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Quantum Entanglement Processing with Atoms

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We discuss practical schemes for triggering evolution of entanglement between qubits. The schemes are especially appealing as they require no experimentally difficult dynamical control and addressing of individual atoms. It is shown that the evolution of a stable or "frozen" entanglement can be triggered by varying the parameters of a given system such as coupling constants between atoms and the field modes or detunings between the atomic and field frequencies. We also address the issue of a controlled (steered) evolution of entanglement between desired pairs of qubits that can be achieved by varying the parameters of a given system. It is of practical importance to know if an initial localized entanglement can be transferred on demand with the perfect fidelity to a particular pair of qubits. In addition, we illustrate how one can achieve a controlled creation of entanglement by spontaneous emission. We use two atoms that directly interact through the dipole-dipole interaction and a collective spontaneous exchange of photons. It is found that time at which an entanglement can be created depends on the initial state of the atoms.

Keywords: Quantum optics, quantum information, entanglement, collective effects.

1 Introduction

The studies of quantum entanglement engineering between trapped atoms, its encoding into atomic states, and transfer from one group of atoms to another are vital to the development of quantum communication and quantum computation [1]. It has been established that trapped and cooled atoms are promising candidates for the realization of all these specific operations that require devices for the control and transmission in the form of long-lived atomic states that are immune to dissipation and decoherence. In order to prevent an encoded logical information from decoherence, the state should belong to a group of states contained inside the so-called decoherence-free subspace that is decoupled from the external environment [2, 3]. Therefore, one of the increasing interests in this context is to find ways to access the internal states of the subspace to trigger a controlled evolution

of the encoded information. It has been proposed to implement a dynamical technique of addressing individual atoms by well stabilized laser fields to achieve controlled manipulation of the internal states of the atoms and evolution of entanglement [4]. However, under realistic conditions the problem of addressing individual atoms poses a substantial experimental challenge. Thus, it is important to investigate alternative methods not involving control over individual atoms.

In this paper, we tackle the problem of triggered evolution of entanglement between two separate two-level atoms serving as qubits. Our focus is on how one could trigger an evolution of a stabile or "frozen" entangled state without the presence of any external fields. We consider three practical schemes where the atoms are coupled to a single mode cavity field or are subject of the interaction with an environment whose the internal modes are in their vacuum states. Microwave and optical cavities have already been proved as useful systems for entanglement creation and a controlled interaction between atoms and field [5]. Although the schemes involving cavities successfully demonstrated the situation for entanglement creation between the atoms, the requirements for controlled evolution of entanglement are not known. We demonstrate that the evolution of an initially "frozen" entanglement can be triggered by a change of a parameter of a given system. The parameters that can be changed are the coupling constants between the atoms and the field modes. Alternatively, one could change frequencies of the cavity modes to detune them from the atomic resonance frequencies. We then apply that ideas to a controlled steering of the evolution of an initial entanglement to a desired pair of qubits including a situation where qubits are completely isolated, such as completely separated cavities each containing a single atom [6–9].

We also address the issue of a controlled creation of entanglement by spontaneous emission. We use motionless qubits (atoms) which are trapped and kept at small distances from each other, so that they directly interact through the dipole-dipole interaction and through a collective spontaneous exchange of photons. We shall see that a crucial parameter in the case of the atoms coupled to the same external environment is the collective damping [10–12]. This parameter has the effect to slow down the spontaneous emission from a collective (antisymmetric) state of the system. As a result, an unbalanced population distribution between the states will occur leading to an entanglement between the atoms.

The paper is organized as follows. We start in Sec. 2 by introducing three different schemes for a controlled evolution of entanglement. We then illustrate in Sec. 3 how a controlled evolution can be triggered in systems which may not be identical in that the cavity frequencies and the coupling constants of the atoms to the cavity modes could be different. The issue of steering the entanglement evolution between remote atoms is discussed in Sec. 4. In Sec. 5, we investigate properties of spontaneously triggered evolution of the atoms that can create a transient entanglement from initially unentangled states. We summarize our results in Sec. 6.

2 Two-Atom Systems

The systems considered here involve two two-level atoms fixed at positions $\vec{r}_i$ $(i = 1, 2)$ with ground states $|g_i\rangle$, excited states $|e_i\rangle$ and transition frequencies ω_0. Both atoms are assumed to be damped with the same rates γ arising from the coupling of the atoms to the external environment and equal to the Einstein A coefficient for spontaneous emission. We discuss in details two particular schemes where the atoms are coupled to a single-mode cavity field of frequency ω_c or are allocated in two separate cavities of frequencies ω_1 and ω_2. In the third scheme considered here, we assume that the atoms are in free space and are coupled to their external environment whose the modes are in the vacuum state. The atoms can be prepared initially in an arbitrary state that can be a separable (product) state or in a superposition (entangled) state. The initial state may evolve in time under the action of the Hamiltonian of the system, or its evolution may be "frozen" for all times.

The dynamics of the atoms can be studied in any complete set of basis states of a given system. It is our purpose to choose the basis that provide equations of motions for populations and coherences of a simple mathematical structure and allow for a particularly transparent physical interpretation.

