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Generalized Convexity and Integral Inequalities

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Abstract: In this paper, we consider a very useful and significant class of convex sets and convex functions that is relative convex sets and relative convex functions which was introduced and studied by Noor [20]. Several new inequalities of Hermite-Hadamard type for relative convex functions are established using different approaches. We also introduce relative h-convex functions and is shown that relative h-convex functions include Noor relative convex functions as special cases. Results obtained in this paper may inspire future research in convex analysis and related optimization fields.

Keywords: convex sets, relative convex sets, convex functions, relative convex functions, relative h-convex functions, fractional integrals, Hermite-Hadamard inequality
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1 Introduction

It is well known [29] that modern analysis directly or indirectly involve the applications of convexity. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions, see [1,2,5,6,7,8,9,17,18,19,20,21,22,24,25,26,33,35]. Inspired and motivated by research going in this dynamic and fascinating field, Noor [20] introduced and studied a new class of convex set and convex function with respect to an arbitrary function; which is called as relative convex set and relative convex function. These relative convex sets and relative convex functions are nonconvex. It is worth mentioning that these relative convex sets and relative convex functions are quite different than those of relative convex sets and relative convex functions considered by Youness [35]. Cristescu et al. [4] has explored the applications of relative convex sets of Noor in the fields of transportation, colored image processing and computer aided design; see example 1 and example 2. Noor [20] has shown that optimality conditions of the differentiable relative convex functions can be characterized by a class of variational inequality. This aspect has motivated Noor [20] to introduce a new class of variational inequality, which is called general variational inequality. For the numerical methods and other aspects of general variational inequalities, see [20] and the references theirin.

The concept of convexity and its variant forms have played a fundamental role in the development of various fields. Hermite (1883) and Hadamard (1896) independently have shown that the convex functions are related to an integral inequality, this inequality is known as Hermite-Hadamard inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a,b \in I$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This double inequality is known as classical Hermite-Hadamard inequality. For different generalizations and extensions of Hermite-Hadamard inequalities interested readers are referred to [1,6,7,8,10,11,12,14,17,18,19,22,24,25,26,28,29,30,31,32,34].

In this paper, we consider the Noor's relative convex functions. We derive several new Hermite-Hadamard inequality for relative convex functions and for its variant forms. Finally we also introduce the concept of relative h-convex function. This class also generalizes the class of relative convex functions. The new ideas and techniques used in this paper may motivate the interested readers to

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explore the applications of relative convex functions in various branches of pure and applied sciences.

2 Basic concepts and results

In this section, we recall the definition of relative convex sets and relative convex functions respectively. We discuss some previously known results for relative convex functions.

Definition 1([20]). *Let K_g be any set in H. The set K_g is said to be relative convex with respect to an arbitrary function $g : H \to H$ such that*

$$(1-t)u + tg(v) \in K_g, \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1].$$

Note that every convex set is relative convex, but the converse is not true, see [20]. If $g = I$, the identity function, then the definition of relative convex set recaptures the definition of classical convex set.
We remark that this type of relative convex sets are called Noor convex sets [3,4] and are distinctly different than that of Youness's relative convex set [35].

Now we give some examples of relative convex sets, which shows the significance of relative convex sets. These examples are mainly due to Cristescu et al [4].

Example 1([4]). One of the most important goals of the International Union of Railways (U.I.C.) is to enable the railway companies to measure the impact of their activity on the environment (see the U.I.C. guide). The environment indicators in the domain of railway transport defined under U.I.C. and presented into the above mentioned guide include the level of noise, which should be, in normal conditions, in the interval $[0,50]db(A)$. The actual noise level produced by wagons is $[125,130]db(A)$. The noise level around the railway stations located in towns is represented by the set $[0,50] \cup [125,130]$. By relocating the railway transport system outside the towns, the resulted level of noise becomes $[0,50]$. Let us denote by $g : \mathbb{R} \to \mathbb{R}$ the function defined by

$$g(x) = \begin{cases} x & if\, x \in [0,50] \\ 0 & otherwise \end{cases} \tag{1}$$

which is the function describing the efforts of kipping the normal level of sound, which works under this project. Then the set $[0,50] \cup [125,130]$ is g-convex.

Other examples are easy to find in the domain of image processing, in which a transformation of the real plane $\mathbb{R}^2$ into a set of grid-points, $\mathbb{Z}^2$ for example, is necessary. In order to present this type of examples we need to choose a transformation of the space, which performs the space digitization. The general definition of this kind of transformations is

Definition 2. *A function $f : \mathbb{R}^n \to \mathbb{Z}^n$ is said to be a method of digitization of $\mathbb{R}^n$ into $\mathbb{Z}^n$ if $f(x) = x$ whenever $x \in \mathbb{Z}^n$.*

In what follows we assume that $n = 2$ and $g = E = f$ is the digitization method used in black and white picture processing by Rosenfeld (1969) and in colored image processing by Chassery (1978). It is defined by $f : \mathbb{R}^2 \to \mathbb{Z}^2$, $f(x,y) = (i,j), i \in \mathbb{Z}, j \in \mathbb{Z}$ *whenever* $(x,y) \in [i-1/2, i+1/2) \times [j-1/2, j+1/2)$.

