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# On the Stability to a Functional Lienard Type Equation with Variable Delay by Fixed Points Theory

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**Abstract:** This paper is devoted to the mathematical analysis of stability of a scalar *Liénard* type equation with variable delay. We use the fixed point technique under an exponentially weighted metric to prove the stability of the zero solution. By this work, we extend and improve some related results in the literature

**Keywords:** Liénard equation, variable delay, stability, fixed points.

## 1 Introduction

For decades, the Lyapunovs second (or direct) method has been very effective in establishing stability, instability, boundedness, global existence, etc. results for a wide variety of ordinary, functional differential and integral equations. However, there is a large set of problems for which it has been ineffective (see, Burton [6]). Probably, the reason leads this fact is that constructing or defining suitable Lyapunov functions or functionals that yield meaningful results for the mentioned qualitative behaviors of ordinary and functional differential equations and integral equations remains as an open problem in the literature by this time. On the same time, in a series of papers and books, many authors have examined particular problems which have offered great difficulties for the mentioned topics and have discussed the qualitative behaviors of solutions by means of various fixed point theorems. Further, while the Lyapunov s direct method usually requires pointwise conditions, fixed point theory needs average conditions. In addition, fixed point theory can be applied directly to study an equation when solutions are being considered on a finite interval. For comprehensive works done on the qualitative behaviors of the equations mentioned, the readers can referee to the books or the papers of Ahmad Rama and Mohana Rao [1], Antosiewicz [2],Becker and Burton [3], Burton [4,5, 6,7,8,9,10], Burton and Furumochi [11], Burton and Hering [12], Burton and Townsend [13], Caldeira-Saraiva [14], Constantin [15], Chen et al. [16], Cherkas and Malysheva [17], *Čžan* [18], Gao and Zhao [19], Graef [20], Hale [21], Heidel [22], Hou and Wu [23], Huang and Yu [24], Jin [25], Jitsuro and Yusuke [26], Kato [27], Liu and Huang [28], Liu [29], Luk [30], Malyseva [31], *Nápoles Valdés* [32], Omari and Zanolin [33], Pi [34], *Qian* ([35],[36]), Sugie and Amano [37], Tunc [38,39,40, 41,42,43,44,45,46], Tunc and Tunc [47], Yoshizawa ([48], [**?**]), Zhang ([50], [51]), Zhou and Jiang [52], Zhou and Xiang [53] and the references cited in these sources. It should be noted that, in mathematics, *a* fixed-point theorem is a result saying that a function $F$ will have at least one fixed point (a point $x$ for which $F(x) = x$), for which for which under some conditions on $F$ that can be stated in general terms. Results of this kind are amongst the most generally useful in mathematics, in many scientific fields and applications; stability, instability, boundedness, periodicity, global existence etc. are fundamental concepts that can be described in terms of fixed points. For example, in economics, Nash equilibrium of a game is a fixed point of the games best response correspondence. However, in physics, more precisely in the theory of Phase Transitions, linearization near an unstable fixed point has led to Wilsons Nobel prize-winning work inventing the renormalization group, and to the mathematical explanation of the term critical phenomenon. In compilers, fixed point computations are used for whole program analysis, which are often required to do code optimization, and the concept of fixed

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point can be used to define the convergence of a function, (see, Becker and Burton [3], Burton [6,7,8,9,10], Burton and Furumochi [11], Pi [34], Zhang [50] and references cited therein).It should be noted that in 2005, Burton [8] considered the scalar *Liénard* type equation with constant delay, $L(>0)$ :

$$x'' + f(t,x,x')x' + b(t)g(x(t-L)) = 0. \qquad (1)$$

Burton [8] obtained conditions for each solution $x(t)$ to satisfy $(x(t),x'(t)) \to (0,0)$ as $t \to \infty$ by using contraction mappings.
Later, in 2011, Pi [34] discussed the stability and properties of solutions to a scalar functional *Liénard* type equation with variable delay, $\tau(t)(>0)$ :

$$x'' + f(t,x,x')x' + b(t)g(x(t-\tau(t))) = 0. \qquad (2)$$

Pi [34] obtained some interesting sufficient conditions ensuring that the zero solution of Eq. (2) is stable and asymptotically stable by using fixed point theory under an exponentially weighted metric. On the other hand, to the best of our knowledge from the literature, the qualitative behaviors of the *Liénard* type equations of the form

$$x'' + \left\{ f(x) + g(x)x' \right\} x' + h(x) = e(t) \qquad (3)$$

and its modified types have been studied by many researches since 1945 by now, due to the fact that these type equations have played very important roles in many scientific areas such as mechanics, engineering, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory, etc. (see, for example, Ahmad and Rama Mohana Rao [1], Antosiewicz [2], Burton ( [4], [5], [10]), Burton and Townsend [13], Caldeira-Saraiva [14], Constantin [15], Chen et al. [16], Cherkas and Malysheva [17], *Čžan* [18], Gao and Zhao [19], Graef [20], Hale [21], Heidel [22], Hou and Wu [23], Huang and Yu [24], Jin [25], Jitsuro andYusuke [26], Kato [27], Liu and Huang [28], Liu [29], Luk [30], Malyseva [31], *Nápoles Valdés* [32], Omari and Zanolin [33], Pi [34], *Qian* ( [35], [36]), Sugie andd Amano [37], Tunc ( [38], [46]), Tunc and Tunc [47], Yoshizawa ( [48], [?]), Zhang ( [50], [51]), Zhou and Jiang [52], Zhou and Xiang [53] and the references cited in these works.
In this paper, instea of the mentioned last equation, we consider the scalar modified *Liénard* type equation with variable delay, $\tau(t)(>0)$ :

$$\begin{aligned} &x'' + \left\{ f_1(t,x,x')x' + f_2(x) \right\} x' + b(t)h(x) \\ &+ b(t)g(x(t-\tau(t))) = 0, \end{aligned}$$

where $t \in \mathfrak{R}^+, \mathfrak{R}^+ = [0,\infty), b : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a bounded and continuous function, $f_1 : \mathfrak{R}^+ \times \mathfrak{R} \times \mathfrak{R}$, $f_2 : \mathfrak{R} \longrightarrow \mathfrak{R}^+$, $h : \mathfrak{R} \to \mathfrak{R}, h(0) = 0, g : \mathfrak{R} \to \mathfrak{R}, g(0) = 0$, and $\tau : \mathfrak{R}^+ \to \mathfrak{R}^+$ are all continuous functions
We can write Eq. (3) in the system form,

$$\begin{aligned} x' &= y \\ y' &= -f_1(t,x,y)y^2 - f_2(x)y - b(t)h(x) \\ &\quad - b(t)g(x(t-\tau(t))). \end{aligned} \qquad (4)$$