For the first scheme of the two atoms located inside a single-mode cavity field, it will prove convenient to study the dynamics of the atoms in the basis of four product (separable) states

$$\begin{aligned} |e\rangle &= |e_1\rangle \otimes |e_2\rangle, \quad |2\rangle = |g_1\rangle \otimes |e_2\rangle, \\ |3\rangle &= |e_1\rangle \otimes |g_2\rangle, \quad |g\rangle = |g_1\rangle \otimes |g_2\rangle. \end{aligned} \tag{2.1}$$

For the second scheme of the atoms located in two separate cavities, the dynamics will involve both the atoms and the cavity modes. Similar to the first scheme, it will be convenient to study the dynamics in the basis of product (separable) states of the form

$$\begin{aligned} |1\rangle &= |e_1\rangle \otimes |g_2\rangle \otimes |0\rangle_1 \otimes |0\rangle_2, \quad |2\rangle = |g_1\rangle \otimes |g_2\rangle \otimes |1\rangle_1 \otimes |0\rangle_2, \\ |3\rangle &= |g_1\rangle \otimes |e_2\rangle \otimes |0\rangle_1 \otimes |0\rangle_2, \quad |4\rangle = |g_1\rangle \otimes |g_2\rangle \otimes |0\rangle_1 \otimes |1\rangle_2. \end{aligned} \tag{2.2}$$

Here, for example, the state $|e_1\rangle \otimes |g_2\rangle \otimes |1\rangle_1 \otimes |0\rangle_2$ represents the atom 1 in the excited state, the atom 2 in the ground, one photon in the cavity 1, and zero photons in the cavity 2.

The third scheme involves two atoms interacting with their external environment. In this case, it will prove convenient to study the dynamics of the atoms in the basis of four collective states, so-called Dicke states, defined as [10, 11]

$$\begin{aligned} |e\rangle &= |e_1\rangle \otimes |e_2\rangle, \\ |s\rangle &= \left(|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle\right)/\sqrt{2}, \\ |a\rangle &= \left(|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle\right)/\sqrt{2}, \end{aligned}$$

$$|g\rangle = |g_1\rangle \otimes |g_2\rangle. \tag{2.3}$$

In this basis, the two-atom system behaves as a single four-level system with the ground energy state $|g\rangle$, two intermediate states $|s\rangle$ and $|a\rangle$, and the upper state $|e\rangle$. For independent atoms, the intermediate states are degenerated and the degeneracy is lifted when the atoms directly interact through the dipole-dipole potential. The magnitude of the shift depends on the distance between the atoms and the orientation of the atomic dipole moments in respect to the interatomic axis.

2.1 Two distant atoms located inside a single-mode cavity

In the first scheme, we assume that the atoms are coupled to a standing-wave cavity mode with the position dependent coupling constants $g_j \equiv g(\vec{r}_j)$, and damped at the rate γ by spontaneous emission to modes other than the preferred cavity mode. We assume that the solid angle subtended by the cavity field is small compared to 4π, that the cavity field can be treated as a one-dimensional field. In addition, we assume that the atoms are separated by the distance r_{12} which is large compare to the resonant wavelength λ_0 of the atomic transitions. Therefore, we may ignore the direct interactions between the atoms. The cavity mode is damped with the rate κ and its frequency ω_c is significantly detuned from the atomic transition frequency ω_0, so there is no direct exchange of photons between the atoms and the cavity mode. The behavior of the total system, the atoms plus the cavity mode, is described by the density operator ρ, which in the interaction picture satisfies the master equation

$$\begin{aligned}\frac{\partial \rho}{\partial t} = &-\frac{i}{\hbar}[H_0, \rho] - \frac{1}{2}\gamma \sum_{j=1}^{2} \left(S_j^+ S_j^- \rho + \rho S_j^+ S_j^- - 2S_j^- \rho S_j^+\right) \\ &- \frac{1}{2}\kappa \left(a^\dagger a \rho + \rho a^\dagger a - 2a\rho a^\dagger\right), \end{aligned} \tag{2.4}$$

where

$$H_0 = \hbar \sum_{j=1}^{2} \left(g_j a S_j^+ \mathrm{e}^{-i\Delta t} + \mathrm{H.c.}\right) \tag{2.5}$$

is the Hamiltonian describing the interaction between the cavity field and the atoms. The operators $S_j^+ = |e_j\rangle\langle g_j|$ and $S_j^- = |g_j\rangle\langle e_j|$ are the raising and lowering operators of the jth atom, and $S_j^z = (|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|)/2$ describes its energy. The boson operators a and $a^\dagger$ are the cavity-mode annihilation and creation operators, $\Delta = \omega_c - \omega_0$ is the detuning of the cavity-mode frequency from the atomic transition frequency, and g_j is the position dependent coupling constant of the jth atom with the cavity mode.

To extract the atomic dynamics from the evolution of the total atoms plus the cavity field system, we introduce the photon number representation for the density operator with respect to the cavity mode, and write the density matrix elements in the basis of the photon

number states of the cavity mode, ρ_{mn}. Since this scheme we will refer explicitly only to single-photon processes, we assume that the cavity mode is strongly detuned from the atomic transition frequency to avoid population of higher energy levels. In this case, we may apply the adiabatic approximation which states that for a large detuning the one-photon coherences ρ_{01} and ρ_{10} vary slowly in time, so we can assume that $\dot{\rho}_{01} \approx 0$ and $\dot{\rho}_{10} \approx 0$. With this approximation, we find a simplified model of the system in the form of the master equation for the reduced (atomic) density operator ρ_A of the form [13]

$$\frac{d\rho_A}{dt} = i\sum_{i=1}^{2}\delta_i\left[S_i^z,\rho_A\right] + i\sum_{i\neq j=1}^{2}\Omega_{ij}\left[S_i^+S_j^-,\rho_A\right] - \frac{1}{2}\gamma\sum_{j=1}^{2}\left(S_j^+S_j^-\rho_A + \rho_A S_j^+S_j^- - 2S_j^-\rho_A S_j^+\right), \tag{2.6}$$

where

$$\delta_i = \frac{|g_i|^2}{\Delta}, \quad \text{and} \quad \Omega_{ij} = \Omega_{ji} = \frac{g_i g_j^*}{\Delta}. \tag{2.7}$$