Example 2([4]). The set $A := B \cup < (i,j),(i+m,j) >$, whenever $i \in \mathbb{Z}, j \in \mathbb{Z}, m \in \mathbb{Z}$ and $B \subseteq [i-1/2, i+m+1/2) \times [j-1/2, j+1/2)$ is an union of triangles having one side $< (i,j),(i+m,j) >$ is relative convex. Indeed, considering two points $x, y \in A$, there are two numbers $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$ such that $x \in [i+k-1/2, i+k+1/2) \times [j-1/2, j+1/2)$ and $y \in [i+l-1/2, i+l+1/2) \times [j-1/2, j+1/2)$. Therefore $g(y) = (i+l, j)$. Then for any $t \in [0,1]$, there is the integer s between k and l such that $tx + (1-t)g(y) \in [i+s-1/2, i+s+1/2) \times [j-1/2, j+1/2) \subseteq A$ since B is an union of triangles having one side $< (i,j),(i+m,j) >$. It means that A is relative convex. In the same manner one can take vertical columns of pixels and obtain relative convex sets.

Definition 3([20]). *A function $f : K_g \to H$ is said to be relative convex, if there exists an arbitrary function $g : H \to H$ such that*

$$f((1-t)u + tg(v)) \le (1-t)f(u) + tf(g(v)), \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1]. \tag{2}$$

Clearly every convex function is relative convex, but the converse is not true. For the properties of relative convex functions, see [18,19,20].
Note that for $t = \frac{1}{2}$ in (2), we have the definition of Jensen's relative convex function. That is

$$f\left(\frac{a+g(b)}{2}\right) \le \frac{f(a)+f(g(b))}{2}.$$

Definition 4([19]). *The function $f : K_g \to (0,\infty)$ is said to be relative logarithmic convex, if there exists an arbitrary function $g : H \to H$ such that*

$$f((1-t)u + tg(v)) \le (f(u))^{1-t}(f(g(v)))^t, \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1]. \tag{3}$$

This implies that

$$\log f((1-t)u + tg(v)) \le (1-t)\log f(u) + t\log f(g(v)), \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1].$$

Definition 5([19]). *The function $f : K_g \to H$ is said to be relative quasi convex, if there exists an arbitrary function $g : H \to H$ such that*

$$f((1-t)u + tg(v)) \le \max\{f(u), f(g(v))\}, \quad \forall u, v \in H : u, g(v) \in K_g, t \in [0,1]. \tag{4}$$

Remark. From inequalities (2), (3) and (4), it follows that

$$\begin{aligned} f((1-t)u+tg(v)) &\leq (f(u))^{1-t}(f(g(v)))^t \\ &\leq (1-t)f(u)+tf(g(v)) \\ &\leq \max\{f(u),f(g(v))\}. \end{aligned}$$

This shows that

$$\begin{array}{r} \text{relative}\log\text{convex function} \Rightarrow \text{relative convex functions} \\ \Downarrow \\ \text{relative quasi convex functions,} \end{array}$$

but the converse is not true.

Definition 6([18]). *Let $K_g = [a, g(b)] \subseteq \mathbb{R}$ be the interval and $g : H \to H$ be any arbitrary function. Then f is relative convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & g(b) \\ f(a) & f(x) & f(g(b)) \end{vmatrix} \geq 0; \quad a \leq x \leq g(b).$$

From the above definition one can obtain the following equivalent forms:

1. f is relative convex function.
2. $f(x) \leq f(a) + \frac{f(g(b))-f(a)}{g(b)-a}(x-a)$.
3. $\frac{f(x)-f(a)}{x-a} \leq \frac{f(g(b))-f(a)}{g(b)-a} \leq \frac{f(g(b))-f(x)}{g(b)-x}$.
4. $\frac{f(a)}{(x-a)(g(b)-a)} + \frac{f(x)}{(g(b)-x)(x-a)} + \frac{f(g(b))}{(g(b)-a)(g(b)-x)} \geq 0$.
5. $(g(b)-x)f(a) + (g(b)-a)f(x) + (x-a)f(g(b)) \geq 0$.

Now we recall the definition of similarly ordered functions.

Definition 7([29]). *Two functions f and w are said to be similarly ordered (f is w-monotone), if*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0, \quad \forall x, y \in \mathbb{R}.$$

Theorem 1. *The product of two relative convex functions f and w is again relative convex provided if f and w are similarly ordered.*

Proof. The proof is obvious. $\square$

We now discuss the optimality condition of the differentiable relative convex functions on the relative convex sets. This result is due to Noor [20]. We include it for the sake of completeness.

Theorem 2([20]). *Let K_g be a relative convex set in H and let f be a differentiable relative convex function. Then, $u \in K_g$ is the minimum of f on K_g, if and only if, $u \in K_g$, satisfies*

$$\langle f'(u), g(v) - u \rangle \geq 0, \qquad \forall v \in H : g(v) \in K_g,$$

where $f'(u)$ is the differential of f at $u \in K_g$. Inequality of the above type is called the general variational inequality.

Now we define generalized Riemann-Liouville fractional integrals, with respect to an arbitrary function g.

Definition 8. *Left-sided and right-sided generalized Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$ are respectively defined as*

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad 0 \leq a < x \leq g(b), \tag{5}$$

and

$$J^{\alpha}_{g(b)^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{g(b)} (t-x)^{\alpha-1} f(t)dt, \quad 0 \leq a < x \leq g(b), \tag{6}$$

where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $x > 0$ is the gamma function.

It is clear from (5) and (6) that $J^{\alpha}_{a^+} f(a) = 0$, and $J^{\alpha}_{g(b)^-} f(g(b)) = 0$. For $g = I$, where I is the identity function then above definition of generalized Riemann-Liouville fractional integrals reduces to the definition of classical Riemann-Liouville fractional integrals, see [13,15,16].

3 Hermite-Hadamard inequalities

In this section we derive some Hermite-Hadamard type of integral inequalities for relative convex functions. From now onward Throughout this section $K_g = [a, g(b)]$ be the interval unless otherwise specified.

