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Graded q-Differential Algebra Approach to Chern-Simons Form

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Abstract: In the present paper we develop noncommutative approach to a connection which is based on a notion of graded q-differential algebra, where q is a primitive Nth root of unity. We define the curvature of connection form and prove Bianchi identity. We construct a graded q-differential algebra to calculate the curvature of connection for any integer $N \geq 2$. Making use of Bianchi identity we introduce the Chern character form of connection form and show that this form is closed. We study the case $N = 3$ which is the first non-trivial generalization because in the case $N = 2$ we have a classical theory. We calculate the curvature of connection form and show that it can be expressed in terms of graded q-commutators, where q is a primitive cubic root of unity. This allows us to prove an infinitesimal homotopy formula, and making use of this formula we introduce the Chern-Simons form.

Keywords: Noncommutative geometry, graded differential algebras, connection form and covariant differential, curvature of connection, Bianchi identity, Chern character form, Chern-Simons form

1. Introduction

A noncommutative approach to a concept of connection is based on a graded differential algebra [16]. One can generalize a notion of graded differential algebra by generalizing a notion of cochain complex. This generalization was proposed in [13], and the basic idea of this generalization is to take a more general equation $d^N = 0, N \geq 2$ than $d^2 = 0$. This generalization of cochain complex is known under the name of N-complex. A generalization of graded differential algebra in which the basic property of differential $d^2 = 0$ is replaced by a more general one $d^N = 0, N \geq 2$ was introduced in [7] and studied in [1, 2, 14, 15]. This generalization of graded differential algebra is called a graded q-differential algebra, where q is a primitive Nth root of unity. Let us mention that a concept of graded q-differential algebra is closely related to a monoidal structure introduced in [13] for a category of N-complexes, and it is proved in [11] that the monoids of the category of N-complexes can be identified as the graded q-differential algebras. It is well known that a connection and its curvature are basic elements of the theory of fiber bundles, and they play an important role not only in a modern differential geometry but also in theoretical physics namely in a gauge field theory. If we consider a grading one element of graded differential algebra as analogous to a connection form then we can develop a noncommutative approach to a concept of connection. We can generalize this approach considering a grading one element of graded q-differential algebra, where q is a primitive Nth root of unity. In the present paper we develop this approach and show that as in the case of graded differential algebra we can define the curvature of connection form in the case of graded q-differential algebra as well. We construct a graded q-differential algebra which is very useful for calculation of the curvature of connection for any integer $N \geq 2$. Moreover we show that the curvature of connection form satisfies the Bianchi identity. Despite the fact that our approach is based on a graded q-differential algebra which is more general structure than a graded differential algebra we have all the basic components of classical theory. The Bianchi identity gives us a possibility to define the Chern character form of connection form and to show that this form is closed. For this purpose making use of N-complex we introduce a notion of trace on a graded q-differential algebra. We study the case $N = 3$ which is the first non-trivial generalization because in the case $N = 2$ we have a classical theory. We calculate the curvature of connection form and show that it can be expressed in terms

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of graded q-commutators, where q is a primitive cubic root of unity. This fact allows us to prove an infinitesimal homotopy formula, and making use of this formula we introduce the Chern-Simons form. It is worth mentioning that for any integer $N > 3$ the curvature of connection form can not be expressed in terms of graded q-commutators, and this is an obstacle to prove an infinitesimal homotopy formula and to define the Chern-Simons form. In the last section we define and study connections constructed by means of graded q-differential algebra on modules.

2. Polynomial algebra

Let $\mathfrak{A} = \oplus_n \mathfrak{A}^n$ be an associative unital graded algebra over $\mathbb{C}$ and d be a graded q-derivation of degree one of this algebra, i.e. $d : \mathfrak{A}^n \to \mathfrak{A}^{n+1}$ is an endomorphism of degree one of the graded vector space of algebra $\mathfrak{A}$ satisfying the graded q-Leibniz rule

$$d(A\,B) = d(A)\,B + q^{|A|}\,A\,d(B), \qquad (1)$$

where $A, B \in \mathfrak{A}$, $|A|$ is the grading of homogeneous element A, q is a complex number different from zero. Let us denote by $\mathscr{L}(\mathfrak{A}) = \oplus_n \mathscr{L}^n(\mathfrak{A})$ the graded algebra of endomorphisms of the graded vector space of algebra $\mathfrak{A}$. Clearly $d \in \mathscr{L}^1(\mathfrak{A})$.

Let $A \in \mathfrak{A}^1$ be an element of grading one of graded algebra $\mathfrak{A}$. The subalgebra of graded algebra $\mathfrak{A}$ generated by the elements $A, dA, d^2A, \ldots$ will be denoted by $\mathfrak{A}_A$. Obviously $\mathfrak{A}_A$ is the graded algebra. Let $L_A \in \mathscr{L}^1(\mathfrak{A})$ be the endomorphism of degree one of graded algebra $\mathfrak{A}$ induced by the left multiplication by A, i.e. $L_A(B) = AB$, where $B \in \mathfrak{A}$. Now the graded q-Leibniz rule (1) can be written in the form

$$d \circ L_A = q^{|A|} L_A \circ d + L_{dA}, \qquad (2)$$

where $\circ$ is the composition of endomorphisms of degree one. Obviously the endomorphisms of degree one d and L_A, where d is a graded q-derivation of degree one of graded algebra $\mathfrak{A}$ and L_A is the endomorphism of left multiplication by an element $A \in \mathfrak{A}^1$, generate the subalgebra of graded algebra of endomorphisms $\mathscr{L}(\mathfrak{A})$, which will be denoted by $\mathscr{L}^d_A(\mathfrak{A})$. The subalgebra of this algebra freely generated by $L_A, L_{dA}, L_{d^2A} \ldots$ will be denoted by $\mathscr{L}_A(\mathfrak{A})$. Let us mention that $\mathscr{L}^d_A(\mathfrak{A})$ is the graded algebra, and its gradation is induced by the graded structure of algebra $\mathscr{L}(\mathfrak{A})$.

The structure of algebra $\mathscr{L}^d_A(\mathfrak{A})$ can be described with the help of new variables

$$\mathfrak{d}, \xi_1, \xi_2, \ldots, \xi_n, \ldots \qquad (3)$$

if we assume that these variables are subjected to the commutation relations

$$\mathfrak{d}\xi_n = q^n\,\xi_n\mathfrak{d} + \xi_{n+1}, \quad n \geq 1. \qquad (4)$$

Let $\mathfrak{P}[\mathfrak{d}, \xi]$ be the polynomial algebra generated by the variables (3) which obey the commutation relations (4). Let us denote by $\mathbb{1}$ the identity element of this algebra. It is worth mentioning that there are no commutation relations between the variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$, and the subalgebra of the algebra $\mathfrak{P}[\mathfrak{d}, \xi]$ freely generated by these variables will be denoted by $\mathfrak{P}[\xi]$.