The first two terms of the master equation (2.6) describe coherent evolution of the atoms, with the parameter δ_i appearing as the frequency shift of the energy levels of the ith atom. It is an analog of a dynamic Stark shift [13]. The other parameter, Ω_{ij} represents the shift in energy separation of the levels of atom i due to its interaction with the atom j through the cavity mode. It is an analog of the familiar dipole-dipole interaction between the atoms [10–12, 14–16]. In the formulation used here this interaction is due to an exchange of virtual photons between the atoms. This shows that one can engineer the interaction between distant atoms simply by the adiabatic elimination of the cavity mode.

Note from Eq. (2.7) that the shift of the atomic levels is determined by the coupling constant g_j and can vary with the position of an atom inside the cavity mode. As a result, the atoms located at different positions inside the cavity mode may experience different shifts. We will analyse this case in details assuming that $g_1 \neq g_2$ such that

$$g_1 = g_0, \quad \text{and} \quad g_2 = g_0\cos\left(kr_{12}\right), \tag{2.8}$$

where $r_{12} = |\vec{r}_2 - \vec{r}_1|$ is the distance between the atoms. This choice of the reference frame corresponds to a situation where atom 1 is kept exactly at an antinode of the standing wave and the atom 2 is moved through successive nodes and antinodes of the standing wave. This choice, of course, involves no loss of generality.

In order to study the dynamics of the system and evolution of an initial entanglement, we use the master equation (2.6) and find the following equations of motion for the density matrix elements

$$\dot{\rho}_{23} = -\left(\gamma - i\delta_{12}\right)\rho_{23} + i\Omega_{12}\left(\rho_{22} - \rho_{33}\right),$$

$$\begin{aligned}
\dot{\rho}_{32} &= -\left(\gamma + i\delta_{12}\right)\rho_{32} - i\Omega_{12}\left(\rho_{22} - \rho_{33}\right), \\
\dot{\rho}_{22} &= -\gamma\rho_{22} + i\Omega_{12}\left(\rho_{23} - \rho_{32}\right), \\
\dot{\rho}_{33} &= -\gamma\rho_{33} - i\Omega_{12}\left(\rho_{23} - \rho_{32}\right), \\
\dot{\rho}_{ee} &= -2\gamma\rho_{ee}.
\end{aligned} \tag{2.9}$$

In Eq. (2.9) the parameter $\delta_{12} = \delta_1 - \delta_2$ is a difference between the single-atom Stark shifts. This parameter is of central importance here as it determines the relative variation of atomic transition frequencies with position of the atoms inside the cavity mode. The parameter δ_{12} is different from zero only when $g_1 \neq g_2$. Otherwise, $\delta_{12} = 0$.

In order to solve the set of the differential equations (2.9), we introduce new variables

$$u = \rho_{23} + \rho_{32}, \quad v = i(\rho_{23} - \rho_{32}), \quad w = \rho_{22} - \rho_{33}, \quad \rho_{dd} = \rho_{22} + \rho_{33}, \tag{2.10}$$

and find that it splits into two independent sets of equations

$$\begin{aligned}
\dot{\rho}_{ee} &= -2\gamma\rho_{ee}, \\
\dot{\rho}_{dd} &= -\gamma\rho_{dd} + 2\gamma\rho_{ee},
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\dot{u} &= -\gamma u + \delta_{12} v, \\
\dot{v} &= -\gamma v - \delta_{12} u - 2\Omega_{12} w, \\
\dot{w} &= -\gamma w + 2\Omega_{12} v.
\end{aligned} \tag{2.12}$$

It is interesting that the equations of motion (2.12) are the exact equivalent of the optical Bloch equations of a two-level system driven by a detuned coherent field, where the dipole-dipole interaction Ω_{12} plays the same role as a driving coherent field, and δ_{12} appears as a detuning of the field from the driven transition [17].

Since we are interested only in one-photon dynamics, it is enough to solve the set of equations (2.12). It is easy to show that the solution, valid for arbitrary initial conditions, is of the form

$$\begin{aligned}
u(t) &= \frac{\mathrm{e}^{-\gamma t}}{\alpha^2}\left[2\Omega_{12}\mathcal{A} + \delta_{12}\left(v_0\alpha\sin\alpha t + \mathcal{B}\cos\alpha t\right)\right], \\
v(t) &= \frac{\mathrm{e}^{-\gamma t}}{\alpha}\left(v_0\alpha\cos\alpha t + \mathcal{B}\sin\alpha t\right), \\
w(t) &= \frac{\mathrm{e}^{-\gamma t}}{\alpha^2}\left\{-\delta_{12}\mathcal{A} + 2\Omega_{12}\left(v_0\alpha\sin\alpha t + \mathcal{B}\cos\alpha t\right)\right\},
\end{aligned} \tag{2.13}$$

where $\alpha = \sqrt{4\Omega_{12}^2 + \delta_{12}^2}$ is the detuned Rabi frequency, and

$$\mathcal{A} = 2\Omega_{12}u_0 - \delta_{12}w_0, \qquad \mathcal{B} = \delta_{12}u_0 + 2\Omega_{12}w_0, \tag{2.14}$$

are given in terms of the initial values $w_0 \equiv w(0)$, $u_0 \equiv u(0)$ and $v_0 \equiv v(0)$ of the population inversion and coherences in the system.

