+j})-E(X_{r:n-1}^i X_{r+1:n-1}^{j-1})\}$$

and

$$E(X_{r:n}^i X_{s:n}^j)=\frac{1}{(n-s+1+\alpha j)}\{j\,E(X_{r:n}^i X_{s1:n}^{j-1})+(n-s+1)E(X_{r:n}^i X_{s-1:n}^j)\}$$

$$+\frac{nP_2}{(n-s+1+\alpha j)}\{E(X_{r:n-1}^i X_{s-1:n-1}^j)-E(X_{r:n-1}^i X_{s:n-1}^j)\}.$$

when $Q=0$, $P=1$ and $\alpha\to 0$ the above results agree with Joshi [11].

Remark 3.2: Substituting $m = -1$ in (3.1) and (3.2), relations for product moments of records can be obtained.

Remark 3.3: Putting $Q = 0$, $P = 1$ (for non-truncation case) in (3.1) and (3.2) we can deduce the following recurrence relation for the product moments of generalized order statistics from the generalized exponential distribution as the form

$$(\alpha j + \gamma_{r+1}) E[X^i(r,n,m,k) X^j(r+1,n,m,k)]$$

$$= jE[X^i(r,n,m,k) X^{j-1}(r+1,n,m,k)] + \gamma_{r+1} E[X^{i+j}(r,n,m,k)] ,$$

and

$$(\alpha j + \gamma_s) E[X^i(r,n,m,k) X^j(s,n,m,k)] = jE[X^i(r,n,m,k) X^{j-1}(s-1,n,m,k)]$$

$$+ \gamma_s E[X^i(r,n,m,k) X^j(s-1,n,m,k)],$$

for $j = i+1$ the above results agree with Saran and Pandey [22].

4 Characterization

Theorem 4.1 Fix a positive integer k and let j be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by equation (1.6) is that

$$E[X^j(r,n,m,k)] = \frac{1}{(\gamma_r + \alpha j)} \{ jE[X^{j-1}(r,n,m,k)] + \gamma_r E[X^j(r-1,n,m,k)] \}$$

$$+ \frac{P_2 \gamma_r K}{(\gamma_r + \alpha j)} \{ E[X^j(r-1,n-1,m,k+m)] - E[X^j(r,n-1,m,k+m)] \} \qquad (4.1)$$

where

$$K = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1} \left(\frac{\gamma_i}{\gamma_i - 1} \right), \quad C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1} \gamma_i^{(n-1,k+m)} ,$$

$$\gamma_i^{(n-1,k+m)} = (k+m) + (n-1-i)(m+1) .$$

Proof The necessary part follows immediately from equation (2.3). On the other hand if the relation in equation (4.1) is satisfied, then on rearranging the terms in equation (4.1) and using (1.2), we have

$$\frac{C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} x^{j-1} (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

$$= \frac{\alpha C_{r-1}}{(r-1)!} \int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

$$+ \frac{\gamma_r C_{r-1}}{j(r-1)!} \int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx$$

$$- \frac{(r-1) C_{r-1}}{j(r-1)!} \int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r + m} f(x) g_m^{r-2}(F(x)) dx$$

$$+ \frac{P_2 (\gamma_r - 1) C_{r-1}}{j(r-1)!} \int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 2} f(x) g_m^{r-1}(F(x)) dx$$

$$- \frac{P_2 (r-1) C_{r-1}}{j(r-1)!} \int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1 + m} f(x) g_m^{r-2}(F(x)) dx$$

$$= \frac{\alpha C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx$$

$$+ \frac{C_{r-1}}{j(r-1)!}\int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r} f(x) g_m^{r-2}(F(x))\left\{\frac{\gamma_r g_m(F(x))}{\bar{F}(x)} - (r-1)(\bar{F}(x))^m\right\}dx$$

$$+ \frac{P_2 C_{r-1}}{j(r-1)!}\int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-2}(F(x))\left\{\frac{(\gamma_r - 1) g_m(F(x))}{\bar{F}(x)} - (r-1)(\bar{F}(x))^m\right\}dx. \quad (4.2)$$