Throughout this paper, we assume that for each $t_0 \geq 0$, $m(t_0) = inf\{s - \tau(s) : s \geq t_0\}$ and $C(t_0) = C([m(t_0),t_0],R)$ with the continuous function norm $||.||$, where
$||\psi|| = sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}$ It will cause no confusion even if we use $||\phi||$ as the supremum on $[m(t_0),\infty)$. It can be seen from Hale [21] that for a given continuous function $\phi$ and a number $y_0$, there exists a solution of the system (4) on an interval $[t_0,T)$, if the solution remains bounded, then $T = \infty$, Let $(x(t),y(t))$ denote the solution $(x(t,\phi,y_0),y(t,\phi,y_0))$.
**Definiton.**The zero solution of the system (4) is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,t_0) > 0$ such that $[\phi \in C(t_0), y_0 \in \mathbb{R}, ||\phi|| + |y_0| < \delta]$ implies that $|x(t,\phi,y_0)| + |y(t,\phi,y_0)| < \varepsilon$ for $t \geq t_0$.
We also suppose throught this paper that $t - \tau(t)$ is strictly increasing and $lim_{n\to\infty}(t - \tau(t)) = \infty$. The inverse of $t - \tau(t)$ exists, denoted by $p(t)$ and $0 \leq b(t) \leq M$, where $M$ is a positive constant.
In the next section, we give some sufficient conditions on the stability of the zero solution of Eq. (3) by the fixed point theory. To achieve our goal, we make use of the contraction mapping principle. Once the correct mapping is constructed, then, the analysis in this paper is similar to the one of Burton [8] and Pi [34]. However, our equation, Eq. (3), is different from those in Burton [8] and Pi [34] and the literature.

## 2 Main Result

Before stating the main result of this paper, we give the expression of the solution of Eq. (3) by the following lemma.
**Lemma1.** Let $\psi : [m(t_0),t_0] \to \mathbb{R}$ be a given continuous function. If *(x(t), y(t))* is the solution of the system (4) on $[t_0,T_1)$ satisfying $x(t) = \psi(t)$, $t \in [m(t_0),t_0]$, and $y(t_0) = x'(t_0)$, then $x(t)$ is the solution of the integral equation

$$\begin{aligned} x(t) &= \psi(t_0)e^{-\int_{t_0}^t \hat{D}(s)ds} \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} B(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} D(u)h(x(u))du \\ &+ \int_{t_0}^t E(t,s)h(x(s))ds \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[ \int_{t_0}^u E(u,s)h(x(s))ds \right] \hat{D}(u)du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \hat{D}(u)[x(u) - g(x(u))]du \end{aligned}$$

$$+\int_{t_0}^{t} E(t,s)g(x(s-\tau(s)))ds$$

$$+\int_{t-\tau(t)}^{t} \hat{D}(s)g(x(s))ds - e^{-\int_{t_0}^{t}\hat{D}(s)ds}$$

$$\times\int_{t_0-\tau(t_0)}^{t_0} \hat{D}(s)g(x(s))ds$$

$$-\int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u} \hat{D}(s)g(x(s))ds\right] e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du$$

$$-\int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u} \hat{D}(s)g(x(s))ds\right] e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du$$

$$\times g(x(s-\tau(s)))ds]\hat{D}(u)du. \quad (5)$$

Conversely, if the continuous function $x(t) = \psi(t)$, $t \in [m(t_0), t_0]$ , is the solution of integeral equation (5) on $[t_0, T_2]$, then $x(t), y(t))$ is the solution of the system (4) on $[t_0, T_2]$

**Proof.** Let

$$f_1(t,x(t),y(t))y(t) + f_2(x(t)) = A(t).$$

Then, Eq. (3) can be stated in the form of

$$x' = y,$$
$$y' = -A(t)y - b(t)h(x) - b(t)g(x(t-\tau(t)))$$

so that

$$y' + A(t)y + b(t)h(x) + b(t)g(x(t-\tau(t))) = 0. \quad (6)$$

Multiplying both sides of (6) by $e^{\int_{t_0}^{t} A(s)ds}$ and then integrating the obtained estimate from $t_0$ to $t$, it follows that

$$y(t) = y(t_0)e^{-\int_{t_0}^{t} A(s)ds} - \int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u))du$$
$$-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)g(x(u-\tau(u)))du$$

so that

$$x'(t) = y(t) = x'(t_0)e^{-\int_u^t A(s)ds} - \int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u))du$$
$$-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)g(x(u-\tau(u)))du. \quad (7)$$

If we choose $x'(t_0)e^{-\int_u^t A(s)ds} = B(t)$ ,them from (7) we have

$$x'(t) = B(t) - \int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u))du$$
$$-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)g(x(u-\tau(u)))du. \quad (8)$$