We can endow the polynomial algebra $\mathfrak{P}[\mathfrak{d}, \xi]$ with a graded structure if we assign grading zero to the identity element $\mathbb{1}$, grading one to variable $\mathfrak{d}$, grading n to variable ξ_n, where $n \geq 1$, and define the grading of any product of variables $\mathfrak{d}, \xi_1, \xi_2, \ldots, \xi_n, \ldots$ as the sum of gradings of its factors. Hence we have

$$|\mathbb{1}| = 0, \quad |\mathfrak{d}| = |\xi_1| = 1, \quad |\xi_n| = n, \quad n \geq 2. \qquad (5)$$

Let us denote by $\mathfrak{P}^n[\mathfrak{d}, \xi]$ the subspace of homogeneous polynomials of grading n of graded algebra $\mathfrak{P}[\mathfrak{d}, \xi]$. For example the subspace $\mathfrak{P}^2[\mathfrak{d}, \xi]$ is spanned by the polynomials $\mathfrak{d}^2, \mathfrak{d}\xi_1, \xi_1\mathfrak{d}, \xi_2, \xi_1^2$. We can write

$$\mathfrak{P}[\mathfrak{d}, \xi] = \oplus_{n\geq 0}\mathfrak{P}^n[\mathfrak{d}, \xi].$$

Take the element $\mathfrak{d} \in \mathfrak{P}[\mathfrak{d}, \xi]$ and define the endomorphism of vector space $\delta : \mathfrak{P}[\xi] \to \mathfrak{P}[\xi]$ by means of graded q-commutator as follows

$$\delta(p) = [\mathfrak{d}, p]_q = \mathfrak{d}\,p - q^{|p|}p\,\mathfrak{d}, \qquad (6)$$

where $p \in \mathfrak{P}[\xi]$ is a homogeneous polynomial of variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ and $|p|$ is the grading of p. Obviously δ is the graded q-derivation of degree one and it follows from the commutation relations (4) of the algebra $\mathfrak{P}[\mathfrak{d}, \xi]$ that $\delta(\xi_n) = \xi_{n+1}$. Next we define the endomorphism of degree one $\Delta : \mathfrak{P}[\xi] \to \mathfrak{P}[\xi]$ by $\Delta = \delta + L_{\xi_1}$, where L_{ξ_1} is the endomorphism of left multiplication by ξ_1. It is easy to show that Δ has the property

$$\Delta(p\,p') = \Delta(p)\,p' + q^{|p|}p\,\delta(p'),$$

where $p, p' \in \mathfrak{P}[\xi]$ are homogeneous polynomials. We define the polynomials $f_0, f_1, \ldots, f_n, \ldots \in \mathfrak{P}[\xi]$ by the recurrent formula

$$f_0 = \mathbb{1}, f_1 = \xi_1, f_{n+1} = \Delta(f_n), \quad n \geq 1. \qquad (7)$$

Clearly $|f_n| = n$.

Before we give an explicit expansion formula for the polynomials $f_0, f_1, \ldots, f_n, \ldots$ let us remind the notions of q-binomial coefficients and composition of integer. Given a complex number $q \neq 0$ one defines the mapping $[\ ]_q : n \in \mathbb{N} \to [n]_q \in \mathbb{C}$ by setting $[0]_q = 0$ and

$$[n]_q = 1 + q + q^2 + \ldots + q^{n-1} = \sum_{k=0}^{n-1} q^k, \quad n \geq 1.$$

The q-factorial of $[n]_q \in K$, where $n \in \mathbb{N}$, is defined by

$$[0]_q! = 1, \quad [n]_q! = [1]_q\,[2]_q\,\ldots[n]_q = \prod_{k=1}^{n}[k]_q, \ n \geq 1.$$

If k, n are integers satisfying $0 \le k \le n, n \ge 1$ then the Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

The Gaussian q-binomial coefficients satisfy the recursion relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{8}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \tag{9}$$

Next we remind a notion of composition of integer. If n is a positive integer then a composition of an integer n is a way of writing n as the sum of strictly positive integers. For example if $n = 3$ then there are three compositions

$$3 = 3,\ 3 = 2+1,\ 3 = 1+2,\ 3 = 1+1+1.$$

Let Ψ_n be the set of all compositions of an integer n. We will write a composition of an integer n in the form of a sequence of strictly positive integers $\sigma = (i_1, i_2, \ldots, i_r)$, where $i_1 + i_2 + \ldots + i_r = n$. Let us denote

$$\begin{aligned} n_1 &= i_1, \\ n_2 &= i_1 + i_2, \\ n_3 &= i_1 + i_2 + i_3, \\ &\ldots \\ n_r &= i_1 + i_2 + \ldots + i_r. \end{aligned}$$

Obviously $n_r = n$. It should be mentioned that the number of elements in the set Ψ_n is 2^{n-1} [12]. The following proposition gives an explicit formula for the polynomials f_n:

Theorem 1. *For any integer $n \ge 2$ we have the following expansion of power of the operator D and the expansion of a polynomial f_n in terms of generators ξ_i:*

$$\Delta^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q f_i\, \delta^{n-i}, \tag{10}$$

$$f_n = \sum_{\sigma \in \Psi_n} \begin{bmatrix} n_1 - 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} n_2 - 1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3 - 1 \\ n_2 \end{bmatrix}_q \cdots \cdots \begin{bmatrix} n_r - 1 \\ n_{r-1} \end{bmatrix}_q \xi_{i_1} \xi_{i_2} \ldots \xi_{i_r}, \tag{11}$$

where $\sigma = (i_1, i_2, \ldots, i_r)$ is a composition of an integer n.

Proof. We will prove the expansion formulae of this theorem by the method of mathematical induction. First we prove the power expansion formula (10) for D^n. Taking into account the definition $D = d + \xi_1$ we see that in the case of $n = 1$ the expansion formula (10) is correct. Indeed in this case we have

$$\Delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q f_0\, \delta + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q f_1 = \delta + \xi_1.$$

Now we assume that the first expansion formula holds for some integer $n > 1$ and show that it also holds when $n+1$ is substituted for n. It is easy to see that for any two homogeneous polynomials $p, p' \in \mathfrak{P}[\xi]$ it holds

$$\Delta(pp') = \Delta(p)p' + q^{|p|} p\, \delta(p'). \tag{12}$$

Making use of the property (12) of the operator Δ and the recurrent formula for q-binomial coefficients (9) we have

$$\begin{aligned} \Delta^{n+1} &= \Delta(\Delta^n) = \Delta\Big(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q f_i\, \delta^{n-i}\Big) \\ &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q \Big(\Delta(f_i)\, \delta^{n-i} + q^i f_i \delta^{n+1-i}\Big) \\ &= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q \Big(f_{i+1}\, \delta^{n-i} + q^i f_i \delta^{n+1-i}\Big) \\ &= f_{n+1} + \delta^{n+1} + \\ &\quad + \sum_{i=1}^{n} f_i\, \delta^{n+1-i} \Big(\begin{bmatrix} n \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} n \\ i \end{bmatrix}_q \Big) \\ &= f_{n+1} + \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q f_i\, \delta^{n+1-i} + \delta^{n+1} \\ &= \sum_{i=0}^{n+1} \begin{bmatrix} n+1 \\ i \end{bmatrix}_q f_i\, \delta^{n+1-i}. \end{aligned}$$

Thus the expansion formula (10) is proved. Now if we apply the both sides of the proved formula to ξ_1 we obtain

$$f_{n+1} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q f_i\, \xi_{n+1-i}, \tag{13}$$

and this is the recurrent formula for the polynomials f_n which we will use in order to prove the expansion formula (11) for f_n.