Theorem 3([19]). *Let $f : K_g \to \mathbb{R}$ be a relative convex function. Then, we have*

$$f\left(\frac{a+g(b)}{2}\right) \leq \frac{1}{g(b)-a} \int_a^{g(b)} f(x)dx \leq \frac{f(a)+f(g(b))}{2}. \tag{7}$$

The inequality (7) is the extension of classical Hermite-Hadamard inequality for relative convex functions. For $g = I$ where I is identity function, inequality (7) reduces to classical Hermite-Hadamard inequality.

Theorem 4. *Let $f, w : K_g \to \mathbb{R}$ be relative convex functions. Then for all $t \in [0,1]$, we have*

$$\begin{aligned} &2f\left(\frac{a+g(b)}{2}\right) w\left(\frac{a+g(b)}{2}\right) \\ &\quad - \left[\frac{1}{6}M(f,w;a,g(b)) + \frac{1}{2}N(f,w;a,g(b))\right] \\ &\leq \frac{1}{g(b)-a} \int_a^{g(b)} f(x)w(x)dx \\ &\leq \frac{1}{3}M(f,w;a,g(b)) + \frac{1}{6}N(f,w;a,g(b)), \end{aligned}$$

where

$$M(f,w;a,g(b)) = f(a)w(a) + f(g(b))w(g(b)), \tag{8}$$

and

$$N(f,w;a,g(b)) = f(a)w(g(b)) + f(g(b))w(a). \tag{9}$$

Proof. Let

$$\begin{aligned}
& f\left(\frac{a+g(b)}{2}\right) w\left(\frac{a+g(b)}{2}\right) \\
&= f\left(\frac{ta+(1-t)g(b)+(1-t)a+tg(b)}{2}\right) \\
&\qquad \times w\left(\frac{ta+(1-t)g(b)+(1-t)a+tg(b)}{2}\right) \\
&\leq \frac{1}{2}[f(ta+(1-t)g(b))+f((1-t)a+tg(b))] \\
&\quad \times \frac{1}{2}[w(ta+(1-t)g(b))+w((1-t)a+tg(b))] \\
&= \frac{1}{4}[f(ta+(1-t)g(b))w(ta+(1-t)g(b)) \\
&\quad +f((1-t)a+tg(b))w((1-t)a+tg(b))] \\
&\quad +\frac{1}{4}[f(ta+(1-t)g(b))w((1-t)a+tg(b)) \\
&\quad +f((1-t)a+tg(b))w(ta+(1-t)g(b))] \\
&\leq \frac{1}{4}[f(ta+(1-t)g(b))w(ta+(1-t)g(b)) \\
&\quad +f((1-t)a+tg(b))w((1-t)a+tg(b))] \\
&\quad +\frac{1}{4}[tf(a)+(1-t)f(g(b)))((1-t)w(a)+tw(g(b))) \\
&\quad +((1-t)f(a)+tf(g(b)))(tw(a)+(1-t)w(g(b)))] \\
&= \frac{1}{4}[f(ta+(1-t)g(b))w(ta+(1-t)g(b)) \\
&\quad +f((1-t)a+tg(b))w((1-t)a+tg(b))] \\
&\quad +\frac{1}{4}[2t(1-t)(f(a)w(a)+f(g(b))w(g(b))) \\
&\quad +(t^2+(1-t)^2)(f(g(b))w(a)+f(a)w(g(b)))\Big].
\end{aligned}$$

Integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}
& f\left(\frac{a+g(b)}{2}\right) w\left(\frac{a+g(b)}{2}\right) \\
&\leq \frac{1}{4}\left[\frac{2}{g(b)-a}\int_a^{g(b)} f(x)w(x)dx\right] \\
&\qquad +\frac{1}{2}\left[\frac{1}{6}M(f,w;a,g(b))+\frac{1}{3}N(f,w;a,g(b))\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
& 2f\left(\frac{a+g(b)}{2}\right) w\left(\frac{a+g(b)}{2}\right) \\
&\qquad -\left[\frac{1}{6}M(f,w;a,g(b))+\frac{1}{3}N(f,w;a,g(b))\right] \\
&\leq \frac{1}{g(b)-a}\int_a^{g(b)} f(x)w(x)dx.
\end{aligned}$$

Also

$$\begin{aligned}
& f(ta+(1-t)g(b))w(ta+(1-t)g(b)) \\
&\leq [tf(a)+(1-t)f(g(b))][tw(a)+(1-t)w(g(b))].
\end{aligned}$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
& \frac{1}{g(b)-a}\int_a^{g(b)} f(x)w(x)dx \\
&\leq \frac{1}{3}M(f,w;a,g(b))+\frac{1}{6}N(f,w;a,g(b)).
\end{aligned}$$

This completes the proof. □

Theorem 5. *Let $f,w : K_g \to \mathbb{R}$ be similarly ordered and relative convex functions on K_g. Then for all $t \in [0,1]$, we have*

$$\begin{aligned}
& 2f\left(\frac{a+g(b)}{2}\right) w\left(\frac{a+g(b)}{2}\right) - \frac{1}{4}M(f,w;a,g(b)) \\
&\leq \frac{1}{g(b)-a}\int_a^{g(b)} f(x)w(x)dx \leq \frac{1}{2}M(f,w;a,g(b)),
\end{aligned}$$

where $M(f,w;a,g(b))$ is given by (8).