Let

$$e^{-\int_u^t A(s)ds} b(u) = C(t,u),$$

$$\int_{t_0}^{\infty} C(u+t-t_0,t)du = D(t) \geq 0,$$

$$\frac{D(t)}{1-\tau'(t)} = \tilde{D}(t),$$

$$\tilde{D}(p(t)) = \hat{D}(t),$$

$$\sup_{t\geq 0} D(t) \leq \sup_{t\geq 0} \hat{D}(t)$$

and

$$\int_{t_0+t-s}^{\infty} C(u+t-t_0,t)du = E(t,s) \geq 0.$$

Hence, it follows from Eq.(8)

$$x'(t) = B(t) - h(x(t))\int_{t_0}^{\infty} C(u+t-t_0,t)du$$
$$+\frac{d}{dt}\int_{t_0}^{t} E(t,s)h(x(s))ds$$
$$-g(x(t-\tau(t)))\int_{t_0}^{\infty} C(u+t-t_0,t)du$$
$$+\frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-\tau(s)))ds$$

so that

$$x'(t) = B(t) - h(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)h(x(s))ds$$
$$-g(x(t-\tau(t)))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-\tau(s)))ds$$
$$= B(t) - h(x(t))D(t) + \frac{d}{dt}\int_{t_0}^{t} E(t,s)h(x(s))ds$$
$$-\tilde{D}(p(t))g(x(t))$$
$$+\frac{d}{dt}\int_{t-\tau(t)}^{t} \tilde{D}(p(t))g(x(t))$$
$$+\frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-\tau(s)))ds.$$

Then

$$x'(t) + \hat{D}(t)x(t) = B(t) - h(x(t))D(t)$$
$$+\frac{d}{dt}\int_{t_0}^{t} E(t,s)h(x(s))ds$$
$$+\hat{D}(t)[x(t) - g(x(t))]$$
$$+\frac{d}{dt}\int_{t-\tau(t)}^{t} \hat{D}(s)g(x(s))ds$$
$$+\frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-\tau(s)))ds. \quad (9)$$

Multiply both sides of (9) by $e^{\int_{t_0}^t \hat{D}(s)ds}$ and then integrate for all $t \in [t_0, T_1]$, we have

$$\begin{aligned} x(t) &= \psi(t_0)e^{-\int_{t_0}^t \hat{D}(s)ds} \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} B(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} D(u)h(x(u))du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\frac{d}{du}\int_{t_0}^u E(u,s)h(x(s))ds\right] du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \hat{D}(u)\left[x(u) - g(x(u))\right] du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\frac{d}{du}\int_{u-\tau(u)}^u \hat{D}(u)g(x(s))ds\right] du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\frac{d}{du}\int_{t_0}^u E(u,s)g(x(s-\tau(s)))ds\right] du. \end{aligned}$$

By integrating the fourth, sixth and seventh terms on the right hand side of the last estimate, it follows that

$$\begin{aligned} x(t) =& \psi(t_0)e^{-\int_{t_0}^t \hat{D}(s)ds} + \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} B(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} D(u)h(x(u))du + \int_{t_0}^t E(t,s)h(x(s))ds \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\int_{t_0}^u E(u,s)h(x(s))ds\right] \hat{D}(u)du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \hat{D}(u)[x(u) - g(x(u))]du \\ &+ \int_{t_0}^t E(t,s)g(x(s-\tau(s)))ds \\ &+ \int_{t-\tau(t)}^t \hat{D}(s)g(x(s))ds - e^{-\int_{t_0}^t \hat{D}(s)ds} \\ &\times \int_{t_0-\tau(t_0)}^{t_0} \hat{D}(s)g(x(s))ds \\ &- \int_{t_0}^t \left[\int_{u-\tau(u)}^u \hat{D}(s)g(x(s))ds\right] e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\int_{t_0}^u E(u,s)g(x(s-\tau(s)))ds\right] \hat{D}(u)du. \end{aligned}$$

This estimate completes the proof of the first part of Lemma 1.
Conversely, we assume that the existence of a continuous function $x(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ such that it satisfies the integral equation on $t \in [t_0, T_2]$ Then, it is differentiable on $[t_0, T_2]$ Hence, it is only needed to differentiate the integral equation. When we differentiate the above integral equation, we can easily conclude the desired result.
Let $(C, ||.||)$ be the Banach space of bounded continuous functions on $[m(t_0), \infty)$ with the supremum norm $||\phi|| = sup\{|\phi(t)| : t \in [m(t_0), \infty)\}$ for $\phi \in C$. Let $p$ denote the supremum metric, $p(\phi_1, \phi_2) = ||\phi_1 - \phi_2||$, where $\phi_1, \phi_2 \in C$ Next, let $\psi : [m(t_0), t_0] \to R$ be a given continuous initial function.
Define the set $S \subset C$ by

$$S = \{\phi : [m(t_0), \infty) \to R | \phi \in C, \phi(t) = \psi(t), t \in [m(t_0), t_0]\},$$

and its subset

$$\begin{aligned} S^l = \{&\phi : [m(t_0), \infty) \to R | \phi \in C, \quad \phi(t) = \psi(t), t \in [m(t_0), t_0] \\ &and |\phi(t)| \le l, t \ge m(t_0)\}, \end{aligned}$$

where $\psi : [m(t_0), t_0] \to [-l, l]$ is a given intial function, $l$ is positive constant. Define a mapping $P : S^l \to S^l$ by

$$(P\phi)(t) = \psi(t), \ \ if \ \ t \in [m(t_0), t_0]$$

and if $t > t_0$, then

$$\begin{aligned} (P\phi)(t) =& \psi(t_0)e^{-\int_{t_0}^t \hat{D}(s)ds} + \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} B(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} D(u)h(\phi(u))du \\ &+ \int_{t_0}^t E(t,s)h(\phi(s))ds \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\int_{t_0}^u E(u,s)h(\phi(s))ds\right] \hat{D}(u)du \\ &+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \hat{D}(u)[\phi(u) - g(\phi(u))]du \\ &+ \int_{t_0}^t E(t,s)g(\phi(s-\tau(s)))ds \\ &+ \int_{t-\tau(t)}^t \hat{D}(s)g(\phi(s))ds \\ &- e^{-\int_{t_0}^t \hat{D}(s)ds} \times \int_{t_0-\tau(t_0)}^{t_0} \hat{D}(s)g(\phi(s))ds \\ &- \int_{t_0}^t \left[\int_{u-\tau(u)}^u \hat{D}(s)g(\phi(s))ds\right] e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du \\ &- \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} [\int_{t_0}^u E(u,s) \\ &\times g(\phi(s-\tau(s)))ds]\hat{D}(u)du. \end{aligned}$$