The density matrix elements oscillate in time with the Rabi frequency α, and all are equally damped with the rate γ due to the decay of the population to the ground state $|g\rangle$. In the subsequent sections, we will use the solution (2.13) to study the time evolution of entanglement between the atoms.

2.2 Two separated single mode cavities

Our second model considered in this paper consists of two separated single mode cavities each containing a single two-level atom. The atoms are coupled to the cavity modes with coupling constants g_1 and g_2 that, in general, may not be identical. In addition, the cavity frequencies ω_1 and ω_2 may differ from the atomic transition frequency ω_0.

The Hamiltonian for the system in the electric-dipole and rotating-wave approximations is of the form

$$\hat{H} = \hat{H}_F + \hat{H}_A + \hat{H}_{int}, \tag{2.15}$$

where

$$\hat{H}_F = \hbar\omega_1 \left(a_1^\dagger a_1 + \frac{1}{2}\right) + \hbar\omega_2 \left(a_2^\dagger a_2 + \frac{1}{2}\right) \tag{2.16}$$

is the Hamiltonian of the cavity fields,

$$\hat{H}_A = \hbar\omega_0 S_1^z + \hbar\omega_0 S_2^z \tag{2.17}$$

is the Hamiltonian of the atoms, and

$$\hat{H}_{int} = \hbar g_1 \left(a_1^\dagger S_1^- + a_1 S_1^\dagger\right) + \hbar g_2 \left(a_2^\dagger S_2^- + a_2 S_2^\dagger\right) \tag{2.18}$$

is the interaction Hamiltonian between the atoms and the cavity modes. Here, as before in the first scheme, the S_i^+ and S_i^- operators are respectively the raising and lowering operators of the ith atom, and $a_j^\dagger$ (a_j) are the creation (annihilation) operators for the mode of the jth cavity.

We assume that the system is completely isolated from the environment. In this case, the evolution is purely coherent and the state of the system is a pure state whose the time evolution is found by solving the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Phi(t)\rangle = \hat{H}|\Phi(t)\rangle, \tag{2.19}$$

where

$$|\Phi(t)\rangle = d_1(t)|1\rangle + d_2(t)|2\rangle + d_3(t)|3\rangle + d_4(t)|4\rangle, \tag{2.20}$$

which is a linear combination of the product states (2.2) of the atoms and the cavity modes at time t. The coefficient $d_i(t)$ determines the probability amplitude of the ith state at time t. It is easy to show that the coefficients satisfy the differential equations

$$\begin{aligned} \frac{d}{dt}d_1 &= -ig_1 d_2, \quad \frac{d}{dt}d_2 = 2i\Delta_1 d_2 - ig_1 d_1, \\ \frac{d}{dt}d_3 &= -ig_2 d_4, \quad \frac{d}{dt}d_4 = 2i\Delta_2 d_4 - ig_2 d_3, \end{aligned} \tag{2.21}$$

where $2\Delta_j = (\omega_0 - \omega_j)$ is the detuning of the jth cavity frequency from the atomic transition frequency.

Equations (2.21) form decoupled pairs of simple differential equations that can be solved by using e.g. the Laplace transform technique. A solution of the equations, valid for an arbitrary initial state is easily written in the form

$$\begin{aligned} d_1(t) &= \mathrm{e}^{i\Delta_1 t}\left\{d_1(0)\cos(\Omega_1 t) - i\left[\delta_1 d_1(0) + \beta_1 d_2(0)\right]\sin(\Omega_1 t)\right\}, \\ d_2(t) &= \mathrm{e}^{i\Delta_1 t}\left\{d_2(0)\cos(\Omega_1 t) + i\left[\delta_1 d_2(0) - \beta_1 d_1(0)\right]\sin(\Omega_1 t)\right\}, \\ d_3(t) &= \mathrm{e}^{i\Delta_2 t}\left\{d_3(0)\cos(\Omega_2 t) - i\left[\delta_2 d_3(0) + \beta_2 d_4(0)\right]\sin(\Omega_2 t)\right\}, \\ d_4(t) &= \mathrm{e}^{i\Delta_2 t}\left\{d_4(0)\cos(\Omega_2 t) + i\left[\delta_2 d_4(0) - \beta_2 d_3(0)\right]\sin(\Omega_2 t)\right\}, \end{aligned} \tag{2.22}$$

where $\Omega_i = \sqrt{g_i^2 + \Delta_i^2}$ $(i = 1, 2)$ is a detuned Rabi frequency, $d_j(0)$ $(j = 1, 2, 3, 4)$ are the initial values of the probability amplitudes, $\delta_i = \Delta_i/\Omega_i$, and $\beta_i = g_i/\Omega_i$ are scaled (dimensionless) detunings and coupling constants, respectively.

The probability amplitudes oscillate sinusoidally with the Rabi frequency Ω_i, and their dynamics is strongly affected by the modulation term that depends on the detuning δ_i and the coupling constant β_j between the atom and the corresponding cavity mode. In general, the time evolution is quite complicated and not easy to interpret. However, one can see from Eqs. (2.22) that the detuning enters the solutions in an antisymmetric way, whereas the coupling strength enters the solutions in a symmetric way. This difference will be evident in the features of the time evolution of entanglement in the system.