Proof. Using the fact that f and w are similarly ordered functions, proof follows from Theorem 4. □

Theorem 6. *Let f be relative convex function, then for all $\lambda \in (0,1)$, we have*

$$\begin{aligned}
f\left(\frac{a+g(b)}{2}\right) \leq \Delta_1(\lambda) &\leq \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \\
&\leq \Delta_2(\lambda) \leq \frac{f(a)+f(g(b))}{2},
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\Delta_1(\lambda) &= \lambda f\left(\frac{(2-\lambda)a+\lambda g(b)}{2}\right) \\
&\quad +(1-\lambda)f\left(\frac{(1-\lambda)a+(1+\lambda)g(b)}{2}\right),
\end{aligned} \tag{11}$$

and

$$\Delta_2(\lambda) = \frac{f((1-\lambda)a+\lambda g(b))+\lambda f(a)+(1-\lambda)f(g(b))}{2}. \tag{12}$$

Now essentially using the technique of [12] one can prove following result. This result plays a key role in proving our next result.

Lemma 1. *If $f^{(n)}(x)$ for $n \in \mathbb{N}$ exists and is integrable on $[a,g(b)]$, then*

$$\begin{aligned}
& \Xi(a,g(b);k;n;f) \\
&= \frac{(g(b)-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)g(b))dt,
\end{aligned}$$

where

$$\begin{aligned}
\Xi(a,g(b);k;n;f) &= \frac{f(a)+f(g(b))}{2} - \frac{1}{g(b)-a}\int_a^{g(b)} f(x)dx \\
&\quad -\sum_{k=2}^{n-1}\frac{(k-1)(g(b)-a)^k}{2(k+1)!}f^{(k)}(a)
\end{aligned}$$

Theorem 7. *Let $f : K_g \to \mathbb{R}$ be n-times differentiable and integrable on K_g. If $|f^{(n)}|^q$ where $q > 1$ is relative convex function, then*

$$|\Xi(a,g(b);k;n;f)| \le \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\mathscr{G}_1|f^{(n)}(a)|^q+\mathscr{G}_2|f^{(n)}(g(b))|^q\right)^{\frac{1}{q}},$$

where

$$\mathscr{G}_1 = \frac{n^2-2}{(n+1)(n+2)},$$

and

$$\mathscr{G}_2 = \frac{n}{(n+1)(n+2)},$$

respectively.

Proof. Using Lemma 1, power-mean inequality and the fact that $|f^{(n)}|^q$ is relative convex function, we have

$$\begin{aligned}
&|\Xi(a,g(b);k;n;f)| \\
&= \left|\frac{(g(b)-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)g(b))dt\right| \\
&\le \frac{(g(b)-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)g(b))|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1 t^{n-1}(n-2t)[t|f^{(n)}(a)|^q+(1-t)|f^{(n)}(g(b))|^q]dt\right)^{\frac{1}{q}} \\
&= \frac{(g(b)-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{n^2-2}{(n+1)(n+2)}|f^{(n)}(a)|^q+\frac{n}{(n+1)(n+2)}|f^{(n)}(g(b))|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 1. *Under the assumptions of Theorem 7, if $q = 1$, then*

$$\begin{aligned}
&|\Xi(a,g(b);k;n;f)| \\
&\le \frac{(g(b)-a)^n}{2n!} \\
&\quad\times\left(\frac{n^2-2}{(n+1)(n+2)}|f^{(n)}(a)|+\frac{n}{(n+1)(n+2)}|f^{(n)}(g(b))|\right).
\end{aligned}$$

Now we derive Hermite-Hadamard type of inequalities for relative logarithmic convex functions.

Theorem 8. *Let $f : K_g \to (0,\infty)$ be relative logarithmic convex function, then for all $t \in [0,1]$, we have*

$$\begin{aligned}
f\left(\frac{a+g(b)}{2}\right) &\le \exp\left[\frac{1}{g(b)-a}\int_a^{g(b)}\log f(x)dx\right] \\
&\le \sqrt{f(a)f(g(b))}.
\end{aligned}$$

Theorem 9. *Let $f : K_g \to (0,\infty)$ be relative logarithmic convex function, then for all $t \in [0,1]$, we have*

$$\begin{aligned}
f\left(\frac{a+g(b)}{2}\right) &\le \exp\left[\frac{1}{g(b)-a}\int_a^{g(b)}\log f(x)dx\right] \\
&\le \frac{1}{g(b)-a}\int_a^{g(b)}G(f(x),f(a+g(b)-x))dx \\
&\le \frac{1}{g(b)-a}\int_a^{g(b)}f(x)dx \\
&\le L[f(g(b)),f(a)] \\
&\le A[f(a),f(g(b))],
\end{aligned}$$

where

$$L[f(g(b)),f(a)] = \frac{f(g(b))-f(a)}{\log f(g(b))-\log f(a)},$$

is the logarithmic mean,

$$A[f(a),f(g(b))] = \frac{f(a)+f(g(b))}{2},$$

is the arithmetic mean, and

$$G[f(a),f(g(b))] = \sqrt{f(a)f(g(b))},$$

is the Geometric mean respectively.