Let

$$h_t(x) = -(\bar{F}(x))^t g_m^{r-1}(F(x)). \quad (4.3)$$

Differentiating both the sides of equation (4.3), we get

$$h_t'(x) = (\bar{F}(x))^t f(x) g_m^{r-2}(F(x))\left\{\frac{t\, g_m(F(x))}{\bar{F}(x)} - (r-1)(\bar{F}(x))^m\right\}.$$

Thus,

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx = \frac{\alpha C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx$$

$$+ \frac{C_{r-1}}{j(r-1)!}\int_{Q_1}^{P_1} x^j h_{\gamma_r}'(x)dx + \frac{P_2 C_{r-1}}{j(r-1)!}\int_{Q_1}^{P_1} x^j h_{\gamma_r - 1}'(x)dx \quad (4.4)$$

.

Integrating the last two terms of RHS in (4.4) by parts and using the values of $h_{\gamma_r}(x)$ and $h_{\gamma_r - 1}(x)$ from (4.3), we get

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx$$

$$= \frac{\alpha C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx + \frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx$$

$$+ \frac{P_2 C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r - 1} g_m^{r-1}(F(x))dx$$

which reduces to

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}(\bar{F}(x))^{\gamma_r - 1} g_m^{r-1}(F(x))\big(f(x) - \alpha x f(x) - \bar{F}(x) - P_2\big)dx = 0. \quad (4.5)$$

Applying the extension of Müntz-Szász Theorem, (see for example Hwang and Lin, [9]), to equation (4.5), we get

$$f(x)(1 - \alpha x) = P_2 + \bar{F}(x).$$

which proves that $f(x)$ has the form as in equation (2.1).

Now we shall use recurrence relation in (2.6), $(P_2 = 0)$, to characterize the non-truncated generalized exponential distribution by the following theorem.

Theorem 4.2 Let X be a non-negative random variable having an absolutely continuous distribution function $F_1(x)$ with $F_1(0)=0$ and $0<F_1(x)<1$ for all $x>0$, then

$$E[X^j(r,n,m,k)]=\frac{1}{(\gamma_r+\alpha j)}\{jE[X^{j-1}(r,n,m,k)]+\gamma_r E[X^j(r-1,n,m,k)]\}, \tag{4.6}$$

if and only if $\bar{F}_1(x)=(1-\alpha x)^{1/\alpha}$, $0\le x\le 1/\alpha$, $0\le\alpha<1$.

Proof The necessary part follows immediately from equation (2.3). On the other hand if the relation in equation (4.6) is satisfied, then on rearranging the terms in equation (4.6) and using (1.2), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^{j-1}(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx$$

$$+\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r+m}f_1(x)g_m^{r-2}(F_1(x))dx-\frac{\alpha jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx \tag{4.7}$$

Integrating the second integrals on the right hand side of equation (4.7), by parts and simplifying the resulting equation, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx$$

$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^{j-1}(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx-\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^{j-1}(\bar{F}_1(x))^{\gamma_r}g_m^{r-1}(F_1(x))dx$$

$$+\frac{C_{r-1}}{(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx-\frac{\alpha jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^j(\bar{F}_1(x))^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx$$

which reduces to

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{1/\alpha} x^{j-1}(\bar{F}_1(x))^{\gamma_r-1}g_m^{r-1}(F_1(x))\big(f(x)-\bar{F}(x)-\alpha x f(x)\big)dx=0. \tag{4.8}$$

Now applying a generalization of Müntz-Szász Theorem (Hwang and Lin, [9]) to equation (4.8), we get

$\frac{f_1(x)}{\bar{F}_1(x)}=\frac{1}{(1-\alpha x)}$ which prove that $\bar{F}_1(x)=(1-\alpha x)^{1/\alpha}$, $0\le x\le 1/\alpha$, $0\le\alpha<1$.

5. Applications

Recurrence relations are interesting in their own right. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterization distributions, which in important area, permitting the identification of population distribution from the properties of the sample. The results established in this paper can be used to determine the mean, variance and coefficients of skewness and kurtosis. The moments can also be used for finding best linear unbiased estimator (BLUE) for parameter and conditional moments.

6. Conclusion

This paper deals with the generalized order statistics from the doubly truncated generalized exponential distribution. Some recurrence relations between the single and product moments are derived. Two characterizing results of doubly truncated generalized exponential distribution have been obtained using a recurrence relation for single moments. Special cases are also deduced.

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