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R-Sets and Metric Dimension of Necklace Graphs

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Abstract: The R-set relative to a pair of distinct vertices of a connected graph G is the set of vertices whose distances to these vertices are distinct. In this paper R-sets are used to show that metric dimension $dim(Ne_n) = 3$ when n is odd and 2 otherwise, where Ne_n is the necklace graph of order $2n+2$. It is also shown that the exchange property of the bases in a vector space does not hold for minimal resolving sets of Ne_n if n is even.

Keywords: Metric dimension, R-set, diameter, necklace graph, exchange property.

1 Introduction

Let G be a connected graph. The distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them and the *diameter* diam(G) of G is $\max_{u,v\in V(G)} d(u,v)$.

For a pair $p = \{x,y\}$ of distinct vertices of G, we shall denote by $RS(p)$ or $RS(x,y)$ the set of vertices $z \in V(G)$ such that $d(z,x) \neq d(z,y)$. Such a set will be called the *resolving set* (or the R-set) relative to the pair $\{x,y\}$.

If $d(z,x) \neq d(z,y)$, then z is said to resolve x and y. It is clear that $\{x,y\} \subseteq RS(x,y) \subseteq V(G)$ for any pair $\{x,y\}$.

The metric dimension of a connected graph G was first defined in [8,13,14]. An equivalent definition is the following [6]: Let $W = \{w_1,w_2,...,w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1),d(v,w_2),...,d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *landmarks* [11] for G. In other words, a set of vertices W is a resolving set if every vertex is uniquely determined by its vector of distances to the vertices in W. It is clear that for any pair of distinct vertices $\{x,y\}$ of G there exists a vertex $w_i \in W$ such that $d(x,w_i) \neq d(y,w_i)$. Hence $RS(x,y) \cap W \neq \emptyset$ for any resolving set W.

A resolving set of minimum cardinality is called a *basis* for G and the number of elements in a basis is the *metric dimension* $dim(G)$ of G. The problem of determining whether $dim(G) < k$ is an NP-complete problem [6,7].

Determining whether a given set $W \subseteq V(G)$ is a resolving set of G need only be verified for the vertices in $V(G)\backslash W$, since every vertex $w \in W$ is the only vertex of G whose distance from W is 0.

An excellent survey of results on the metric dimension and its applications appears in [3,4]. Let $\mathscr{F} = (G_n)_{n\geq 1}$ be a family of connected graphs G_n of order $\varphi(n)$ for which $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G) \leq C$ for every $n \geq 1$ then we shall say that $\mathscr{F}$ has bounded metric dimension. Properties of the metric dimension of infinite graphs and extremal properties involving metric dimension and diameter were considered in [3,9,18].

If all graphs in $\mathscr{F}$ have the same metric dimension (which does not depend on n), $\mathscr{F}$ is called a family with constant metric dimension [10]. A connected graph G has $dim(G) = 1$ if and only if G is a path [6]; cycles C_n have metric dimension 2 for every $n \geq 3$. Also generalized Petersen graphs $P(n,2)$, antiprisms A_n and Harary graphs $H_{4,n}$ are families of graphs with constant metric dimension [10].

The following families of graphs have unbounded metric dimension: if W_n denotes a wheel with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately

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deleting n spokes, then $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$, see [2] and $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for every $n \geq 4$, see [16]. An example of a family which has bounded metric dimension is the family of necklaces $(Ne_n)_{n\geq 1}$. The necklace graph, denoted by Ne_n [15] is a cubic Halin graph [12] obtained by joining by a cycle all vertices of degree 1 of a caterpillar (also called comb) having n vertices of degree 3 and $n+2$ vertices of degree 1, denoted by $u_1, u_2, ..., u_n$ and $v_0, v_1, ..., v_{n+1}$, respectively (see Fig. 1). We have

$$V(Ne_n) = \{v_0, ..., v_{n+1}, u_1, ..., u_n\}$$

and

$$E(Ne_n) = \{v_i v_{i+1}; u_i u_{i+1}; u_i v_i : 1 \leq i \leq n\} \cup \{v_0 v_1; v_0 u_1; v_n v_{n+1}; u_n v_{n+1}\}.$$

The metric dimension $dim(Ne_n)$ is bounded but not constant and it depends on the parity of n. This will be shown in the next section. The following lemma is based on the observation that if for a pair $\{x,y\}$ of vertices, the distance $d(x,y)$ is even, then the middle vertex v of a shortest path between x and y has equal distances to x and to y, hence $v \notin RS(x,y)$.