We start the proof of expansion formula (11) with the base case $n = 2$. From the definition $\Delta = \delta + \xi_1$ it follows

$$f_2 = \Delta(f_1) = \Delta(\xi_1) = \delta\xi_1 + \xi_1^2 = \xi_2 + \xi_1^2.$$

On the other hand if $n = 2$ then there are two compositions $2 = 2,\ 2 = 1 + 1$, and the expansion formula (11) gives

$$f_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \xi_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \xi_1^2 = \xi_2 + \xi_1^2.$$

Hence in the base case of mathematical induction $n = 2$ the formula (11) is correct. Now we assume that the second expansion formula holds for some positive integer $n > 2$ and show that it also holds when $n+1$ is substituted for n. Let us consider the sum

$$\sum_{\sigma \in \Psi_{n+1}} \begin{bmatrix} n_2 - 1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3 - 1 \\ n_2 \end{bmatrix}_q \cdots \cdots \begin{bmatrix} n \\ n_r \end{bmatrix}_q \xi_{i_1} \xi_{i_2} \ldots \xi_{i_{r+1}}, \tag{14}$$

where $\sigma = (i_1, i_2, \ldots, i_r, i_{r+1})$ is a composition of an integer $n+1$. Hence $i_1 + \ldots + i_r + i_{r+1} = n+1$. Our aim is to show that this sum is equal to the polynomial f_{n+1}. Let us fix an integer $i \in \{0, 1, \ldots, n\}$ and a generator ξ_{n+1-i}. It is clear that if we select the compositions of an integer $n+1$ which have the form $(i_1, i_2, \ldots, i_r, n+1-i)$, i.e. the last integer of each composition is previously fixed integer $n+1-i$, and we remove in each composition the last integer then the set of compositions $(i_1, i_2, \ldots, i_r)$ is the set of all compositions of an integer i, i.e. $\{(i_1, i_2, \ldots, i_r)\} = \Psi_i$. Indeed we have

$$i_1 + i_2 + \ldots + i_r + n + 1 - i = n + 1,$$

which implies $i_1 + i_2 + \ldots + i_r = i$. Consequently if we select all terms of the sum (14) with $i_{r+1} = n+1-i$ (i.e. containing a generator ξ_{n+1-i} at the end of a product of generators) then we get the sum

$$\sum_{\sigma \in \Psi_{n+1}} \begin{bmatrix} n_2 - 1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3 - 1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n \\ i \end{bmatrix}_q \times \\ \times \xi_{i_1} \xi_{i_2} \ldots \xi_{i_r} \xi_{n+1-i}, \tag{15}$$

where the sum is taken over the compositions of integer $n+1$ which have the form $\sigma = (i_1, i_2, \ldots, i_r, n+1-i) \in \Psi_{n+1}$. We would like to point out that the product of binomial coefficients of each term in this sum contains the factor

$$\begin{bmatrix} n \\ i \end{bmatrix}_q .$$

Hence we can write the sum (15) as follows

$$\begin{bmatrix} n \\ i \end{bmatrix}_q \Big(\sum_{\tau \in \Psi_i} \begin{bmatrix} n_2 - 1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3 - 1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} i - 1 \\ n_{r-1} \end{bmatrix}_q \\ \times \xi_{i_1} \xi_{i_2} \ldots \xi_{i_r} \Big) \xi_{n+1-i},$$

where $\tau = (i_1, i_2, \ldots, i_r) \in \Psi_i$ and the sum is taken over all compositions of integer i. Now we make use of the assumption of an inductive step that the expansion formula for a polynomial f_m holds for each integer $m \in \{1, 2, \ldots, n\}$. Hence the sum in the previous formula is equal to f_i, i.e

$$\sum_{\tau \in \Psi_i} \begin{bmatrix} n_2 - 1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3 - 1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} i - 1 \\ n_{r-1} \end{bmatrix}_q \times \\ \times \xi_{i_1} \xi_{i_2} \ldots \xi_{i_r} = f_i.$$

Thus the sum (15) is equal to

$$\begin{bmatrix} n \\ i \end{bmatrix}_q f_i \, \xi_{n+1-i},$$

and summing up all these terms with respect to i we get the sum (14). Consequently the sum (14) we started with is equal to the sum

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q f_i \, \xi_{n+1-i},$$

which in turn is equal to f_{n+1} (see the recurrent relation (13)). This ends the proof.

For the first values of n the formula (11) gives the polynomials

$$f_2 = \xi_2 + \xi_1^2, \tag{16}$$
$$f_3 = \xi_3 + \xi_2 \, \xi_1 + [2]_q \, \xi_1 \, \xi_2 + \xi_1^3, \tag{17}$$
$$f_4 = \xi_4 + \xi_3 \, \xi_1 + [3]_q \, \xi_1 \xi_3 + [3]_q \, \xi_2^2 \\ + \xi_2 \xi_1^2 + [3]_q \, \xi_1^2 \xi_2 + [2]_q \, \xi_1 \xi_2 \xi_1 + \xi_1^4. \tag{18}$$

3. Polynomial algebra at Nth root of unity

In this section we will study the structure of the graded polynomial algebra $\mathfrak{P}[\mathfrak{d}, \xi]$ introduced in the previous section in the case when q is a primitive Nth root of unity and prove that in this case the graded polynomial algebra $\mathfrak{P}[\xi]$ can be endowed with a structure of graded q-differential algebra.

Let $N \geq 2$ be an integer and q be a primitive Nth root of unity. A graded vector space $\mathscr{V} = \oplus_n \mathscr{V}^n$ is said to be a cochain N-complex or simply N-complex if it is endowed with an endomorphism of degree one $d : \mathscr{V}^n \to \mathscr{V}^{n+1}$ satisfying $d^N = 0$ [7,8,13]. A graded associative unital algebra $\mathscr{A}$ with graded q-derivation of degree one d is said to be a graded q-differential algebra if d satisfies $d^N = 0$ [6]. A graded q-derivation of degree one d will be referred to as N-differential of graded q-differential algebra. Particularly if $N = 2$ then graded q-differential algebra is a graded differential algebra with graded differential d satisfying $d^2 = 0$. One can construct a graded q-differential algebra with the help of the following theorem [4]

Theorem 2. *Let $\mathscr{A}$ be a graded associative unital algebra $\mathscr{A} = \oplus_k \mathscr{A}^k$, and q be a primitive Nth root of unity. If there exists an element of grading one $v \in \mathscr{A}^1$ which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{A})$, where $\mathscr{Z}(\mathscr{A})$ is the graded center of $\mathscr{A}$, then a graded algebra $\mathscr{A}$ endowed with the inner graded q-derivation $d = \mathrm{ad}_v^q$ is the graded q-differential algebra and d is its N-differential.*

Making use of the theorem (2) we prove the following theorem:

Theorem 3. *If q is a primitive Nth root of unity and the variable $\mathfrak{d}$ satisfies $\mathfrak{d}^N = \lambda \cdot \mathbb{1}$, where λ is any complex number, then for any integer $k > N$ a variable ξ_k vanishes, i.e. the polynomial algebra $\mathfrak{P}[\mathfrak{d}, \xi]$ is generated by the finite set of variables $\{\mathfrak{d}, \xi_k\}_{k=1}^N$ which obey the relations*

$$\mathfrak{d}\xi_1 = q\, \xi_1 \mathfrak{d} + \xi_2, \qquad \mathfrak{d}\xi_2 = q^2\, \xi_2 \mathfrak{d} + \xi_3,$$
$$\ldots$$
$$\mathfrak{d}\xi_{N-1} = q^{N-1}\, \xi_{N-1} \mathfrak{d} + \xi_N, \qquad \mathfrak{d}\xi_N = \xi_N \mathfrak{d},$$
$$\mathfrak{d}^N = \lambda \cdot \mathbb{1}.$$

The graded q-derivation $\delta = [\mathfrak{d}, \ ]_q : \mathfrak{P}[\xi] \to \mathfrak{P}[\xi]$ is the N-differential, i.e. $d^N = 0$, and the graded polynomial algebra $\mathfrak{P}[\xi]$ is the graded q-differential algebra.