By contrast, we will show that asymmetric cavities and cavity-atom detunings can prove advantageous in enabling a control of entanglement. Our objective then is to include $\omega_1 \neq \omega_2 \neq \omega_0$ and $g_1 \neq g_2$, which also has the purpose of better modelling a real experimental situation, where it may be difficult to produce identical cavities. The unequal coupling constants for example may arise when atoms are not in equivalent positions inside the cavities.

2.3 Two atoms in free space coupled to the same external environment

We now turn to the third scheme that differs from the previous schemes in not involving a cavity. In this scheme we consider two atoms located at distances comparable or even

smaller than the resonant wavelength and simultaneously coupled to a multimode vacuum field. The atoms can now directly interact with each other so that we fully incorporate the collective interactions and study in detail the time evolution of the concurrence starting from an initially separable state. The atoms radiate spontaneously and their radiation field exerts a strong dynamical influence on one another through the vacuum field modes. The time evolution of the system is studied using the Lehmberg–Agarwal [11,12,15,16] master equation, which reads as

$$\begin{aligned}\frac{\partial \rho_A}{\partial t} &= -i\omega_0 \sum_{i=1}^{2} \left[S_i^z, \rho_A\right] - i \sum_{i\neq j=1}^{2} \Omega_{ij} \left[S_i^+ S_j^-, \rho_A\right] \\ &\quad -\frac{1}{2} \sum_{i,j=1}^{2} \gamma_{ij} \left(\left[\rho_A S_i^+, S_j^-\right] + \left[S_i^+, S_j^- \rho_A\right]\right), \end{aligned} \tag{2.23}$$

where $\gamma_{ii} \equiv \gamma$ are the spontaneous decay rates of the atoms caused by their direct coupling to the vacuum field. The master equation (2.23) already contains the interatomic interaction between the atoms that is determined by the parameters γ_{ij} and Ω_{ij} $(i \neq j)$. The interaction parameters depend on the distance between the atoms and describe the collective damping and the dipole-dipole interaction defined, respectively, by

$$\begin{aligned}\gamma_{ij} &= \frac{3}{2}\gamma \left\{ \left[1 - \left(\hat{\mu} \cdot \hat{r}_{ij}\right)^2\right] \frac{\sin\left(k_0 r_{ij}\right)}{k_0 r_{ij}} \right. \\ &\quad \left. + \left[1 - 3\left(\hat{\mu} \cdot \hat{r}_{ij}\right)^2\right] \left[\frac{\cos\left(k_0 r_{ij}\right)}{\left(k_0 r_{ij}\right)^2} - \frac{\sin\left(k_0 r_{ij}\right)}{\left(k_0 r_{ij}\right)^3}\right] \right\}, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\Omega_{ij} &= \frac{3}{4}\gamma \left\{ -\left[1 - \left(\hat{\mu} \cdot \hat{r}_{ij}\right)^2\right] \frac{\cos\left(k_0 r_{ij}\right)}{k_0 r_{ij}} \right. \\ &\quad \left. + \left[1 - 3\left(\hat{\mu} \cdot \hat{r}_{ij}\right)^2\right] \left[\frac{\sin\left(k_0 r_{ij}\right)}{\left(k_0 r_{ij}\right)^2} + \frac{\cos\left(k_0 r_{ij}\right)}{\left(k_0 r_{ij}\right)^3}\right] \right\}, \end{aligned} \tag{2.25}$$

where $k_0 = \omega_0/c$, and $r_{ij} = |\vec{r}_j - \vec{r}_i|$ is the distance between the atoms.

The master equation (2.23) allows us to study the dynamics of the atoms interacting with each other through the coupling to the common vacuum field. Equation (2.23) leads to a closed set of four equations of motion for the populations of the collective states of the system and one-photon coherences between them. The set of the equations is easily solved by direct integration and the solution, valid for an arbitrary initial condition, is of the form

$$\begin{aligned}\rho_{ee}(t) &= \rho_{ee}(0)\, \mathrm{e}^{-2\gamma t}, \\ \rho_{ss}(t) &= \rho_{ss}(0)\, \mathrm{e}^{-(\gamma+\gamma_{12})t} + \rho_{ee}(0) \frac{\gamma+\gamma_{12}}{\gamma-\gamma_{12}} \left(\mathrm{e}^{(\gamma-\gamma_{12})t} - 1\right) \mathrm{e}^{-2\gamma t}, \\ \rho_{aa}(t) &= \rho_{aa}(0)\, \mathrm{e}^{-(\gamma-\gamma_{12})t} + \rho_{ee}(0) \frac{\gamma-\gamma_{12}}{\gamma+\gamma_{12}} \left(\mathrm{e}^{(\gamma+\gamma_{12})t} - 1\right) \mathrm{e}^{-2\gamma t}, \end{aligned}$$

$$\rho_{sa}(t) \quad = \quad \rho_{sa}(0)\,\mathrm{e}^{-(\gamma+2i\Omega_{12})t}, \tag{2.26}$$

and $\rho_{gg}(t) = 1 - \rho_{ee}(t) - \rho_{ss}(t) - \rho_{aa}(t)$. Note that the full solution for the density matrix elements exhibits the effect of the collective damping γ_{12} and the dipole-dipole interaction Ω_{12}.

2.4 Measure of entanglement between two atoms

In order to determine the amount of entanglement between the atoms and the entanglement dynamics, we use concurrence that is the widely accepted measure of entanglement. The concurrence introduced by Wootters [18] is defined as

$$\mathcal{C} = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \tag{2.27}$$

where $\{\lambda_i\}$ are the the eigenvalues of the matrix

$$R = \rho_A \tilde{\rho}_A, \tag{2.28}$$

with $\tilde{\rho}_A$ given by

$$\tilde{\rho}_A = \sigma_y \otimes \sigma_y\, \rho_A^*\, \sigma_y \otimes \sigma_y, \tag{2.29}$$

and σ_y is the usual Pauli matrix given by

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.30}$$

The systems are entangled for $C > 0$ with the maximum possible entanglement given by $C = 1$, while $C = 0$ implies separability.