Lemma 1.[17] *If $RS(x,y) = V(G)$ for a pair $\{x,y\}$ of distinct vertices of a connected graph G then $d(x,y)$ is odd.*

Lemma 2.[17] *If $RS(x,y) = \{x,y\}$ for each pair $\{x,y\}$ of distinct vertices of a connected graph G then G is a complete graph.*

Theorem 1.[17] *If G has n vertices and $diam(G) = 2$ then the number of pairs $\{x,y\}$ such that $RS(x,y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$. This bound is reached only for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

If a graph G of order n has $diam(G) = n-1$ then G is a path P_n and every pair of vertices $\{x,y\}$ of P_n such that $d(x,y)$ is odd satisfies $RS(x,y) = V(P_n)$; the number of such pairs equals $\lfloor n^2/4 \rfloor$.
By joining by an edge the centers of the stars $K_{1,\lfloor n/2 \rfloor -1}$ and $K_{1,\lceil n/2 \rceil -1}$ for any $n \geq 4$ the resulting graph G has diameter 3 and the number of pairs such that $RS(x,y) = V(G)$ equals $\lfloor n^2/4 \rfloor$. This property also holds for even cycles.
These facts lead to the following conjecture:
Conjecture [17]. *For any connected graph G of order $n \geq 2$ the number of pairs $\{x,y\}$ such that $RS(x,y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$.*

2 Metric dimension of necklace graph

Since necklace graph Ne_n is not a path we have $dim(Ne_n) \geq 2$ for any $n \geq 1$. Ne_1 is K_4, so $dim(Ne_1) = 3$ and also $dim(Ne_2) = 2$.

Theorem 2. *For every $n \geq 1$ we have*

$$dim(Ne_n) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. **a**) Let $n = 2k$. In this case a resolving set of Ne_n is $W = \{v_0, v_k\}$ since the representations of the vertices with respect to W are the following:

$$r(v_i|W) = \begin{cases} (i, k-i), & \text{for } 0 \leq i \leq k; \\ (n-i+2, i-k), & \text{for } k+1 \leq i \leq n+1. \end{cases}$$

and

$$r(u_i|W) = \begin{cases} (i, k-i+1), & \text{for } 1 \leq i \leq k; \\ (n-i+2, i-k+1), & \text{for } k+1 \leq i \leq n. \end{cases}$$

Since all vertices have distinct representations we obtain $dim(Ne_n) = 2$ in this case.
b) When $n = 2k+1$ we show that $W = \{v_0, v_{k+1}, u_k\}$ is a resolving set. The representations of the vertices of Ne_n with respect to $U = \{v_0, v_{k+1}\}$ are the following:

$$r(v_i|U) = \begin{cases} (i, k-i+1), & 0 \leq i \leq k+1; \\ (n-i+2, i-k-1), & k+2 \leq i \leq n+1. \end{cases}$$

and

$$r(u_i|U) = \begin{cases} (i, k-i+2), & 1 \leq i \leq k+1; \\ (n-i+2, i-k), & k+2 \leq i \leq n. \end{cases}$$

U distinguishes all vertices of Ne_n unless u_i and v_{n+2-i} for $1 \leq i \leq k+1$. This can be done by u_k, hence W is a resolving set, implying that $dim(Ne_n) \leq 3$.
We will show that $dim(Ne_n) \geq 3$, by proving that any resolving set has at least three vertices. Suppose that there exists a resolving set W of Ne_n such that $|W| = 2$. We shall prove that this leads to a contradiction.
By taking into account the action of the automorphism group of Ne_n, it is sufficient to consider only the cases when $v_{k+1}, v_{k+2}, ...$ or v_{n+1} belongs to W, where $diam(Ne_n) = k+2$.
A. Let $v_{n+1} \in W$. We get $d(v_k, v_{n+1}) = d(u_k, v_{n+1}) = d(v_{k+1}, v_{n+1}) = d(u_{k+1}, v_{n+1}) = k+1$ (see Fig. 1). Also

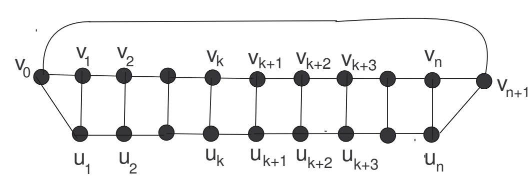

Fig. 1: Ne_n

$$RS(v_k, u_{k+1}) = \{u_{k+2}, u_{k+3}, ..., u_n, v_1, v_2 ..., v_{k-1}\} \cup$$