Proof. The proof of first inequality follows directly from Theorem 8. In order to prove second inequality, we proceed as

$$G(f(x),f(a+g(b)-x)) = \exp\left[\log G(f(x),f(a+g(b)-x))\right].$$

Integrating above inequality with respect to x on $[a,g(b)]$, we have

$$\begin{aligned}
&\frac{1}{g(b)-a}\int_a^{g(b)}G(f(x),f(a+g(b)-x))dx \\
&= \frac{1}{g(b)-a}\int_a^{g(b)}\exp\left[\log G(f(x),f(a+g(b)-x))\right]dx \\
&\ge \exp\left[\frac{1}{g(b)-a}\int_a^{g(b)}\log G(f(x),f(a+g(b)-x)dx\right] \\
&= \exp\left[\frac{1}{g(b)-a}\int_a^{g(b)}\frac{\log f(x)+\log f(a+g(b)-x)}{2}dx\right] \\
&= \exp\left[\frac{1}{g(b)-a}\int_a^{g(b)}\log f(x)dx\right].
\end{aligned}$$

Using $AM-GM$ inequality, we have

$$G(f(x),f(a+g(b)-x))\le\frac{f(x)+f(a+g(b)-x)}{2}.$$

Integrating the above inequality with respect to x on $[a,g(b)]$, we have

$$\frac{1}{g(b)-a}\int_a^{g(b)}G(f(x),f(a+g(b)-x))dx\le\frac{1}{g(b)-a}\int_a^{g(b)}f(x)dx.$$

Since f is relative logarithmic convex function, so for all $t\in[0,1]$, we have

$$f((1-t)a+tg(b))\le[f(a)]^{1-t}[f(g(b))]^t.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}\frac{1}{g(b)-a}\int_a^{g(b)}f(x)dx&\le f(a)\int_0^1\left[\frac{f(g(b))}{f(a)}\right]^t dt\\&=\frac{f(g(b))-f(a)}{\log f(g(b))-\log f(a)}\\&=L[f(g(b)),f(a)]\\&\le\frac{f(a)+f(g(b))}{2}\\&=A[f(a),f(g(b))].\end{aligned}$$

This completes the proof. □

Theorem 10. *Let $f,w:K_g\to(0,\infty)$ be relative logarithmic convex functions, then*

$$\log w\left(\frac{a+g(b)}{2}\right)-\frac{1}{g(b)-a}\int_a^{g(b)}\log w(x)dx$$
$$\le\frac{1}{g(b)-a}\int_a^{g(b)}\log f(x)dx-\log f\left(\frac{a+g(b)}{2}\right).$$

Proof. Let f and g be relative logarithmic convex functions. Then

$$\begin{aligned}&f\left(\frac{a+g(b)}{2}\right)w\left(\frac{a+g(b)}{2}\right)\\&=f\left(\frac{(1-t)a+tg(b)+ta+(1-t)g(b)}{2}\right)\\&\quad w\left(\frac{(1-t)a+tg(b)+ta+(1-t)g(b)}{2}\right)\\&\le[f((1-t)a+tg(b))f(ta+(1-t)g(b))]^{\frac{1}{2}}\\&\quad[w((1-t)a+tg(b))w(ta+(1-t)g(b))]^{\frac{1}{2}}.\end{aligned}$$

Taking log on both sides, we have

$$\begin{aligned}&\log\left[f\left(\frac{a+g(b)}{2}\right)w\left(\frac{a+g(b)}{2}\right)\right]\\&\le\frac{1}{2}[\log f((1-t)a+tg(b))+\log f(ta+(1-t)g(b))\\&\quad+\log w((1-t)a+tg(b))+\log w(ta+(1-t)g(b))].\end{aligned}$$

Integrating both sides of above inequality with respect to t on $[0,1]$, we have the required result. □

Theorem 11. *Let $f,w:K_g\to\mathbb{R}$ be relative logarithmic convex functions, then*

$$\frac{1}{g(b)-a}\int_a^{g(b)}f(x)w(a+g(b)-x)dx$$
$$\le\frac{[A(f(a),f(g(b)))]^2+[A(w(a),w(g(b)))]^2}{2}.$$

Proof. Since f,w be relative logarithmic convex functions, then, we have

$$\begin{aligned}&\frac{1}{g(b)-a}\int_a^{g(b)}f(x)w(a+g(b)-x)dx\\&=\int_0^1 f(ta+(1-t)g(b))w((1-t)a+tg(b))dt\\&\le\int_0^1[f(a)]^t[f(g(b))]^{1-t}[w(a)]^{1-t}[w(g(b))]^t dt\\&=\int_0^1[f(g(b))]\left[\frac{f(a)}{f(g(b))}\right]^t w(a)\left[\frac{w(g(b))}{w(a)}\right]^t dt\\&=f(g(b))w(a)\int_0^1\left[\frac{f(a)w(g(b))}{f(g(b))w(a)}\right]^t dt\\&=f(g(b))w(a)\frac{\frac{f(a)w(g(b))-f(g(b))w(a)}{f(g(b))w(a)}}{\log f(a)w(g(b))-\log f(g(b))w(a)}\\&=\frac{f(a)w(g(b))-f(g(b))w(a)}{\log f(a)w(g(b))-\log f(g(b))w(a)}\\&=L[f(a)w(g(b)),f(g(b))w(a)]\\&\le\frac{f(a)w(g(b))+f(g(b))w(a)}{2}\\&=A[f(a)w(g(b)),f(g(b))w(a)]\\&\le\frac{1}{2}\int_0^1\left\{[f(ta+(1-t)g(b))]^2+[w((1-t)a+tg(b))]^2\right\}dt\\&\le\frac{1}{2}\int_0^1\left\{[f(a)]^t[f(g(b))]^{1-t}\right\}^2 dt\\&\quad+\frac{1}{2}\int_0^1\left\{[w(a)]^{1-t}[w(g(b))]^t\right\}^2 dt\\&=\frac{[f(g(b))]^2}{4}\int_0^2\left[\frac{f(a)}{f(g(b))}\right]^u du\\&\quad+\frac{[w(a)]^2}{4}\int_0^2\left[\frac{w(g(b))}{w(a)}\right]^u du\\&=\frac{1}{4}\frac{[f(a)]^2-[f(g(b))]^2}{\log f(a)-\log f(g(b))}+\frac{1}{4}\frac{[w(a)]^2-[w(g(b))]^2}{\log w(a)-\log w(g(b))}\\&=\frac{1}{2}\left[\frac{f(a)+f(g(b))}{2}\frac{f(a)-f(g(b))}{\log f(a)-\log f(g(b))}\right]\end{aligned}$$