Hence, for $\phi, \varphi \in S$, it can be written that

$$|(P\phi)(t) - (P\varphi)(t)| \le \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} D(u)|h(\phi(u)) - h(\varphi(u))|du$$

$$+ \int_{t_0}^t E(t,s)|h(\phi(s)) - h(\varphi(s))|ds$$

$$+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \left[\int_{t_0}^u E(u,s)|h(\phi(s)) - h(\varphi(s))|ds\right] \hat{D}(u)du$$

$$+ \int_{t_0}^t e^{-\int_u^t \hat{D}(s)ds} \hat{D}(u)|[\phi(u) - g(\phi(u))] - [\varphi(u) - g(\varphi(u))]|du$$

$$+ \int_{t_0}^t E(t,s)|g(\phi(s-\tau(s))) - g(\varphi(s-\tau(s)))|ds$$

$$+\int_{t-\tau(t)}^{t}\hat{D}(s)|g(\phi(s))-g(\varphi(s))|ds$$

$$+\int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(u)|g(\phi(s))-g(\varphi(s))|ds\right]e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du$$

$$+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}$$

$$\times\left[\int_{t_0}^{u}E(u,s)|g(\phi(s-\tau(s)))-g(\varphi(s-\tau(s)))|ds\right]$$

$$\times\hat{D}(u)du.$$

Since $g(x)$ and $h(x)$ satisfy the Lipschitz condition let $L$ denote the common Lipschitz for $g(x)$, $x-g(x)$ and $h(x)$ Then, it follows that

$$\begin{aligned}|(P\phi)(t)-(P\varphi)(t)|\leq&\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du\\&+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\\&\times\left[\int_{t_0}^{u}E(u,s)ds\right]\hat{D}(u)du\\&+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du\\&+2\int_{t_0}^{t}E(t,s)ds\\&+\int_{t-\tau(t)}^{t}\hat{D}(s)ds\\&+\int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(u)ds\right]\\&\times e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du\\&+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\left[\int_{t_0}^{u}E(u,s)ds\right]\\&\times\hat{D}(u)du\times L||\phi-\varphi||.\end{aligned}$$

It is also clear that

$$\begin{aligned}\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du&=e^{-\int_u^t\hat{D}(s)ds}|_{t_0}^{t}\\&=1-e^{-\int_u^t\hat{D}(s)ds}\approx 1,\end{aligned}$$

for large $t$. But since $g(x)$ and $h(x)$ are non-linear, then $L$ may not be small enough. Hence, $P$ may not be a contracting mapping. We can solve this problem by giving an exponentially weight metric.

**Lemma 2.**Suppose that there exists a positive constant $l$ such that $g(x)$ and $h(x)$ satisfy the Lipschitz condition on $[-l,l]$. Then there exists a metric $d$ on $S^l$ such that
**(i)** the metric space $(S^l,d)$ is complete,
**(ii)** $P$ is a contaction mapping on $(S^l,d)$ if $P$ maps $S^l$ into itself.

**Proof.(i)** We change the supremum norm to an exponentially weighted norm $|\phi|_\mu$, which is defined on $S^l$. Let $X$ be the space of all continuous functions $\phi:[m(t_0),\infty)\to R$ such that

$$|\phi|_\mu=sup\{|\phi(t)|e^{-\mu(t)}:t\in[m(t_0),\infty)\}<\infty,$$

where $\mu=kL\int_{t_0}^{t}(\hat{D}(s)+D(s))ds$, $k$ is a positive constat, $k>8$ and $L$ is the common
Lipschitz constant for $g(x)$, $x-g(x)$ and $h(x)$. Then $(X,|.|_\mu)$ is a Banach space. This fact can be verified by using Cauchy's criterion for uniform convergence. Thus $(X,d)$ is a complete metric spaces with $d(\phi,\varphi)=|\phi-\varphi|_\mu$, where $\phi,\varphi\in X$. Under this metric the space $S^l$ is a closed subset of $X$. Thus the metric space $(S^l,d)$ is complete.
**(ii)** Let $P:S^l\to S^l$. It is clear that $D(t)\geq 0$, $\hat{D}(t)\geq 0$ and $E(t,s)\geq 0$. For $\phi,\varphi\in S^l$, then, we get

$$|(P\phi)(t)-(P\varphi)(t)|e^{-\mu(t)}\leq\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)$$

$$\times|h(\phi(u))-h(\varphi(u))|e^{-\mu(t)}du$$

$$+\int_{t_0}^{t}E(t,s)|h(\phi(s))-h(\varphi(s))|e^{-\mu(t)}ds$$

$$+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\left[\int_{t_0}^{u}E(u,s)|h(\phi(s))-h(\varphi(s))|ds\right]\hat{D}(u)e^{-\mu(t)}du$$

$$+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)|[\phi(u)-g(\phi(u))]-[\varphi(u)-g(\varphi(u))]|e^{-\mu(t)}du$$

$$+\int_{t_0}^{t}E(t,s)|g(\phi(s))-g(\varphi(s))|e^{-\mu(t)}ds$$

$$+\int_{t-\tau(t)}^{t}\hat{D}(s)|g(\phi(s)-g(\varphi(s))|e^{-\mu(t)}ds$$

$$+\int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(u)|g(\phi(s))-g(\varphi(s))|e^{-\mu(t)}ds\right]e^{-\int_u^t\hat{D}(s)ds}\hat{D}(u)du$$

$$+\int_{t_0}^{t}e^{-\int_u^t\hat{D}(s)ds}\left[\int_u^t E(u,s)|g(\phi(s-\tau(s)))-g(\varphi(s-\tau(s)))|e^{-\mu(t)}ds\right]$$

$$\times\hat{D}(u)du.$$

For $u\leq t$, since $D(t)\geq 0$, $\hat{D}(t)\geq 0$ then

$$\mu(u)-\mu(t)=-kL\int_u^t\left[\hat{D}(s)+D(s)\right]ds\leq -kL\int_u^t\hat{D}(s)ds$$

or

$$\mu(u)-\mu(t)=-kL\int_u^t\left[\hat{D}(s)+D(s)\right]ds\leq -kL\int_u^t D(s)ds.$$