$\{v_0\} \cup \{v_k, u_{k+1}\}; RS(v_{k+1}, u_k) = \{v_0\} \cup \{v_{k+1}, u_k\} \cup \{u_1, u_2, \ldots, u_{k-1}, v_{k+2}, v_{k+3} \ldots, v_n\}$; $RS(v_k, u_{k+1}) \cap RS(v_{k+1}, u_k) = \{v_0\}$ but $d(v_0, v_k) = d(v_0, u_k) = k$. It follows that there is no resolving set having two vertices including v_{n+1}.

B. Let $v_{k+1} \in W$. As $d(v_{k+2}, v_{k+1}) = d(u_{k+1}, v_{k+1}) = d(v_k, v_{k+1}) = 1$, we get (see Fig. 1):
$RS(v_k, u_{k+1}) = \{v_1, v_2, \ldots, v_{k-1}, u_{k+2}, u_{k+3}, \ldots, u_n\} \cup \{v_0\} \cup \{v_k, u_{k+1}\}$; $RS(v_{k+2}, u_{k+1}) = \{v_{k+2}, u_{k+1}\} \cup \{u_1, u_2, \ldots, u_k, v_{k+3}, \ldots, v_n\} \cup \{v_{n+1}\}$; $RS(v_k, u_{k+1}) \cap RS(v_{k+2}, u_{k+1}) = \{u_{k+1}\}$, but $d(u_{k+1}, v_{k+2}) = 2 = d(u_{k+1}, v_k)$ and v_k, v_{k+2} have unit distances from v_{k+1}. It follows that $v_{k+1} \notin W$, a contradiction.

C. Let $v_{k+2} \in W$. In this case $RS(v_{k+1}, u_{k+2}) = \{v_{n+1}\} \cup \{v_1, , \ldots, v_k, u_{k+3}, \ldots, u_n\} \cup \{v_{k+1}, u_{k+2}\}$;
$RS(v_{k+3}, u_{k+2}) = \{v_0, v_{n+1}\} \cup \{v_{k+4}, , \ldots, v_n, u_2, \ldots, u_{k+1}\} \cup \{v_{k+3}, u_{k+2}\}$ (see Fig. 1).
We have $RS(v_{k+3}, u_{k+2}) \cap RS(v_{k+1}, u_{k+2}) = \{v_{n+1}, u_{k+2}\}$. But $d(v_k, v_{n+1}) = d(u_{k+1}, v_{n+1})$ and $d(v_k, v_{k+2}) = d(u_{k+1}, v_{k+2})$. So W contains v_{k+2} and u_{k+2}. But neither resolves the pair $\{v_{k+1}, v_{k+3}\}$ so $v_{k+2} \notin W$.

D. If $v_i \in W$ and $k+3 \leq i \leq n$ we shall consider its antipodal vertex u_{i-k-1} (for which $d(v_i, u_{i-k-1}) = k+2$) and its neighbors (see Fig. 2). We get

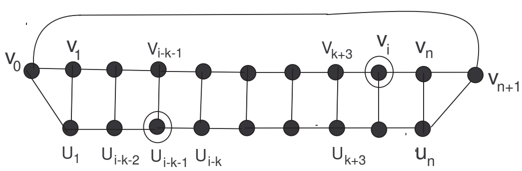

Fig. 2: Ne_n (n odd) with two antipodal vertices

$d(v_i, v_{i-k-1}) = d(v_i, u_{i-k-2}) = d(v_i, u_{i-k}) = k+3$ and $d(v_i, v_1) = d(v_i, u_1) = n - i + 3$.
Since v_{n+1} and v_0 have equal distances to u_1 and v_1, respectively, it follows that $v_0, v_{n+1} \notin W$. Also $d(v_j, v_1) = d(v_j, u_1)$ and $d(u_j, v_1) = d(u_j, u_1)$ for every $k+3 \leq j \leq n$, which implies that

$$(W \setminus \{v_i\}) \cap \{v_0, v_{k+3}, \ldots, v_{n+1}, u_{k+3}, \ldots, u_n\} = \emptyset.$$