Proof. It follows from Theorem 2 that the inner graded q-derivation of polynomial algebra $\mathfrak{P}[\mathfrak{d},\xi]$ determined by the graded q-commutator with the element $\mathfrak{d}$, i.e. the inner graded q-derivation $[\mathfrak{d},\ ]_q : \mathfrak{P}[\mathfrak{d},\xi] \to \mathfrak{P}[\mathfrak{d},\xi]$, is the N-differential of the polynomial algebra $\mathfrak{P}[\mathfrak{d},\xi]$ because the element $\mathfrak{d}$ satisfies $\mathfrak{d}^N = \lambda\cdot\mathbb{1}$ which means that it lies in the graded center of $\mathfrak{P}[\mathfrak{d},\xi]$. Now if we consider the restriction of this inner graded q-derivation of $\mathfrak{P}[\mathfrak{d},\xi]$ to the subalgebra $\mathfrak{P}[\xi]\subset\mathfrak{P}[\mathfrak{d},\xi]$ then we have the graded q-derivation (not inner) δ of the algebra $\mathfrak{P}[\xi]$ which obviously satisfies $\delta^N = 0$. Hence δ is the N-differential of $\mathfrak{P}[\xi]$ and $\mathfrak{P}[\xi]$ is a graded q-differential algebra. Finally δ has the property $\delta(\xi_k) = \xi_{k+1}$ and $\xi_{N+1} = \delta(\xi_N) = \delta^N(\xi_1) = 0$ which proves that for any integer $k > N$ a variable ξ_k vanishes.

From now and until the end of this Section we will assume that q is a primitive Nth root of unity, the conditions of Theorem 3 are satisfied and $\mathfrak{P}[\xi]$ is a graded q-differential algebra with N-differential δ generated by the set of variables $\xi_1,\xi_2,\ldots,\xi_N$. Let us remind that there is the endomorphism $\varDelta$ defined on the algebra $\mathfrak{P}[\xi]$ by the formula $\varDelta = \delta + L_{\xi_1}$. Hence $\varDelta$ is a differential operator of the algebra $\mathfrak{P}[\xi]$ but the following theorem shows that the Nth power of this differential operator is not a differential operator but the endomorphism of left multiplication by the polynomial f_N.

Theorem 4. *The Nth power of endomorphism $\varDelta = \delta + L_{\xi_1} : \mathfrak{P}[\xi]\to\mathfrak{P}[\xi]$ is the endomorphism of left multiplication by the polynomial f_N, i.e. $\varDelta^N = L_{f_N}$.*

Proof. We prove this theorem with the help of expansion formula (10). We have

$$\varDelta^N = \sum_{i=0}^{N}\begin{bmatrix} N\\ i\end{bmatrix}_q f_i\,\delta^{N-i}.$$

The q-binomial coefficients in this expansion formula with $i\in\{1,2,\ldots,N-1\}$ vanish because q is a primitive Nth root of unity. The first term in this expansion also vanishes because δ is an N-differential and satisfies $\delta^N = 0$. Hence $\varDelta^N = L_{f_N}$.

Theorem 5. *The polynomial $f_N\in\mathfrak{P}[\xi]$ satisfies the identity*

$$\delta(f_N) + ad^q_{\xi_1}(f_N) = 0,$$

where $ad^q_{\xi_1} = [\xi_1,\ ]_q : \mathfrak{P}[\xi]\to\mathfrak{P}[\xi]$ is an inner graded q-derivation of algebra $\mathfrak{P}[\xi]$.

Proof. Let us remind that there is the recurrent relation (13) for the polynomials f_k which has the form

$$f_{k+1} = \sum_{i=0}^{k}\begin{bmatrix} k\\ i\end{bmatrix}_q f_i\,\xi_{k+1-i}.$$

Taking $k = N$ in this relation we obtain

$$f_{N+1} = \sum_{i=0}^{N}\begin{bmatrix} N\\ i\end{bmatrix}_q f_i\,\xi_{N+1-i}. \tag{19}$$

For any integer $1\le i\le N-1$ the q-binomial coefficient in (19) labelled with i vanishes because q is a primitive Nth root of unity. Thus the above relation takes the form

$$f_{N+1} = f_0\,\xi_{N+1} + f_N\xi_1.$$

From Theorem 3 follows that $\xi_{N+1} = 0$, and the first term at the right-hand side of the above relation vanishes. Hence

$$\begin{aligned}0 &= f_{N+1} - f_N\,\xi_1 = \varDelta(f_N) - f_N\,\xi_1\\ &= \delta(f_N) + \xi_1\,f_N - f_N\,\xi_1 = \delta(f_N) + \mathrm{ad}^q_{\xi_1}(f_N) = 0.\end{aligned}$$

4. Chern-Simons forms

The aim of this section is to develop a noncommutative analog of Chern-Simons form with the help of graded q-differential algebra. We will introduce the Chern character form and prove the infinitesimal homotopy formula for the Chern character form.

Let us remind that $\mathfrak{A}$ is a graded algebra with q-derivation d, where q is a complex number different from zero, and A is a grading one element of this algebra. Let $\mathfrak{A}_A$ be the subalgebra of algebra $\mathfrak{A}$ freely generated by the elements $A, dA, d^2A,\ldots$. It is clear that the algebras $\mathfrak{A}_A$ and $\mathscr{L}_A(\mathfrak{A})$ are free algebras, and they are isomorphic if we identify their generators as follows

$$d^nA \leftrightarrow L_{d^nA} \leftrightarrow \xi_{n+1},\quad n\ge 0. \tag{20}$$

Moreover the algebra $\mathscr{L}^d_A(\mathfrak{A})$ is isomorphic to the algebra $\mathfrak{P}[\mathfrak{d},\xi]$ if in addition to (20) we identify the generator d viewed as the element of $\mathscr{L}^d_A(\mathfrak{A})$ with the generator $\mathfrak{d}$. This follows immediately from the graded q-Leibniz rule (1) and the commutation relations (4). Hence considering d as an element of algebra $\mathscr{L}^d_A(\mathfrak{A})$ we can identify the inner graded q-derivation $\mathrm{ad}^q_d = [d,\ ]_q$ of $\mathscr{L}^d_A(\mathfrak{A})$ with the inner graded q-derivation $\mathrm{ad}^q_{\mathfrak{d}} = [\mathfrak{d},\ ]$ of $\mathfrak{P}[\mathfrak{d},\xi]$. Let us remind that the restriction of the inner graded q-derivation $\mathrm{ad}^q_{\mathfrak{d}} = [\mathfrak{d},\ ]$ to the subalgebra $\mathfrak{P}[\xi]$ generated by ξ_n is the graded q-derivation δ of $\mathfrak{P}[\xi]$. If we restrict the inner graded q-derivation $\mathrm{ad}^q_d = [d,\ ]_q$ to the subalgebra $\mathscr{L}_A(\mathfrak{A})$ which is isomorphic to the algebra $\mathfrak{P}[\xi]$ and according to (20) replace each generator L_{d^nA} of $\mathscr{L}_A(\mathfrak{A})$ with corresponding generator d^nA of $\mathfrak{A}_A$ then the restriction of inner graded q-derivation $\mathrm{ad}^q_{\mathfrak{d}} = [\mathfrak{d},\ ]$ to the subalgebra $\mathfrak{P}[\xi]$ can be identified with graded q-derivation d of $\mathfrak{A}_A$. This immediately follows from the graded q-Leibniz rule written in the form