$$+\frac{1}{2}\left[\frac{w(a)+w(g(b))}{2}\frac{w(a)-w(g(b))}{\log w(a)-\log w(g(b))}\right]$$
$$=\frac{1}{2}[A[f(a),f(g(b))]L[f(a),f(g(b))]]$$
$$+\frac{1}{2}[A[w(a),w(g(b))]L[w(a),w(g(b))]]$$
$$\leq\frac{1}{2}\left[\frac{f(a)+f(g(b))}{2}\frac{f(a)+f(g(b))}{2}\right]$$
$$+\frac{1}{2}\left[\frac{w(a)+w(g(b))}{2}\frac{w(a)+w(g(b))}{2}\right]$$
$$=\frac{[A(f(a),f(g(b)))]^2+[A(w(a),w(g(b)))]^2}{2},$$

which is the required result. □

Theorem 12. *Let $f,w:K_g\to(0,\infty)$ be increasing and relative logarithmic convex functions on K_g. Then, we have*

$$f\left(\tfrac{a+g(b)}{2}\right)L[w(a),w(g(b))]+w\left(\tfrac{a+g(b)}{2}\right)L[f(a),f(g(b))]$$
$$\leq\tfrac{1}{g(b)-a}\int_a^{g(b)}f(x)w(x)dx+L[f(a)w(a),f(g(b))w(g(b))].$$

Proof. Let f and w are relative logarithmic convex functions. Then, we have

$$f(ta+(1-t)g(b))\leq[f(a)]^t[f(g(b))]^{1-t}$$
$$w(ta+(1-t)g(b))\leq[w(a)]^t[w(g(b))]^{1-t}.$$

Now, using $\langle x_1-x_2,x_3-x_4\rangle\geq0$, $(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2<x_3<x_4$, we have

$$f(ta+(1-t)g(b))[w(a)]^t[w(g(b))]^{1-t}$$
$$+w(ta+(1-t)g(b))[f(a)]^t[f(g(b))]^{1-t}$$
$$\leq f(ta+(1-t)g(b))w(ta+(1-t)g(b))$$
$$+[f(a)]^t[f(g(b))]^{1-t}[w(a)]^t[w(g(b))]^{1-t}.$$

Integrating above inequalities with respect to t on $[0,1]$, we have

$$\int_0^1 f(ta+(1-t)g(b))[w(a)]^t[w(g(b))]^{1-t}dt$$
$$+\int_0^1 w(ta+(1-t)g(b))[f(a)]^t[f(g(b))]^{1-t}dt$$
$$\leq\int_0^1 f(ta+(1-t)g(b))w(ta+(1-t)g(b))dt$$
$$+\int_0^1[f(a)]^t[f(g(b))]^{1-t}[w(a)]^t[w(g(b))]^{1-t}dt.$$

Now, since f and w are increasing, we have

$$\int_0^1 f(ta+(1-t)g(b))dt\int_0^1[w(a)]^t[w(g(b))]^{1-t}dt$$
$$+\int_0^1 w(ta+(1-t)g(b))dt\int_0^1[f(a)]^t[f(g(b))]^{1-t}dt$$
$$\leq\int_0^1 f(ta+(1-t)g(b))w(ta+(1-t)g(b))dt$$
$$+\int_0^1[f(a)]^t[f(g(b))]^{1-t}[w(a)]^t[w(g(b))]^{1-t}dt.$$

Now computing the simple integration, we have

$$\frac{1}{g(b)-a}\int_a^{g(b)}f(x)dxL[w(a),w(g(b))]$$
$$+\frac{1}{g(b)-a}\int_a^{g(b)}w(x)dxL[f(a),f(g(b))]$$
$$\leq\frac{1}{g(b)-a}\int_a^{g(b)}f(x)w(x)dx+L[f(a)w(a),f(g(b))w(g(b))].$$

Now, using the left hand side of Hermite-Hadamard's inequality for relative logarithmic convex functions, we have

$$f\left(\frac{a+g(b)}{2}\right)L[w(a),w(g(b))]$$
$$+w\left(\frac{a+g(b)}{2}\right)L[f(a),f(g(b))]$$
$$\leq\frac{1}{g(b)-a}\int_a^{g(b)}f(x)w(x)dx+L[f(a)w(a),f(g(b))w(g(b))].$$

The desired result. □

4 Fractional Hermite-Hadamard inequalities

In this section, we give some Hermite-Hadamard type inequalities for relative convex functions via Riemann-Liouville fractional integrals.