The concurrence depends on the density matrix of a given system. In the first scheme, we consider time evolution of an initial entanglement between the atoms without presence of an external fields. In this case, the density matrix of the atoms written in the basis of the states (2.1) is in a simple block diagonal form

$$\rho(t) = \begin{pmatrix} \rho_{ee}(t) & 0 & 0 & 0 \\ 0 & \rho_{33}(t) & \rho_{32}(t) & 0 \\ 0 & \rho_{23}(t) & \rho_{22}(t) & 0 \\ 0 & 0 & 0 & \rho_{gg}(t) \end{pmatrix}, \tag{2.31}$$

in which we put all the coherences, except the atom-atom coherences $\rho_{23}(t)$ and $\rho_{32}(t)$, equal to zero. Knowledge of the density matrix of the system allows to find the concurrence, which with the density matrix of the form (2.31) has a simple analytical form

$$\mathcal{C}(t) = 2\max\left\{0,\, |\rho_{23}(t)| - \sqrt{\rho_{gg}(t)\rho_{ee}(t)}\right\}, \tag{2.32}$$

or alternatively, we may study the concurrence (2.32) in terms of the components $u(t)$ and $v(t)$ of the Bloch vector as

$$\mathcal{C}(t) = \max\left\{0,\, |u(t) - iv(t)| - \, 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}\right\}. \tag{2.33}$$

It is evident that a non-zero coherence between the $|2\rangle$ and $|3\rangle$ states is the necessary condition for entanglement, but not in general sufficient one since there is also a threshold term in the concurrence, as seen from Eqs. (2.32) and (2.33), involving the populations $\rho_{ee}(t)$ and $\rho_{gg}(t)$.

For the second scheme, we will calculate the concurrence between the atoms by taking trace over the cavity modes to obtain the reduced density matrix, from which the eigenvalues can be obtained. Thus, in the basis of the states (2.2), the reduced density matrix of the atoms, after tracing over the cavity modes, is of the form

$$\rho(t) = \begin{pmatrix} |d_1(t)|^2 & d_1(t)d_3^*(t) \\ d_1^*(t)d_3(t) & |d_{33}(t)|^2 \end{pmatrix}, \tag{2.34}$$

for which the concurrence is

$$\mathcal{C}_{\alpha\beta}(t) = \max\left\{0,\, 2|d_1(t)||d_3(t)|\right\}, \tag{2.35}$$

where the subscript $\alpha\beta$ indicates the concurrence between the atoms labeled as α and β.

In the same fashion, we may define the concurrence measures of another possible pairs of qubits of the system, such as $\mathcal{C}_{ab}(t)$, cavity 1 mode $-$ cavity 2 mode; $\mathcal{C}_{\alpha a}(t)$, atom $\alpha -$ cavity mode 1; $\mathcal{C}_{\alpha b}(t)$, atom $\alpha -$ cavity mode 2; $\mathcal{C}_{\beta a}(t)$, atom $\beta -$ cavity mode 1; $\mathcal{C}_{\beta b}(t)$, atom $\beta -$ cavity mode 2. These concurrences are found to be evaluated as

$$\begin{aligned}
\mathcal{C}_{ab}(t) &= \max\left\{0,\, 2|d_2(t)||d_4(t)|\right\}, \quad \mathcal{C}_{\alpha a}(t) = \max\left\{0,\, 2|d_1(t)||d_2(t)|\right\}, \\
\mathcal{C}_{\alpha b}(t) &= \max\left\{0,\, 2|d_1(t)||d_4(t)|\right\}, \quad \mathcal{C}_{\beta a}(t) = \max\left\{0,\, 2|d_3(t)||d_2(t)|\right\}, \\
\mathcal{C}_{\beta b}(t) &= \max\left\{0,\, 2|d_3(t)||d_4(t)|\right\}.
\end{aligned} \tag{2.36}$$

Note that there is no the threshold term when only one-photon states are involved in the dynamics of the system [19].

The third scheme involves dynamics of two interacting atoms coupled to an external environment that is a vacuum field. In this case the dynamics of the atoms are determined by the density operator which written in the basis of the collective states (2.3) has the following form

$$\rho(t) = \begin{pmatrix} \rho_{ee}(t) & 0 & 0 & 0 \\ 0 & \rho_{ss}(t) & \rho_{sa}(t) & 0 \\ 0 & \rho_{as}(t) & \rho_{aa}(t) & 0 \\ 0 & 0 & 0 & \rho_{gg}(t) \end{pmatrix}. \tag{2.37}$$

With the density matrix (2.36), the concurrence has a simple analytical form

$$\mathcal{C}(t) = \max\left\{0, \tilde{\mathcal{C}}(t)\right\}, \tag{2.38}$$

with

$$\tilde{\mathcal{C}}(t) = \sqrt{\left[\rho_{ss}(t) - \rho_{aa}(t)\right]^2 - \left[\rho_{sa}(t) - \rho_{as}(t)\right]^2} - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}. \tag{2.39}$$

We see that the problem of entanglement creation and evolution in the two qubit system can be determined in terms of populations and coherences between the collective levels. In addition, the concurrence shows the threshold behavior that depends on the distribution of the population between the entangled and separable states. Notice that the threshold depends on the population of the upper state $|e\rangle$. Thus, no threshold features can be observed in entanglement creation by spontaneous emission for qubits initially prepared in a single photon state. As we shall see in Sec. 5, there are distinguished differences in the way how entanglement is created from initial separable one-photon and two-photon states.