$$\mathrm{ad}^q_d(L_A) = [d, L_A]_q = d\circ L_A - q^{|A|}L_A\circ d = L_{dA}.$$

Hence if we identify each generator ξ_n of algebra $\mathfrak{P}[\xi]$ with corresponding generator $d^{n-1}A$ of isomorphic algebra $\mathfrak{A}_A$ then the graded q-derivation δ is identified with the graded q-derivation d of $\mathfrak{A}_A$. Consequently the calculus developed in Section 2 for the algebra $\mathfrak{P}[\xi]$ can be applied to the algebra $\mathfrak{A}_A$ if we replace $\xi_{n+1}\to d^nA, \delta\to d$. For instant we can introduce an endomorphism $D:\mathfrak{A}_A\to\mathfrak{A}_A$

which can be viewed as an analog of $\Delta = \delta + L_{\xi_1}$ giving it by the formula $D = d + L_A$. It is worth mentioning that D has the remarkable property

$$D(P\,P') = D(P)\,P' + q^{|P|}P\,d(P'), \tag{21}$$

where $P, P' \in \mathfrak{A}_A$. We can also introduce the polynomials $F_n \in \mathfrak{A}_A$ defining them by the recurrent formula

$$F_0 = \mathbb{1}, \quad F_1 = A, \quad F_{n+1} = D(F_n). \tag{22}$$

It is obvious that the polynomials F_n can be viewed as analogs of polynomials f_n and we can calculate a polynomial F_n by replacing each generator ξ_k in f_n with generator $d^{k-1}A$. It follows from Theorem 1 that the following expansion formulae for D and the polynomials F_n hold

$$D^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q F_i\, d^{n-i},$$

$$F_n = \sum_{\sigma\in\Psi_n} \begin{bmatrix} n_1-1 \\ 0 \end{bmatrix}_q \begin{bmatrix} n_2-1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_3-1 \\ n_2 \end{bmatrix}_q \cdots \cdots \begin{bmatrix} n_r-1 \\ n_{r-1} \end{bmatrix}_q d^{i_1-1}Ad^{i_2-1}A\ldots d^{i_r-1}A.$$

If $n = 2,3,4$ then we get the polynomials

$$F_2 = dA + A^2, \tag{23}$$

$$F_3 = d^2A + dA\,A + [2]_q\,A\,dA + A^3, \tag{24}$$

$$F_4 = d^3A + d^2A\,A + [3]_q\,A\,d^2A + [3]_q\,(dA)^2 + dA\,A^2 + [3]_q\,A^2\,dA + [2]_q\,A\,dA\,A + A^4. \tag{25}$$

It should be pointed out that from the beginning of this section and up to now q is a complex number different from zero. From now and until the end of this section we will assume that q is a primitive Nth root of unity. In this case we can prove the following theorem:

Theorem 6. *If q is a primitive Nth root of unity then an algebra $\mathfrak{A}$ with graded q-derivation d is a graded q-differential algebra with N-differential d if and only if for any element of grading one $A \in \mathfrak{A}$ the generator $\mathfrak{d}$ of algebra $\mathfrak{P}[\xi]$ satisfies $\mathfrak{d}^N = \lambda\cdot\mathbb{1}$, where λ is any complex number.*

In what follows we will assume that the necessary and sufficient condition of Theorem 6 is satisfied, and hence $\mathfrak{A}$ is a graded q-differential algebra with N-differential d. Since a graded differential algebra can be viewed as an analog of algebra of differential forms on a manifold or in a noncommutative case as an analog of algebra of matrix-valued differentials forms we can use a terminology of modern differential geometry interpreting an element of grading one $A \in \mathfrak{A}$ as analogous to a connection form [16]. Then the endomorphism $D = d + L_A$ satisfying (21) will be referred to as a covariant N-differential. It follows from Theorem 4 that the Nth power of covariant N-differential D is the endomorphism of left multiplication by the polynomial F_N. Thus we can view the polynomial F_N as analogous to the curvature of connection form A, and the polynomial F_N will be referred to as the curvature of connection form A. It follows from Theorem 5 that the curvature F_N satisfies the identity

$$d(F_N) + [A, F_N]_q = 0, \tag{26}$$

this identity will be referred to as Bianchi identity for the curvature of connection form A.

Particularly if $N = 2$ then q is a primitive quadratic root of unity, i.e. $q = -1$, and an algebra $\mathfrak{A}$ is a graded differential algebra with differential d which satisfies $d^2 = 0$. In this case the curvature F_2 of connection form A is given by the formula $F_2 = dA + A^2 = dA + \frac{1}{2}[A, A]$, where $[\ ,\ ]$ is the graded commutator.

If $N = 3$ then q is a primitive cubic root of unity satisfying the identity $1+q+q^2 = 0$. In this case $\mathfrak{A}$ is a graded q-differential algebra with 3-differential d which has the property $d^3 = 0$. The curvature F_3 of connection form A is given by $F_3 = d^2A + dA\,A + [2]_q\,A\,dA + A^3$. Making use of the identity $1 + q + q^2 = 0$ we get $[2]_q = 1 + q = -q^2$. Thus the curvature can be written in the form

$$F_3 = d^2A + dA\,A - q^2\,A\,dA + A^3 = d^2A + [dA, A]_q + A^3, \tag{27}$$

where $[\ ,\ ]_q$ is the graded q-commutator.

Let us denote by $[\mathfrak{A},\mathfrak{A}]_q$ the subspace of $\mathfrak{A}$ spanned by graded q-commutators $[P, P']_q$, where P, P' are homogeneous elements of graded q-differential algebra $\mathfrak{A}$. Let $\mathscr{V} = \oplus_n \mathscr{V}^n$ be an N-complex with an endomorphism of degree one $\hat{d} : \mathscr{V}^n \to \mathscr{V}^{n+1}$ satisfying $\hat{d}^N = 0$. A homogeneous degree zero homomorphism of vector spaces $\tau : \mathfrak{A}^n \to \mathscr{V}^n$ which satisfies

$$\hat{d}\circ\tau = \tau\circ d, \quad \tau([\mathfrak{A},\mathfrak{A}]_q) = 0, \tag{28}$$

will be referred to as a trace on a graded q-differential algebra $\mathfrak{A}$. For any positive integer n the element $\tau(F_N^n/n!)$ of N-complex $\mathscr{V}$ will be referred to as the Chern character form of connection form A.