Further for $s \le t$, it follows that

$$\mu(s-\tau(s))-\mu(t) = -kL\int_{s-\tau(s)}^{t}[\hat{D}(u)+D(u)]du \le -kL\int_{s}^{t}D(u)du.$$

On the other hand, since $E(t,s) \ge 0$, it is clear that

$$E(t,s) = \int_{t-s-t_0}^{\infty} C(u+s-t_0,s)du \le \int_{t_0}^{\infty} C(u+s-t_0,s)du = D(s).$$

Then, it can be seen that

$$\begin{aligned}|(P\phi)(t)-(P\varphi)(t)|e^{-\mu(t)} &\le L|\phi-\varphi|_{\mu} \times \int_{t_0}^{t} e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)e^{\mu(u)-\mu(t)}du \\ &+ \int_{t_0}^{t} E(t,s)e^{\mu(s)-\mu(t)}ds \\ &+ \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)e^{\mu(s)-\mu(t)}ds\right]e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du \\ &+ \int_{t_0}^{t} e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)e^{\mu(u)-\mu(t)}du \\ &+ \int_{t_0}^{t} E(t,s)e^{\mu(s)-\mu(t)}ds + \int_{t-\tau(t)}^{t}\hat{D}(s)e^{\mu(s)-\mu(t)}ds \\ &+ \int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(s)e^{\mu(s)-\mu(t)}ds\right]e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du \\ &+ \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)e^{\mu(s-\tau(s))-\mu(t))}ds\right]e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du.\end{aligned}$$

By an esay calculation, we can obtain

$$|(P\phi)(t)-(P\varphi)(t)|e^{-\mu(t)} \le \{\frac{2}{kL+1}+\frac{3}{kL}+\frac{3}{kL} \times \int_{t_0}^{t} e^{-\int_u^t \hat{D}(s)ds}\hat{D}(u)du\} \times L|\phi-\varphi|_{\mu}.$$

Thus, we have

$$|(P\phi)(t)-(P\varphi)(t)|e^{-\mu(t)} \le \{\frac{8}{k}L|\phi-\varphi|_{\mu}, t > t_0.$$

For

$$t \in [m(t_0),t_0], (P\phi)(t) = (P\varphi)(t) = \theta(t).$$

Hence,

$$d(P\phi,P\varphi) \le \{\frac{8}{k}d(\phi-\varphi), (k > 8).$$

Therefore, $P$ is a contraction mapping on $(S^l,d)$.

**Theorem.** We assume that the following assumptions hold:

**(i)** There exists a positive constant $l$ such that $g$ and $h$ satisfy the Lipschitz condition on $[-l,l]$ and $g$ and $h$ are odd and they are strictly increasing on $[-l,l]$, and $x-g(x)$ are non-decreasing on $[-l,l]$.

**(ii)** There exists an $\alpha \in (0,1)$ and a continuous function $a(t) : [0,\infty) \to [0,\infty)$ such that

$$a(t) : [0,\infty) \to [0,\infty) \; such \;\; that$$

$$f_1(t,x,y)y + f_2(x) \ge a(t)$$

for $t \ge 0$, $x \in R$, $y \in R$, and

$$\begin{aligned}&g(l)\{2\sup_{t\ge 0}\int_{t}^{p(t)}\int_{0}^{\infty} e^{-\int_s^{w+s}a(v)dv}b(s)dwds \\ &+ 2\sup_{t\ge 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{w+s}a(v)dv}b(s)dwds\} \\ &+ 2h(l)\sup_{t\ge 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_s^{w+s}a(v)dv}b(s)dwds \\ &\le \alpha g(l) + (1-\alpha)h(l).\end{aligned}$$

**(iii)**There exist constant $a_0 > 0$ and $Q > 0$ such that for each $t \geqslant 0$ , if $J \geqslant Q$, then

$$\int_{t}^{t+J} a(v)dv \geqslant a_0 J.$$

Then there exists $\delta \in (0,l)$ such that for each initial function $\psi : [m(t_0),t_0] \to \mathbb{R}$ and $x'(t_0)$ satisfying $|x'(t_0)| + \|\psi\| \le \delta$, and there is a unique continuous function $x : [m(t_0),\infty) \to \mathbb{R}$ satisfying $x(t) = \psi(t)$, $t \in [m(t_0),t_0]$, which is a solution of Eq. (3) on $[t_0,\infty)$. Moreover, the zero solution of Eq. (3) is stable.