3 Triggered Entanglement Evolution

We now proceed to analyze the problem how one could trigger an evolution of of stable or "frozen" entangled state. We consider the problem for two of the three schemes introduced above that involve the cavity field.

For the first scheme, suppose that the system is initially prepared in a pure superposition state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|g_1\rangle|e_2\rangle + |e_1\rangle|g_2\rangle\right), \tag{3.1}$$

which corresponds to the atoms initially maximally entangled.

For this state, the initial conditions for the Bloch vector components are $u_0 = 1$, $v_0 = w_0 = 0$. Then we can easily find from Eqs. (2.13) and (2.33) that the concurrence $\mathcal{C}(t) = 1$ at $t = 0$, and its time evolution is of the form

$$\mathcal{C}(t) = \mathrm{e}^{-\gamma t}\left|1 - \frac{2\delta_{12}^2}{\alpha^2}\sin^2\left(\frac{1}{2}\alpha t\right) - i\frac{\delta_{12}}{\alpha}\sin\alpha t\right|. \tag{3.2}$$

We see from Eq. (3.2) that the concurrence varies periodically in time only if $\delta_{12} \neq 0$. In the absence of δ_{12}, i.e. when the atoms are in equivalent positions inside the cavity mode, the entanglement remains stabile that its oscillation in time is completely suppressed.

Figure 3.1 display the concurrence C as a function of time and the position of the atom "2" inside the mode of the cavity field. One see that the initial entanglement does not evolve in time when the atom is located at the antinode of the cavity field. This situation corresponds to the case of both atoms located in equivalent positions inside the standing wave of the cavity field that results in equal coupling constants $g_1 = g_2$. In other words, it

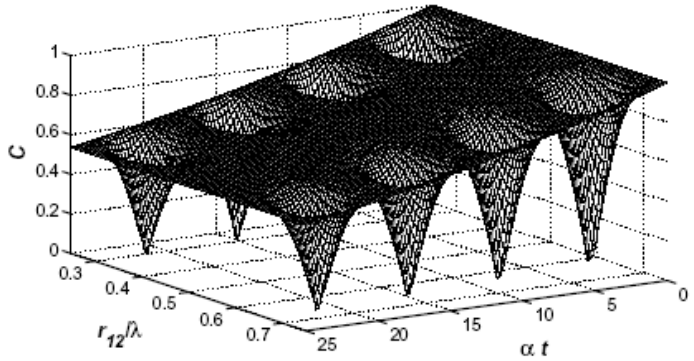

Figure 3.1: Concurrence as a function of the normalized time αt and the position of the second atoms inside the standing-wave cavity mode. The overall slow decay of the concurrence is due to spontaneous emission with the rate $\gamma/\alpha = 0.1$.

corresponds to the atoms located precisely at antinodes of the standing wave. By breaking the symmetry between the coupling constants, i.e. by a dislocation of the atom "2" from the antinode of the standing wave, it is possible to trigger an evolution of the "frozen" entanglement.

Let us now consider the second scheme and illustrate the effect of the detuning $\Delta_1 = \Delta_2 = \Delta$ on the evolution of an initial entanglement. Assume that initially the system was in a superposition state with a uniform population distribution over the available energy states (2.2). The state can be written as

$$|\Psi_0\rangle = \frac{1}{2}\left(|1\rangle + e^{i\theta}|2\rangle \pm (|3\rangle - e^{i\phi}|4\rangle)\right), \tag{3.3}$$

where θ and ϕ are arbitrary phase factors.

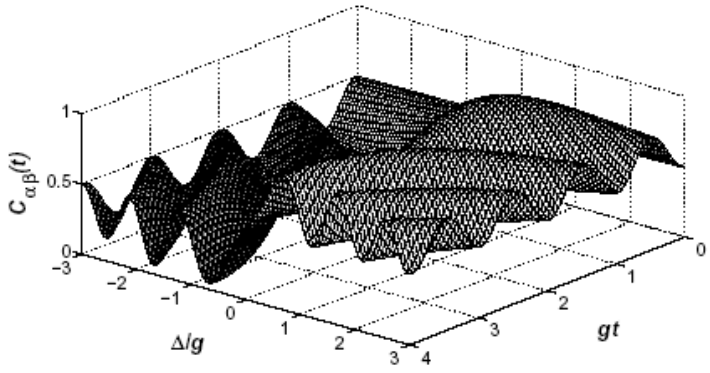

Figure 3.2: Concurrence as a function of the normalized time gt and the detuning of the cavity modes from the atomic resonances for $g_\beta/g_\alpha = 1$.

It is easy to see from Eq. (2.22) that with the initial state (3.3) and zero detunings, all of the two-qubit entanglement between pairs remains constant with time. In this case, the entanglement is maximally shared between the six qubit pairs, so $C_{\alpha\beta}(0) = C_{ab}(0) = C_{\alpha a}(0) = C_{\alpha b}(0) = C_{\beta a} = C_{\beta b}(0) = 1/4$. It is interesting to note that the entanglement remains stabile independent of whether $g_1 = g_2$ or $g_1 \neq g_2$.