Theorem 7. *The Chern character form of connection form A is closed form, i.e.*

$$\hat{d}\{\tau(\frac{F_N^n}{n!})\} = 0. \tag{29}$$

Proof. Let us show that F_N^n satisfies the identity

$$d(F_N^n) + [A, F_N^n]_q = 0. \tag{30}$$

In the case of $N = 1$ we have the Bianchi identity. Now assuming that F_N^{n-1} satisfies (30) we will show that F_N^n satisfies the same identity. We have

$$\begin{aligned} d(F_N^n) + [A, F_N^n]_q &= d(F_N\,F_N^{n-1}) + [A, F_N^n]_q \\ &= d(F_N)F_N^{n-1} + q^N\,F_N\,d(F_N^{n-1}) + A\,F_N^n - F_N^n\,A \\ &= \{d(F_N) + [A, F_N]_q\}\,F_N^{n-1} + \\ &\quad + F_N\,\{d(F_N^{n-1}) + [A, F_N^{n-1}]_q\} = 0. \end{aligned}$$

Thus

$$\hat{d}\{\tau(\frac{F_N^n}{n!})\} = \frac{1}{n!}\tau\{d(F_N^n)\} = \frac{1}{n!}\tau(-[A, F_N^n]_q) = 0.$$

Theorem 8.*Let q be a primitive cubic root of unity and $A(t) = t\,A$ be a family of grading one elements of graded q-differential algebra $\mathfrak{A}$. Then the following infinitesimal homotopy formula holds*

$$\frac{\partial}{\partial t}\{\frac{\tau(F_3^n(t))}{n!}\} = \hat{d}^2\{\frac{\tau(\dot{A}(t)F_3^{n-1}(t))}{n-1!}\}. \tag{31}$$

*Proof.*If $n = 1$ then the infinitesimal homotopy formula takes the form

$$\frac{\partial}{\partial t}\{\tau(F_3(t))\} = \hat{d}^2\{\tau(\dot{A}(t))\}. \tag{32}$$

Making use of formulae (27) and

$$A^3 = \frac{[A,[A,A]_q]_q}{(1-q)(1-q^2)}, \tag{33}$$

we can write the left-hand side of (32) as follows

$$\frac{\partial}{\partial t}\{\tau(F_3(t))\} = \frac{\partial}{\partial t}\{\tau(d^2A(t) + [dA(t),A(t)]_q + \\ + \frac{[A(t),[A(t),A(t)]_q]_q}{(1-q)(1-q^2)})\} = \hat{d}^2\{\tau(\dot{A}(t))\}.$$

We show how to prove the infinitesimal homotopy formula in the case $n = 2$ and for integers $n > 2$ this formula can be proved in a similar way. If $n = 2$ then the formula (31) takes the form

$$\frac{1}{2}\frac{\partial}{\partial t}\{\tau(F_3^2(t))\} = \hat{d}^2\{\tau(\dot{A}(t)F_3(t))\}. \tag{34}$$

For $A(t) = t\,A$ we calculate the polynomial of t in the left-hand side of infinitesimal homotopy formula

$$\frac{1}{2}\frac{\partial}{\partial t}\{\tau(F_3^2(t))\} = \sum_{k=1}^{5} P_k\,t^k, \tag{35}$$

where

$$P_1 = (d^2A)^2,$$

$$P_2 = \frac{3}{2}d^2A\,dA\,A - \frac{3q^2}{2}d^2A\,A\,dA + \frac{3}{2}dA\,A\,d^2A - \\ -\frac{3q^2}{2}d^2A\,dA\,d^2A,$$

$$P_3 = 2\,(d^2A\,A^3 + dA\,A\,dA\,A - q^2\,dA\,A^2\,dA - \\ -q^2\,A\,(dA)^2A - q^2\,A\,dA\,A\,dA + A^3d^2A),$$

$$P_4 = \frac{5}{2}(dA\,A^4 - q^2\,A\,dA\,A^3 + A^3\,dA\,A - q^2\,A^4dA),$$

$$P_5 = 3\,A^6.$$

Obviously there are no terms with t^4 and t^5 in the right-hand side of formula (34), and the polynomials P_4, P_5 have to be expressed in terms of graded q-commutators. Indeed we have

$$P_4 = \frac{5}{2}(2[dA,A^4]_q + [A^3,dA\,A]_q - q^2[A\,dA,A^3]_q),$$

$$P_5 = \frac{3}{(1-q)(1-q^2)}[[A^2,A^2]_q,A^2]_q.$$

Now we calculate the right-hand side of formula (34). We obtain

$$\dot{A}(t)F_3(t) = t\,A\,d^2A + t^2\,A\,dA\,A - q^2t^2\,A^2\,dA + t^3\,A^4.$$

Differentiating twice by d and comparing with the left-hand side we conclude that the formula (34) holds if three polynomials can be expressed in terms of graded q-commutators. Because of the limited space of the present article we will show only one of these polynomials and represent it by graded q-commutators. This polynomial is

$$q^2\,A\,dA\,A^2 + q\,A^2\,d^2A\,A + A^3\,d^2A + d^2\,A^3,$$

and this polynomial can be written in the form

$$(1-\frac{q^2}{3})[A^2,d^2A\,A]_q + \frac{1}{3}[A^2,A\,d^2A]_q + \\ +\frac{2}{3}(q^2-1)[A^2d^2A,A]_q - \frac{4}{3}q^2[A,[A^2,d^2A]]_q.$$

Integrating both sides of the formula (34) we obtain

$$\frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\{\tau(F_3^2(t))\}\,dt = \int_0^1 \hat{d}^2\{\tau(\dot{A}(t)F_3(t))\}\,dt,$$

or

$$\frac{1}{2}\tau(F_3^2(t)) = \hat{d}^2\int_0^1 \tau\{\dot{A}(t)\,F_3(t)\}\,dt. \tag{36}$$

The integral at the right-hand side of the previous formula will be referred to as the Chern-Simons form and it will be denoted by $\mathrm{CS}_q^3(A,\tau)$, i.e.

$$\mathrm{CS}_q^3(A,\tau) = \int_0^1 \tau\{\dot{A}(t)\,F_3(t)\}\,dt. \tag{37}$$

Since $A(t) = t\,A$ we can find an explicit formula for the Chern-Simons form

$$\mathrm{CS}_q^3(A,\tau) = \frac{1}{2}\tau(A\,d^2A + \frac{2}{3}A\,dA\,A - \frac{2q^2}{3}A^2\,dA + \frac{1}{2}\,A^4).$$

5. Connection on module

We begin this section by recalling the notion of Ω-connection given in [9]. Suppose that $\mathfrak{A}$ is an unital associative algebra over the field of complex numbers and $\mathcal{E}$ is a left module over $\mathfrak{A}$. Let Ω be a graded differential algebra with differential d, such that $\Omega^0 = \mathfrak{A}$, it means that the map $d : \mathfrak{A} \to \Omega^1$ is a differential calculus over $\mathfrak{A}$. Since an subspace of elements of grading one can be viewed as a $(\mathfrak{A},\mathfrak{A})$-bimodule, the tensor product $\Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ clearly has the structure of left $\mathfrak{A}$-module.

Definition 1.*A linear map $\nabla : \mathcal{E} \to \Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ is called an* Ω-connection *if it satisfies*

$$\nabla(us) = du \otimes_{\mathfrak{A}} s + u\nabla(s)$$

for any $u \in \mathfrak{A}$ and $s \in \mathcal{E}$.

Similarly to the case of connections on vector bundles, this map has a natural extension $\nabla : \Omega \otimes_{\mathfrak{A}} \mathcal{E} \to \Omega \otimes_{\mathfrak{A}} \mathcal{E}$ by setting

$$\nabla(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + (-1)^p \omega \nabla(s),$$

where $\omega \in \Omega^p$ and $s \in \mathcal{E}$.