Theorem 13. *Let f be positive and relative convex function also $f\in L[a,g(b)]$, then we have the following inequality*

$$f\left(\frac{a+g(b)}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(g(b)-a)^\alpha}[J^\alpha_{a^+}f(g(b))+J^\alpha_{g(b)^-}f(a)]$$
$$\leq\frac{f(a)+f(g(b))}{2}.$$

Proof. Since f is relative convex function, then, we have

$$2f\left(\frac{a+g(b)}{2}\right)\leq f(ta+(1-t)g(b))+f((1-t)a+tg(b)).$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and then integrating it with respect to t on $[0,1]$, we have

$$\frac{2}{\alpha}f\left(\frac{a+g(b)}{2}\right)$$
$$\leq\int_0^1 t^{\alpha-1}f(ta+(1-t)g(b))dt+\int_0^1 t^{\alpha-1}f((1-t)a+tg(b))dt.$$

Let $u=g(b)-t(g(b)-a)$ and $v=a+t(g(b)-a)$. Then, from above inequality, we have

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+g(b)}{2}\right)\\
&\leq \int_a^{g(b)}\left(\frac{g(b)-u}{g(b)-a}\right)^{\alpha-1}f(u)\frac{du}{g(b)-a}\\
&\quad+\int_a^{g(b)}\left(\frac{v-a}{g(b)-a}\right)^{\alpha-1}f(v)\frac{dv}{g(b)-a}\\
&=\frac{1}{(g(b)-a)^{\alpha}}\left\{\int_a^{g(b)}(g(b)-u)^{\alpha-1}f(u)du\right.\\
&\qquad\left.+\int_a^{g(b)}(v-a)^{\alpha-1}f(v)dv\right\}\\
&=\frac{\Gamma(\alpha)}{(g(b)-a)^{\alpha}}\left\{\frac{1}{\Gamma(\alpha)}\int_a^{g(b)}(g(b)-u)^{\alpha-1}f(u)du\right.\\
&\qquad\left.+\frac{1}{\Gamma(\alpha)}\int_a^{g(b)}(v-a)^{\alpha-1}f(v)dv\right\}\\
&=\frac{\Gamma(\alpha)}{(g(b)-a)^{\alpha}}[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{g(b)^-}f(a)]. \qquad (13)
\end{aligned}$$

Also

$$\begin{aligned}
&f(ta+(1-t)g(b))+f((1-t)a+tg(b))\\
&\leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b))\\
&=f(a)+f(g(b)),
\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}f(ta+(1-t)g(b))dt+\int_0^1 t^{\alpha-1}f((1-t)a+tg(b))dt\\
&\leq\{f(a)+f(g(b))\}\int_0^1 t^{\alpha-1}dt,
\end{aligned}$$

Now by simple computation and using the definitions of Riemann-Liouville integrals, we have

$$\frac{\Gamma(\alpha)}{(g(b)-a)^{\alpha}}[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{g(b)^-}f(a)]\leq\frac{f(a)+f(g(b))}{\alpha}. \qquad (14)$$

After combining (13) and (14) and simple rearrangements, we have the required result. □

Now using the techniques of [31,34], one can prove the following results respectively.

Lemma 2. *Let $f:K_g\to\mathbb{R}$ be a differentiable function on K_g°. If $f'\in L[a,g(b)]$, then*

$$\begin{aligned}
&\frac{f(a)+f(g(b))}{2}-\frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{(g(b))^-}f(a)\right]\\
&=\frac{g(b)-a}{2}\int_0^1((1-t)^{\alpha}-t^{\alpha})f'(ta+(1-t)g(b))dt.
\end{aligned}$$

Lemma 3. *Let $f:K_g\to\mathbb{R}$ be twice differentiable function on K_g°. If $f''\in L[a,g(b)]$, then*

$$\begin{aligned}
&\frac{f(a)+f(g(b))}{2}-\frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{(g(b))^-}f(a)\right]\\
&=\frac{(g(b)-a)^2}{2}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''(ta+(1-t)g(b))dt.
\end{aligned}$$

Theorem 14. *Let $f:K_g\to\mathbb{R}$ be a differentiable function on K_g° and $f'\in L[a,g(b)]$. If $|f'|$ is relative convex on K_g, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(g(b))}{2}-\frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{(g(b))^-}f(a)\right]\right|\\
&\leq\frac{g(b)-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[|f'(a)|+|f'(g(b))|].
\end{aligned}$$

Proof. Using Lemma 2 and the fact that $|f'|$ is relative convex, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(g(b))}{2}-\frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(g(b))+J^{\alpha}_{(g(b))^-}f(a)\right]\right|\\
&=\left|\frac{g(b)-a}{2}\int_0^1((1-t)^{\alpha}-t^{\alpha})f'(ta+(1-t)g(b))dt\right|\\
&\leq\frac{g(b)-a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}||f'(ta+(1-t)g(b))|dt\\
&\leq\frac{g(b)-a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|[t|f'(a)|+(1-t)|f'(g(b))|]dt\\
&\leq\frac{g(b)-a}{2}\left[\int_0^{\frac12}[(1-t)^{\alpha}-t^{\alpha}][t|f'(a)|+(1-t)|f'(g(b))|]dt\right.\\
&\qquad\left.+\int_{\frac12}^1[(t^{\alpha}-(1-t)^{\alpha})][t|f'(a)|+(1-t)|f'(g(b))|]dt\right]\\
&=\frac{g(b)-a}{2}[I_1+I_2]. \qquad (15)
\end{aligned}$$

Now

$$\begin{aligned}
I_1&=\int_0^{\frac12}[(1-t)^{\alpha}-t^{\alpha}][t|f'(a)|+(1-t)|f'(g(b))|]dt\\
&=|f'(a)|\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{(\frac12)^{\alpha+1}}{\alpha+1}\right]\\
&\quad+|f'(g(b))|\left[\frac{1}{\alpha+2}-\frac{(\frac12)^{\alpha+1}}{\alpha+1}\right], \qquad (16)
\end{aligned}$$

and

$$I_2=\int_{\frac12}^1[t^{\alpha}(1-t)^{\alpha}][t|f'(a)|+(1-t)|f'(g(b))|]dt$$

$$= |f'(a)| \left[\frac{1}{\alpha+2} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right]$$
$$+ |f'(g(b))| \left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right]. \tag{17}$$