**Proof.**Choosing $\psi : [m(t_0),t_0] \to \mathbb{R}$ and $x'(t_0)$ such that

$$\left(Q+\frac{e^{-a_0 Q}}{a_0}\right)|x'(t_0)| + \delta + 2h(l) + g(\delta)\int_{t_0-\tau(t_0)}^{t_0}\hat{D}(s)ds \le (1-\alpha)g(l) + \alpha h(l).$$

By noting the assumption $g(0) = 0$ of the theorem, it is clear that $g(l) \le l$. Since $g(x)$ satisfies Lipschitz condition on $[-l,l]$, and $g(x)$ is continuous function on $[-l,l]$,there

exists a constant $\delta$ such that $\delta < l$.
Thus, from the expression for $(P\phi)(t)$, it follows that

$$
\begin{aligned}
|(P\phi)(t)| \le\ & \delta + \int_{t_0}^{t} e^{-\int_{t_0}^{t}\hat{D}(s)ds}|x'(t_0)|e^{-\int_{t_0}^{u}A(s)ds}du \\
& + \int_{t_0}^{t} e^{-\int_{u}^{t}\hat{D}(s)ds}\hat{D}(u)h(l)du + \int_{t_0}^{t} E(t,s)h(l)ds \\
& + \int_{t_0}^{t} e^{-\int_{u}^{t}\hat{D}(s)ds}\left[\int_{t_0}^{u} E(u,s)h(l)ds\right]\hat{D}(u)du \\
& + \int_{t_0}^{t} e^{-\int_{u}^{t}\hat{D}(s)ds}\hat{D}(u)(l-g(l))du \\
& + \int_{t_0}^{t} E(t,s)g(l)ds \\
& + \int_{t-\tau(t)}^{t}\hat{D}(s)g(l)ds + \int_{t_0-\tau(t_0)}^{t_0}\hat{D}(s)g(\delta)ds \\
& + \int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(u)g(l)ds\right]e^{-\int_{u}^{t}\hat{D}(s)ds}\hat{D}(u)du \\
& + \int_{t_0}^{t} e^{-\int_{u}^{t}\hat{D}(s)ds}\left[\int_{t_0}^{u} E(u,s)g(l)ds\right]\hat{D}(u)du.
\end{aligned}
$$

Further, subject to the assumptions of the theorem, it follows that

$$
\begin{aligned}
\int_{t_0}^{t} E(t,s)ds &= \int_{t_0}^{t}\int_{t_0+t-s}^{\infty} e^{-\int_{s}^{u+s-t_0}A(v)dv}b(s)duds \\
&= \int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{u+s}A(v)dv}b(s)duds \\
&\le \sup_{t\geqslant 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{u+s}A(v)dv}b(s)duds,
\end{aligned}
$$

$$
\begin{aligned}
\int_{t-\tau(t)}^{t}\hat{D}(s)ds &= \int_{t-\tau(t)}^{t}\hat{D}(p(s))ds \\
&= \int_{t-\tau(t)}^{t}\frac{\hat{D}(p(s))}{1-\tau'(t)}ds \\
&= \int_{t}^{p(t)} D(s)ds = \int_{t}^{p(t)}\int_{0}^{\infty} e^{-\int_{s}^{w+s}A(v)dv} \\
&\times b(s)dwds \\
&\le \sup_{t\geqslant 0}\int_{t}^{p(t)}\int_{0}^{\infty} e^{-\int_{s}^{w+s}A(v)dv}b(s)dwds,
\end{aligned}
$$

$$
\begin{aligned}
& h(l)\int_{t_0}^{t} E(t,s)ds + \int_{t_0}^{t} e^{-\int_{u}^{t}\hat{D}(s)ds} \\
& \quad\times\left[\int_{t_0}^{u} E(u,s)h(l)ds\right]\hat{D}(u)du \\
& \quad + g(l)\int_{t_0}^{t} E(t,s)ds + \int_{t_0}^{t}\left[\int_{u-\tau(u)}^{u}\hat{D}(s)g(l)ds\right] \\
& \quad\times e^{-\int_{u}^{t}\hat{D}(s)ds}\hat{D}(u)du \\
& \quad + \int_{t-\tau(t)}^{t}\hat{D}(s)g(l)ds + \int_{t_0}^{t}\left[\int_{t_0}^{u} E(u,s)g(l)ds\right] \\
& \quad\times e^{-\int_{u}^{t}\hat{D}(s)ds}\hat{D}(u)du
\end{aligned}
$$

$$
\begin{aligned}
& \le g(l)\{2\sup_{t\geqslant 0}\int_{t}^{p(t)}\int_{0}^{\infty} e^{-\int_{s}^{w+s}A(v)dv}b(s)dwds \\
& + 2\sup_{t\geqslant 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{w+s}A(v)dv}b(s)dwds\} \\
& + 2h(l)\sup_{t\geqslant 0}\int_{0}^{t}\int_{t-s}^{\infty} e^{-\int_{s}^{w+s}A(v)dv}b(s)dwds \\
& \le \alpha g(l) + (1-\alpha)h(l).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|(P\phi)(t)| \le\ & \delta + g(\delta)\int_{t_0-\tau(t_0)}^{t_0}\hat{D}(s)ds + (l-g(l)) \\
& + \alpha g(l) + (1-\alpha)h(l) \\
& + h(l) + \int_{t_0}^{t} e^{-\int_{t_0}^{t}\hat{D}(s)ds}|x'(t_0)|e^{-\int_{t_0}^{u}A(s)ds}du \\
\le\ & \delta + g(\delta)\int_{t_0-\tau(t_0)}^{t_0}\hat{D}(s)ds + (l-g(l)) + h(l) \\
& + \alpha g(l) + (1-\alpha)h(l) \\
& + \int_{t_0}^{t}|x'(t_0)|e^{-\int_{t_0}^{u}A(s)ds}du.
\end{aligned}
$$

Using condition (iii) of the theorem, we get

$$
\begin{aligned}
\int_{t_0}^{t} e^{-\int_{t_0}^{u}A(s)ds}du &= \int_{t_0}^{t_0+Q} e^{-\int_{t_0}^{u}A(s)ds}du \\
& + \int_{t_0+Q}^{t} e^{-\int_{t_0}^{u}A(s)ds}du \\
& \le \int_{t_0}^{t_0+Q}du + \int_{t_0+Q}^{t} e^{-a_0(u-t_0)}du \le Q \\
& + \frac{e^{-a_0Q}}{a_0}.
\end{aligned}
$$