Now our aim is to generalize a notion of Ω-connection taking graded q-differential algebra instead of graded differential algebra Ω. Let $\mathfrak{A}$ be an unital associative $\mathbb{C}$-algebra, Ω_q is a graded q-differential algebra with N-differential d and $\mathfrak{A} = \Omega_q^0$. Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. Considering algebra Ω_q as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product of left $\mathfrak{A}$-modules $\Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$ which clearly has the structure of left $\mathfrak{A}$-module. To shorten the notation, we denote this left $\mathfrak{A}$-module by $\mathfrak{F}$. Taking into account that an algebra Ω_q can be viewed as the direct sum of $(\mathfrak{A}, \mathfrak{A})$-bimodules Ω_q^i we can split the left $\mathfrak{A}$-module $\mathfrak{F}$ into the direct sum of the left $\mathfrak{A}$-modules $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathfrak{A}} \mathcal{E}$, i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, which means that $\mathfrak{F}$ inherits the graded structure of algebra Ω_q, and $\mathfrak{F}$ is the graded left $\mathfrak{A}$-module. It is worth noting that the left $\mathfrak{A}$-submodule $\mathfrak{F}^0 = \mathfrak{A} \otimes_{\mathfrak{A}} \mathcal{E}$ of elements of grading zero is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$, where isomorphism $\varphi : \mathcal{E} \to \mathfrak{F}^0$ can be defined for any $s \in \mathcal{E}$ by

$$\varphi(s) = e \otimes_{\mathfrak{A}} s, \tag{38}$$

where e is the identity element of algebra $\mathfrak{A}$. Since a graded q-differential algebra Ω_q can be viewed as the (Ω_q, Ω_q)-bimodule, the left $\mathfrak{A}$-module $\mathfrak{F}$ can be also considered as the left Ω_q-module, and we will use this structure to describe a concept of N-connection. Let us mention that multiplication by elements of Ω^i, where $i \neq 0$, does not preserve the graded structure of the left Ω_q-module $\mathfrak{F}$.

The tensor product $\mathfrak{F}$ has also the structure of the vector space over $\mathbb{C}$ where this vector space is the tensor product of the vector spaces Ω_q and $\mathcal{E}$. It is evident that $\mathfrak{F}$ is a graded vector space, i.e. $\mathfrak{F} = \oplus_i \mathfrak{F}^i$, where $\mathfrak{F}^i = \Omega_q^i \otimes_{\mathbb{C}} \mathcal{E}$. Due to the structure of vector space of $\mathfrak{F}$ we can introduce the notion of linear operator on $\mathfrak{F}$. We denote the vector space of linear operators on $\mathfrak{F}$ by $\mathfrak{L}(\mathfrak{F})$. The structure of the graded vector space of $\mathfrak{F}$ induces the structure of a graded vector space on $\mathfrak{L}(\mathfrak{F})$, and we shall denote the subspace of homogeneous linear operators of degree i by $\mathfrak{L}^i(\mathfrak{F})$.

Definition 2.*An* N-connection *on the left Ω_q-module $\mathfrak{F}$ is a linear operator $\nabla_q : \mathfrak{F} \to \mathfrak{F}$ of degree one satisfying the condition*

$$\nabla_q(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + q^{|\omega|}\, \omega\, \nabla_q(s), \tag{39}$$

where $\omega \in \Omega_q^k$, $s \in \mathcal{E}$, and $|\omega|$ is the degree of the homogeneous element of algebra Ω_q.

Making use of the previously introduced notations we can write $\nabla_q \in \mathfrak{L}^1(\mathfrak{F})$. It is worth pointing out that if $N = 2$ then $q = -1$, and in this particular case the Definition 2 gives us the algebraic analog of a classical connection. We see that connection on vector bundle can be viewed as a linear map on a left module of sections of vector bundle, taking values in algebra of differential 1-forms with values in this vector bundle, which clearly has a structure of a left module over an algebra of smooth functions on a base manifold. Hence a concept of a N-connection can be viewed as a generalization of a classical connection.

One can define an N-connection on right modules. If $\mathcal{E}^R$ is a right $\mathfrak{A}$-module, a N-connection on $\mathfrak{G} = \mathcal{E}^R \otimes_{\mathfrak{A}} \Omega_q$ is a linear map $\nabla_q : \mathfrak{G} \to \mathfrak{G}$ of degree one such that $\nabla_q(\xi \otimes_{\mathfrak{A}} \omega) = \xi \otimes_{\mathfrak{A}} d\omega + q^{\omega} \nabla_q(\xi)\omega$ for any $\xi \in \mathcal{E}^R$ and homogeneous element $\omega \in \Omega_q$.

Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. The set of all homomorphisms of $\mathcal{E}$ into $\mathfrak{A}$ has the structure of the dual module of the left $\mathfrak{A}$-module $\mathcal{E}$ and is denoted by $\mathcal{E}^*$. It is evident that $\mathcal{E}^*$ is a right $\mathfrak{A}$-module.

Definition 3.*A linear map $\nabla_q^* : \mathcal{E}^* \to \mathcal{E}^* \otimes_{\mathfrak{A}} \Omega_q^1$ defined as follows*

$$\nabla_q^*(\eta)(\xi) = d(\eta(\xi)) - \eta(\nabla_q(\xi)),$$

where $\xi \in \mathcal{E}$, $\eta \in \mathcal{E}^*$ and ∇_q is an N-connection on $\mathcal{E}$, is said to be the dual connection of ∇_q.

It is easy to verify that ∇_q^* has a structure of N-connection on the right module $\mathcal{E}^*$. Indeed, for any $f \in \mathfrak{A}$, $\eta \in \mathcal{E}^*$, $\xi \in \mathcal{E}$ we have

$$\begin{aligned}
\nabla_q^*(\eta f)(\xi) &= d(\eta f(\xi)) - (\eta f)(\nabla_q \xi) \\
&= d(\eta(\xi) f) - \eta(\nabla_q \xi) f \\
&= d(\eta(\xi)) f + \eta(\xi) \otimes_{\mathfrak{A}} df - \eta(\nabla_q \xi) f \\
&= \eta(\xi) \otimes_{\mathfrak{A}} df + \nabla_q^*(\eta(\xi)) f.
\end{aligned}$$

In order to define a Hermitian structure on a right $\mathfrak{A}$-module $\mathcal{E}$ we assume $\mathfrak{A}$ to be a graded q-differential algebra with involution $*$ such that the largest linear subset contained in the convex cone $C \in \mathfrak{A}$ generated by a^*a is equal to zero, i.e. $C \cap (-C) = 0$. The right $\mathfrak{A}$-module $\mathcal{E}$ is called a Hermitian module if $\mathcal{E}$ is endowed with a sesquilinear map $h : \mathcal{E} \times \mathcal{E} \to \mathfrak{A}$ which satisfies

$$\begin{aligned}
&h(\xi\omega, \xi\omega') = \omega^* h(\xi, \xi')\omega', \quad \forall \omega, \omega' \in \mathfrak{A},\ \forall \xi, \xi' \in \mathcal{E}, \\
&h(\xi, \xi) \in C, \quad \forall \xi \in \mathcal{E} \ \text{ and } \ h(\xi, \xi) = 0 \Rightarrow \xi = 0.
\end{aligned}$$

We have used the convention for sesquilinear map to take the second argument to be linear, therefore we define a Hermitian structure on right modules. In a similar manner one can define a Hermitian structure on left modules.

Definition 4.*An N-connection ∇_q on a Hermitian right $\mathfrak{A}$-module $\mathcal{E}$ is said to be* consistent with a Hermitian structure of *$\mathcal{E}$ if it satisfies*

$$dh(\xi, \xi') = h(\nabla_q(\xi), \xi') + h(\xi, \nabla_q(\xi')),$$

where $\xi, \xi' \in \mathcal{E}$.