Combining (15), (16) and (17) completes the proof. □

Theorem 15. *Let $f : K_g \to \mathbb{R}$ be twice differentiable function on K_g° and $f'' \in L[a, g(b)]$. If $|f''|$ is relative convex on K_g, then*

$$\left| \frac{f(a)+f(g(b))}{2} - \frac{\Gamma(\alpha+1)}{2(g(b)-a)^\alpha} \left[J^\alpha_{a^+} f(g(b)) + J^\alpha_{(g(b))^-} f(a) \right] \right|$$
$$= \frac{\alpha(g(b)-a)^2}{2(\alpha+1)(\alpha+2)} \left(\frac{|f''(a)| + |f''(g(b))|}{2} \right).$$

Proof. Using Lemma 3, the proof directly follows from Theorem 14. □

5 Relative h-convex functions

In this section, we investigate the class of relative h-convex functions and also discuss some special cases.

Definition 9. *A function $f : K_g \to H$ is said to be relative h-convex function with respect to two functions $h : [0,1] \to (0,\infty)$ and $g : H \to H$ such that K_g is a relative convex set, we have*

$$f((1-t)x + tg(y)) \leq h(1-t)f(x) + h(t)f(g(y)),$$
$$\forall x, y \in H : x, g(y) \in K_g, t \in (0,1). \tag{18}$$

Special cases:

I. If we take $h(t) = t$ then we have the definition of relative convex function.
II. If we take $h(t) = t^{-1}$ in (18), then the definition of relative h-convex function reduces to the definition of relative Godunova-Levin function.

Definition 10. *A function $f : K_g \to H$ is said to be relative Godunova-Levin function with respect to $g : H \to H$, if*

$$f((1-t)x + tg(y)) \leq (1-t)^{-1} f(x) + t^{-1} f(g(y)),$$
$$\forall x, y \in H : x, g(y) \in K_g, t \in (0,1). \tag{19}$$

III. If we take $h(t) = t^s$ in (18), then the definition of relative h-convex function reduces to the definition of relative s-convex function.

Definition 11. *A function $f : K_g \to [0,\infty)$ is said to be relative s-convex function where $s \in (0,1]$ with respect to function $g : H \to H$, such that*

$$f((1-t)x + g(y)) \leq (1-t)^s f(x) + t^s f(g(y))$$
$$\forall x, y \in [0,\infty) : x, g(y) \in K_g, t \in [0,1]. \tag{20}$$

IV. If we take $h(t) = 1$ in (18), then the definition of relative h-convex function reduces to the definition of relative P-function.

Definition 12. *A function $f : K_g \to H$ is said to be relative P-function with respect to function $g : H \to H$, such that*

$$f((1-t)x + tg(y)) \leq f(x) + f(g(y))$$
$$\forall x, y \in K_g : x, g(y) \in K_g, t \in [0,1]. \tag{21}$$

V. If we take $h(t) = t^{-s}$ in (18), then the definition of relative h-convex function reduces to the definition of relative s-Godunova-Levin function.

Definition 13. *A function $f : K_g \to H$ is said to be relative s-Godunova-Levin function where $s \in [0,1]$ with respect to function $g : H \to H$, such that*

$$f((1-t)x + tg(y)) \leq (1-t)^{-s} f(x) + t^{-s} f(g(y))$$
$$\forall x, y \in K_g : x, g(y) \in K_g, t \in [0,1]. \tag{22}$$

VI. If we take $g = I$ in (18), then the definition of relative h-convex function reduces to the definition of h-convex function [33].

We now discuss some Hermite-Hadamard inequalities related to relative h-convex functions.

Theorem 16. *Let $f : K_g \to \mathbb{R}$ be a relative h-convex function, such that $h(\frac{1}{2}) \neq 0$, then, we have*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+g(b)}{2} \right) \leq \frac{1}{g(b)-a} \int_a^{g(b)} f(x)dx$$
$$\leq [f(a) + f(g(b))] \int_0^1 h(t)dt.$$

Remark. For $h(t) = t^{-1}, h(t) = t^s$, $h(t) = 1$ and $h(t) = t^{-s}$ in Theorem 16 we have results for relative Godunova-Levin functions, relative s-convex functions, relative P-functions and relative s-Godunova-Levin functions, respectively.

Theorem 17. *Let f be relative h_1-convex function and w be relative h_2-convex function such that $h_1(\frac{1}{2}) \neq 0$ and $h_2(\frac{1}{2}) \neq 0$, then*

$$\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+g(b)}{2} \right) w\left(\frac{a+g(b)}{2} \right) \right]$$
$$- \left[M(f,w;a,g(b)) \int_0^1 h_1(t)h_2(1-t)dt \right.$$
$$\left. + N(f,w;a,g(b)) \int_0^1 h_1(t)h_2(t)dt \right]$$
$$\leq \frac{1}{g(b)-a} \int_a^{g(b)} f(x)w(x)dx$$
$$\leq \left[M(f,w;a,g(b)) \int_0^1 h_1(t)h_2(t)dt \right.$$
$$\left. + N(f,w;a,g(b)) \int_0^1 h_1(t)h_2(1-t)dt \right],$$

where, $M(f,w;a,g(b)) = f(a)w(a) + f(g(b))w(g(b))$ *and* $N(f,w;a,g(b)) = f(a)w(g(b)) + f(g(b))w(a)$.

Proof. The proof follows directly from Theorem 4.

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