Hence,

$$|(P\phi)(t)| \leq \delta + g(\delta)\int_{t_0-\tau(t_0)}^{t_0} \hat{D}(s) + (l-g(l)) + h(l) + \alpha g(l) + (1-\alpha)h(l) + Q|x'(t_0)| + \frac{e^{-a_0Q}}{a_0}|x'(t_0)|.$$

Since

$$\left(Q+\frac{e^{-a_0Q}}{a_0}\right)|x'(t_0)| + \delta + 2h(l) + g(\delta)\int_{t_0-\tau(t_0)}^{t_0} \hat{D}(s)ds \leq (1-\alpha)g(l) + \alpha h(l),$$

then it follows that

$$|(P\phi)(t)| \leq (\alpha-1)g(l) + l + (1-\alpha)g(l) + \alpha h(l) - \alpha h(l)$$

so that

$$|(P\phi)(t)| \leq l.$$

Thus, it is clear that if $t \in [m(t_0), t_0]$, then it can be seen that $(P_2\phi)(t) = \psi(t)$, $|(P_2\phi)(t)| \leq l$, $t \in [m(t_0), \infty)$. Therefore, $P\phi : S^l \to S^l$. Since $P$ is contraction mapping $P$ has unique fixed point $x(l)$ such that $|x(t)| \leq l$.
From Eq. (7),

$$y(t) = x^{'}(t_0)e^{-\int_{t_0}^t A(s)ds} - \int_{t_0}^t e^{-\int_u^t A(s)ds} b(u)h(x(u))du - \int_{t_0}^t e^{-\int_u^t A(s)ds} b(u)g(x(u-\tau(u)))du,$$

it can be obtained that

$$\begin{aligned}
|y(t)| &\leq |x^{'}(t_0)| + M\int_{t_0}^t e^{-\int_u^t A(s)ds}|h(x(u))|du \\
&+ M\int_{t_0}^t e^{-\int_u^t A(s)ds}|g(x(u-\tau(u)))|du \\
&\leq \delta + M\int_{t_0}^t e^{-\int_u^t A(s)ds}|h(x(u))|du \\
&+ M\int_{t_0}^t e^{-\int_u^t A(s)ds}|g(x(u-\tau(u)))|du \\
&\leq \delta + M\int_{t_0}^t e^{-\int_u^t A(s)ds}|(x(u))|du \\
&+ M\int_{t_0}^t e^{-\int_u^t A(s)ds}|x(u-\tau(u))|du \\
&\leq l + lM\int_{t_0}^t e^{-\int_u^t A(s)ds}du \\
&+ lM\int_{t_0}^t e^{-\int_u^t A(s)ds}du \\
&\leq l\left[+2M\left(Q+\frac{e^{-a_0Q}}{a_0}\right)\right].
\end{aligned}$$

We can conclude that

$$|x(t)| + |y(t)| < 2l\left[1 + M(Q + \frac{e^{-a_0Q}}{a_0})\right]$$

This completes the proof of the theorem.

## 3 Conclusion

A functional *Liénard* type equation with variable delay is considered. The stability of the zero solution of the equation is discussed. In proving our main result, we use the fixed points theory by defining an exponentially weight metric. Our result extends and improves some recent results in the literature.

## References


[1] Ahmad, S.; Rama Mohana Rao, M., Theory of Ordinary differential Equations. With Applications in Biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, (1999).

[2] Antosiewicz, H. A., On non-linear differential equations of the second order with integrable forcing term. J. London Math. Soc., **30**, 64-67 (1955).

[3] Becker, L. C.; Burton, T. A., Stability, fixed points and inverses f delays. Proc. Roy. Soc. Edinburgh Sect. A, **136**, 245-275.

[4] Burton, T. A., On the equation $x^{''} + f(x)h(x^{'})x^{'} + g(x) = e(t)$ Ann. Mat. Pura Appl., **85**, 277-285 (1970).

[5] Burton, T. A., Stability and periodic solutions of ordinary and functional differential equations, Academic Press, Orlando, (1985).

[6] Burton, T. A., Stability by fixed point theory or Liapunov theory: A comparison. Fixed Point Theory, **4**, 15-32 (2003).

[7] Burton, T. A., Fixed points and stability of a nonconvolution equation. Proc. Amer. Math. Soc., **132**, 3679-3687 (2004).

[8] Burton, T. A., Fixed points, stability, and exact linearization. Nonlinear Anal., **61**, 857-870 (2005).

[9] Burton, T. A.,The case for stability by fixed point theory. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., suppl., **13B**, 253-263 (2006).

[10] Burton, T. A., Stability by fixed point theory for functional differential equations. Dover Publications, Inc., Mineola, NY, (2006).

[11] Burton, T. A.; Furumochi, T., Fixed points and problems in stability theory for ordinary and functional differential equations. Dynam. Systems Appl., **10**, 89-116 (2001).

[12] Burton, T. A.; Hering, R. H., Liapunov theory for functional-differential equations. 20th Midwest ODE Meeting (Iowa City, IA, 1991). Rocky Mountain J. Math., **24**, 3-17 (1994).

[13] Burton, T. A.; Townsend, C. G., On the generalized *Liénard* equation with forcing function. J. Differential Equations, **4**, 620-633 (1968).

[14] Caldeira-Saraiva, F., The boundedness of solutions of a *Liénard* equation arising in the theory of ship rolling. IMA J. Appl. Math., **36**, 129-139 (1986).