Our next aim is to define a curvature of N-connection. Following [3] we start with

Proposition 1. *The N-th power of any N-connection ∇_q is the endomorphism of degree N of the left Ω_q-module $\mathfrak{F}$.*

Proof. It suffices to verify that for any homogeneous element $\omega \in \Omega_q$ an endomorphism $\nabla_q \in \mathfrak{L}^1(\mathfrak{F})$ satisfies the condition

$$\nabla_q^N(\omega \otimes_{\mathfrak{A}} s) = \omega\, \nabla_q^N(s).$$

We expand the k-th power of ∇_q as follows

$$\nabla_q^k(\omega \otimes_{\mathfrak{A}} s) = \sum_{m=0}^{k} q^{m|\omega|} \begin{bmatrix} k \\ m \end{bmatrix}_q d^{k-m}\omega\, \nabla_q^m(s), \quad (40)$$

where $\begin{bmatrix} k \\ m \end{bmatrix}_q$ are the q-binomial coefficients. Since d is the N-differential of a graded q-differential algebra Ω_q we have $d^N\omega = 0$. According to $\begin{bmatrix} N \\ m \end{bmatrix}_q = 0$ for $1 \leq m \leq N-1$, we see that in the case of $k = N$ the expansion (40) takes the following form

$$\nabla_q^N(\omega \otimes_{\mathfrak{A}} s) = q^{N|\omega|}\omega\, \nabla_q^N(s) = \omega\, \nabla_q^N(s) \quad (41)$$

and this clearly shows that ∇_q^N is the endomorphism of the left Ω_q-module $\mathfrak{F}$.

This proposition allows us to define the curvature of N-connection as follows

Definition 5. *The endomorphism $F = \nabla_q^N$ of degree N of the left Ω_q-module $\mathfrak{F}$ is said to be the* curvature of a N-connection ∇_q.

Suppose that $\mathfrak{L}(\mathfrak{F})$ is the graded vector space. We proceed to show that $\mathfrak{L}(\mathfrak{F})$ has a structure of graded algebra. To this end, we take the product $A \circ B$ of two linear operators A, B of the vector space $\mathfrak{F}$ as an algebra multiplication. If $A : \mathfrak{F} \to \mathfrak{F}$ is a homogeneous linear operator than we can extend it to the linear operator $L_A : \mathfrak{L}(\mathfrak{F}) \to \mathfrak{L}(\mathfrak{F})$ on the whole graded algebra of linear operators $\mathfrak{L}(\mathfrak{F})$ by means of the graded q-commutator as follows

$$L_A(B) = [A, B]_q = A \circ B - q^{|A||B|} B \circ A, \quad (42)$$

where B is a homogeneous linear operator. It makes allowable to extend an N-connection ∇_q to the linear operator on the vector space $\mathfrak{L}(\mathfrak{F})$

$$\nabla_q(A) = [\nabla_q, A]_q = \nabla_q \circ A - q^{|A|} A \circ \nabla_q, \quad (43)$$

where A is a homogeneous linear operator. As it follows from the Definition (2), ∇_q is the linear operator of degree one on the vector space $\mathfrak{L}(\mathfrak{F})$, i.e. $\nabla_q : \mathfrak{L}^i(\mathfrak{F}) \to \mathfrak{L}^{i+1}(\mathfrak{F})$, and ∇_q satisfies the graded q-Leibniz rule with respect to the algebra structure of $\mathfrak{L}(\mathfrak{F})$. Consequently the curvature F of an N-connection can be viewed as the linear operator of degree N on the vector space $\mathfrak{F}$, i.e. $F \in \mathfrak{L}^N(\mathfrak{F})$. Therefore one can act on F by N-connection ∇_q, and it holds that

Proposition 2. *For any N-connection ∇_q the curvature F of this connection satisfies the Bianchi identity $\nabla_q(F) = 0$.*

Proof. We have

$$\nabla_q(F) = \nabla_q \circ F - q^N F \circ \nabla_q = \nabla_q^{N+1} - \nabla_q^{N+1} = 0.$$

The following theorem shows that not every left $\mathfrak{A}$-module admits an N-connection [5]. In analogy with the theory of Ω-connection [9] we can prove that there is an N-connection on every projective module. Let us first prove the following proposition.

Proposition 3. *If $\mathcal{E} = \mathfrak{A} \otimes V$ is a free $\mathfrak{A}$-module, where V is a $\mathbb{C}$-vector space, then $\nabla_q = d \otimes I_V$ is N-connection on $\mathcal{E}$ and this connection is flat, i.e. its curvature vanishes.*

Proof. Indeed, $\nabla_q : \mathfrak{A} \otimes V \to \Omega_q^1 \otimes (\mathfrak{A} \otimes V)$ and

$$\begin{aligned} \nabla_q(f(g \otimes v)) &= (d \otimes I_V)(f(g \otimes v)) = d(fg) \otimes v = \\ &= (dfg) \otimes v + f(dg \otimes v) \\ &= df \otimes_{\mathfrak{A}} (g \otimes v) + f\nabla_q(g \otimes v), \end{aligned}$$

where $f, g \in \mathfrak{A}$, $v \in V$. Since d satisfies $d^N = 0$ and q is a primitive Nth root of unity, we get

$$\nabla_q^N(f(g \otimes v)) = \sum_{k+m=N} \begin{bmatrix} N \\ m \end{bmatrix}_q d^k f(d^m g \otimes v) = 0,$$

i. e. the curvature of such a N-connection vanishes.

Theorem 9. *Every projective module admits an N-connection.*

Proof. Let $\mathcal{P}$ be a projective module. From the theory of modules it is known that a module $\mathcal{P}$ is projective if and only if there exists a module $\mathcal{N}$ such that $\mathcal{E} = \mathcal{P} \oplus \mathcal{N}$ is a free module. A free left $\mathfrak{A}$-module $\mathcal{E}$ can be represented as the tensor product $\mathfrak{A} \otimes V$, where V is a $\mathbb{C}$-vector space. A linear map $\nabla_q = \pi \circ (d \otimes I_V) : \mathcal{P} \to \Omega_q^1 \otimes_{\mathfrak{A}} \mathcal{P}$ is a N-connection on a projective module $\mathcal{P}$, where $d \otimes I_V$ is a N-connection on a left $\mathfrak{A}$-module $\mathcal{E}$, π is the projection on the first summand in the direct sum $\mathcal{P} \oplus \mathcal{N}$ and $\pi(\omega \otimes_{\mathfrak{A}} (g \otimes v)) = \omega \otimes_{\mathfrak{A}} \pi(g \otimes v) = \omega \otimes_{\mathfrak{A}} m$, where $\omega \in \Omega_q^1$, $g \in \mathfrak{A}$, $v \in V$, $m \in \mathcal{P}$. Taking into account Proposition 3 we get

$$\begin{aligned} \nabla_q(fm) &= \pi((d \otimes I_V)(fm)) = \pi(df \otimes_{\mathfrak{A}} m + fdm) = \\ &= df \otimes_{\mathfrak{A}} \pi(m) + f\nabla_q(m) \\ &= df \otimes_{\mathfrak{A}} m + f\nabla_q(m), \end{aligned}$$

where $f \in \mathfrak{A}$, $m \in \mathcal{P}$.

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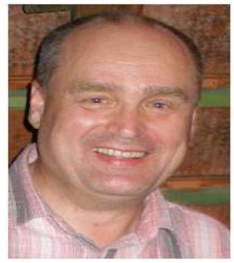

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