For any $x, y \in V(Ne_n)$ and $x \neq y$ denote by $RS'(x,y)$ the set $RS(x,y) \setminus \{v_0, v_{k+3}, \ldots, v_{n+1}, u_{k+3}, \ldots, u_n\}$.
We deduce $RS'(u_{i-k-2}, v_{i-k-1}) = \{v_{i-k}, \ldots, v_{k+2}, u_1, \ldots, u_{i-k-1}\} \cup \{u_{i-k-2}, v_{i-k-1}\}$; $RS'(u_{i-k}, v_{i-k-1}) = \{v_1, \ldots, v_{i-k-2}, u_{i-k+1}, \ldots, u_{k+2}\} \cup \{u_{i-k}, v_{i-k-1}\}$ hence $RS'(u_{i-k-2}, v_{i-k-1}) \cap RS'(u_{i-k}, v_{i-k-1}) = \{v_{i-k-1}\}$. It follows that $W = \{v_i, v_{i-k-1}\}$. But vertices u_{i-k-2} and u_{i-k} have equal distances from both vertices of W, a contradiction. Consequently, every resolving set of $V(Ne_n)$ has at least three vertices for n odd.

3 Exchange property

We have seen that a subset W of vertices of a graph G is a resolving set if every vertex in G is uniquely determined by its distances to the vertices of W. Resolving sets behave like bases in a vector space in that each vertex in the graph can be uniquely identified relative to the vertices of these sets. But though resolving sets do share some of the properties of bases in a vector space, they do not always have the exchange property from linear algebra. Resolving sets are said to have the exchange property in G if whenever S and R are minimal resolving sets for G and $r \in R$, then there exists $s \in S$ so that $S - \{s\} \cup \{r\}$ is a minimal resolving set [1].
If the exchange property holds for a graph G, then every minimal resolving set for G has the same size and algorithmic methods for finding the metric dimension of G are more feasible. Thus to show that the exchange property does not hold in a given graph, it is sufficient to show two minimal resolving sets of different size. However, since the converse is not true, knowing that the exchange property does not hold does not guarantee that there are minimal resolving sets of different size.
The following results concerning exchange property for resolving sets were deduced in [1]:

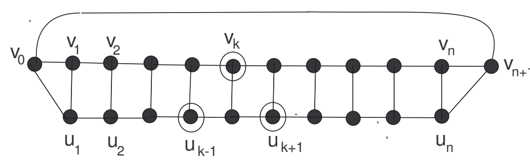

Fig. 3: Ne_n (n even) with a minimal resolving set

Theorem 3.[1] *The exchange property holds for resolving sets in trees.*

Theorem 4.[1] *For $n \geq 8$, resolving sets do not have the exchange property in n-wheels W_n.*

The following question was proposed by Boutin in [1].
Question [1]: In which planar graphs does the exchange property hold for resolving sets?
In this section we study the exchange property for resolving sets in necklace graphs, which are planar graphs.

Theorem 5.*For $n \geq 4$, n even, resolving sets of the necklace graph Ne_n do not have the exchange property.*

*Proof.*If $n = 2k$, we have seen that a minimal resolving set, which is also minimum is $\{v_0, v_{n/2}\}$. We shall prove that $W = \{v_k, u_{k-1}, u_{k+1}\}$ is another minimal resolving set containing three vertices. The representations of the vertices not belonging to W with respect to W are the following:

$$r(v_i|W) = \begin{cases} (k, k-1, k+1), & i = 0; \\ (k-i, k-i, k-i+2), & 1 \le i \le k-1; \\ (i-k, i-k+2, i-k), & \\ & k+1 \le i \le n-1; \\ (k, k+1, k), & i = n; \\ (k+1, k, k), & i = n+1. \end{cases}$$

and

$$r(u_i|W) = \begin{cases} (k-i+1, k-i-1, k-i+1), & \\ & 1 \le i \le k-2; \\ (1,1,1), & i = k; \\ (i-k+1, i-k+1, i-k-1), & \\ & k+2 \le i \le n. \end{cases}$$

We deduce that these representations are different, which implies that W is a resolving set.
W is also minimal, since $A = \{v_{n/2}, u_{n/2-1}\} \subset W$, $B = \{v_{n/2}, u_{n/2+1}\} \subset W$ and $C = \{u_{n/2-1}, u_{n/2+1}\} \subset W$ are not resolving sets: vertices $v_{n/2-1}$ and $u_{n/2}$ have equal distances to vertices of A, $v_{n/2+1}$ and $u_{n/2}$ to vertices of B and v_0 and v_1 to vertices of C. This concludes the proof.

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