[15] Constantin, A., A note on a second-order nonlinear differential system. Glasg. Math. J., **42**, 195-199 (2000).
[16] Chen, T.; Liu, W.; Hu, Z., New results on the existence of periodic solutions for a higher-order *Liénard* type p - Laplacian differential equation. Math. Methods Appl. Sci., **34**, 2189-2196 (2011).
[17] Cherkas, L. A.; Malysheva, O. N., On the estimation of the number of limit cycles in *Liénard* systems with a small parameter. Differ. Uravn., **47**, 225-230 (2011); translation in Differ. Equ., **47**, 224-230 (2011).
[18] *Čžan*, Pan-gin, On stability with arbitrary initial perturbations of the solutions of a system of two differential equations. Acta Math. Sinica, **9**, 442-445 (1959).
[19] Gao, S. Z.; Zhao, L. Q., Global asymptotic stability of generalized *Liénard* equation. Chinese Sci. Bull., **40**, 105-109 (1995).
[20] Graef, John R., On the generalized Linard equation with negative damping. J. Differential Equations, **12**, 34-62 (1972).
[21] Hale, J., Theory of functional differential equations. Second edition. Applied Mathematical Sciences, Springer-Verlag, New York-Heidelberg, **3**, (1977).
[22] Heidel, J. W., Global asymptotic stability of a generalized *Liénard* equation. SIAM J. Appl. Math., **19**, 629-636 (1970).
[23] Hou, X.; Wu, Z., Existence and uniqueness of periodic solutions for a kind of *Liénard* equation with multiple deviating arguments. J. Appl. Math. Comput., **38**, 181-193 (2012).
[24] Huang, L. H.; Yu, J. S., On boundedness of solutions of generalized $Liénard's$ system and its application. Ann. Differential Equations, **9**, 311-318 (1993).
[25] Jin, Z., Boundedness and convergence of solutions of a second-order nonlinear differential system. J. Math. Anal. Appl., **256**, 360-374 (2001).
[26] Jitsuro, S.; Yusuke, A., Global asymptotic stability of non-autonomous systems of *Liénard* type. J. Math. Anal. Appl., **289**, 673-690 (2004).
[27] Kato, J., On a boundedness condition for solutions of a generalized *Liénard* equation. J. Differential Equations, **65**, (1986).
[28] Liu, B.; Huang, L., Boundedness of solutions for a class of retarded *Liénard* equation. J. Math. Anal. Appl., **286**, (2003).
[29] Liu, Z. R., Conditions for the global stability of the *Liénard* equation. Acta Math. Sinica, **38**, 614-620 (1995).
[30] Luk, W. S., Some results concerning the boundedness of solutions of *Liénard* equations with delay. SIAM J. Appl. Math., **30**, 768-774 (1976).
[31] Malyseva, I. A., Boundedness of solutions of a *Liénard* differential equation. Differetial'niye Uravneniya, **15**, 1420-1426 (1979).
[32] Npoles *Valdés*, J. E., Boundedness and global asymptotic stability of the forced *Liénard* equation. Rev. Un. Mat. Argentina, **41**, 47-59 (2001).
[33] Omari, P.; Zanolin, F., On the existence of periodic solutions of forced *Liénard* differential equations. Nonlinear Anal., **11**, 275-284 (1987).
[34] Pi, D., Study the stability of solutions of functional differential equations via fixed points. Nonlinear Anal., **74**, 639-651 (2011).
[35] Qian, C. X., Boundedness and asymptotic behaviour of solutions of a second-order nonlinear system. Bull. London Math. Soc., **24**, 281-288 (1992).
[36] Qian, C. X., On global asymptotic stability of second order nonlinear differential systems. Nonlinear Anal., **22**, 823-833 (1994).
[37] Sugie, J.; Amano, Y., Global asymptotic stability of non-autonomous systems of *Liénard* type. J. Math. Anal. Appl., **289**, 673-690 (2004).
[38] Tunc, C., Some new stability and boundedness results of solutions of *Liénard* type equations with deviating argument. Nonlinear Anal. Hybrid Syst., **4**, 85-91 (2010).
[39] Tunc, C., Boundedness analysis for certain two-dimensional differential systems via a Lyapunov approach. Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **53**, 61-68 (2010).
[40] Tunc, C., A note on boundedness of solutions to a class of non-autonomous differential equations of second order. Appl. Anal. Discrete Math., **4**, 361-372 (2010).
[41] Tunc, C., Boundedness results for solutions of certain nonlinear differential equations of second order. J. Indones. Math. Soc., **16**, 110-113 (2010).
[42] Tunc, C., New stability and boundedness results of solutions of *Liénard* type equations with multiple deviating arguments. J. Contemp. Math. Anal., **45**, 47-56 (2010).
[43] Tunc, C., On the stability and boundedness of solutions of a class of nonautonomous differential equations of second order with multiple deviating arguments. Afr. Mat., **23**, 249-259 (2012) .
[44] Tunc, C., Stability and uniform boundedness results for non-autonomous *Liénard* -type equations with a variable deviating argument. Acta Math. Vietnam, **37**, 311-326 (2012).
[45] Tunc, C., Stability to vector *Liénard* equation with constant deviating argument. Nonlinear Dynam., **73**, 1245-1251 (2013).
[46] T Allahviranloo, S Salahshour, Bounded and symmetric solutions of fully fuzzy linear systems in dual form, Procedia Computer Science, Elsevier, (2011).
[47] Tunc, C.; Tunc, E., On the asymptotic behavior of solutions of certain second-order differential equations. J. Franklin Inst., **344**, 391-398 (2007).
[48] Yoshizawa, T., Asymptotic behavior of solutions of a system of differential equations. Contributions to Differential Equations, **1**, 371-387 (1963).
[49] Yoshizawa, T., Stability theory by Liapunov's second method. Publications of the Mathematical Society of Japan, no. 9, The Mathematical Society of Japan, Tokyo, (1966).
[50] Zhang, B., Fixed points and stability in differential equations with variable delays Nonlinear Anal., **63**, e233-e242 (2005).
[51] Zhang, Y., Eventually vanished solutions of a forced *Liénard* system. Appl. Math. Mech., **30**, 1335-1344 (2009).
[52] Zhou, X.; Jiang, W., Stability and boundedness of retarded *Liénard* -type equation. Chinese Quart. J. Math., **18**, 7-12 (2003).
[53] Zhou, J.; Xiang, L., On the stability and boundedness of solutions for the retarded *Liénard* -type equation. Ann. Differential Equations, **15**, 460-465 (1999).

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