A nonzero detuning Δ triggers an evolution of the entanglement. This is illustrated in Fig. 3.2, where we plot the concurrence $C_{\alpha\beta}$ as a function of time and the detuning Δ for equal coupling constants $g_1 = g_2 = g$. We see that the atomic concurrence varies in time only for nonzero detuning and the time evolution of the concurrence is not symmetric with respect to the sign of the detunings. A large and even *maximal* entanglement between the atoms can be created when the detuning is *positive*, whereas the initial atomic entanglement is reduced and can even be suppressed when the detuning is *negative*. When the atom-atom entanglement is maximum, $C_{\alpha\beta} = 1$, then the entanglement between the other qubit pairs is zero and vice versa, when the atom-atom entanglement is zero then entanglement between one of the qubit pairs is maximal. This result also implies a possibility of a controlled evolution of the system towards the maximum entanglement between one of the qubit pairs. It is also interesting that the maximum atomic entanglement $C_{\alpha\beta} = 1$ is created only for a particular value of the detuning $\Delta = g$. Otherwise, the atomic entanglement is reduced.

4 Steered Entanglement Transfer

The possibility of triggering an asymmetric evolution of an initial entanglement by changing the sign of the detuning Δ implies that one can also engineer the direction of evolution of entanglement by controlling the detuning. By this we mean that an initial entanglement can be transferred to a desired "localized" atom-atom entanglement by a suitable choice of the detunings. As it is clear from Fig. 3.2, positive detuning can channel entanglement entirely into the atoms, at appropiate times. "Localized" in this context means that the entanglement exists solely between the two atoms. Therefore, the entanglement transfer can be controlled by varying the frequency of the cavity mode.

Let us now consider an another steering mechanism for entanglement transfer; an asymmetry in the coupling constants g_1 and g_2. As before, in Sec. 3, we will discuss how to achieve a steered evolution of an initial entanglement to a desired pair of qubits using the second scheme involving two cavities each containing a single two-level atom. We assume that initially the system is prepared in a superposition state such that

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |3\rangle\right), \tag{4.1}$$

with the states $|1\rangle$ and $|3\rangle$ being the atomic excitation states of the basis (2.2). It is easy to check that with this initial state, the atoms are maximally entangled, $\mathcal{C}_{\alpha\beta}(0) = 1$, at $t = 0$ and the other pairs of qubits are disentangled.

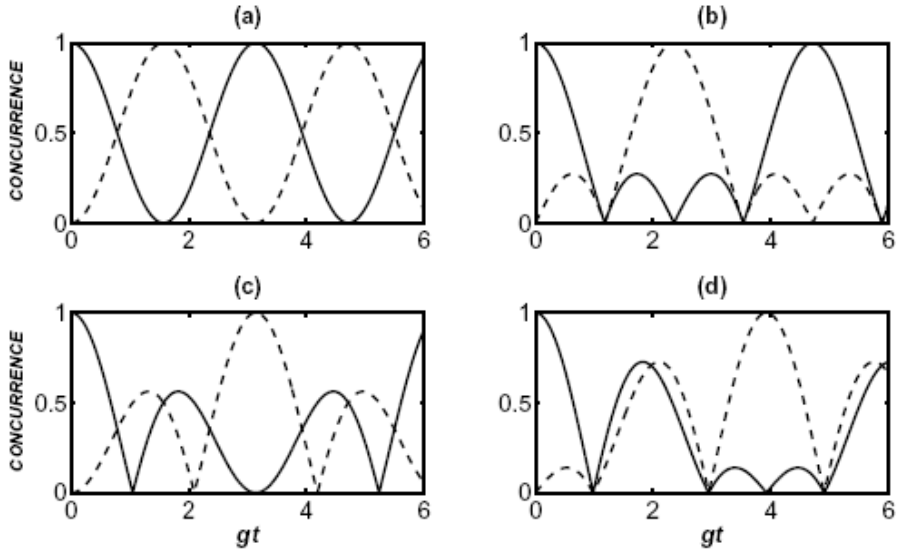

Figure 4.1: Concurrence for qubit pairs plotted as a function of the normalized time $gt = \frac{1}{2}(g_1+g_2)t$ for $\Delta = 0$ and different ratios of the coupling constants g_2/g_1: (a) $g_2/g_1 = 1$, (b) $g_2/g_1 = 2$, (c) $g_2/g_1 = 3$, (d) $g_2/g_1 = 4$. In all figures the solid line is for the atom-atom concurrence $C_{\alpha\beta}$. The dashed line is for the concurrence measure (a) C_{ab}, (b) $C_{\beta a}$, (c) C_{ab} and (d) $C_{\beta a}$.

Consider the time evolution of the concurrence measures. Using Eqs. (2.22), we find that the concurrence measures (2.35) for the case of exact resonances $\Delta_1 = \Delta_2 = 0$ and unequal coupling constants, $g_1 \neq g_2$ are given by

$$\begin{aligned} C_{\alpha\beta}(t) &= \big|\cos(g_1 t)\big|\big|\cos(g_2 t)\big|, \quad C_{ab}(t) = \big|\sin(g_1 t)\big|\big|\sin(g_2 t)\big|, \\ C_{\alpha a}(t) &= \big|\cos(g_1 t)\big|\big|\sin(g_1 t)\big|, \quad C_{\alpha b}(t) = \big|\cos(g_1 t)\big|\big|\sin(g_2 t)\big|, \\ C_{\beta a}(t) &= \big|\cos(g_2 t)\big|\big|\sin(g_1 t)\big|, \quad C_{\beta b}(t) = \big|\sin(g_2 t)\big|\big|\cos(g_2 t)\big|. \end{aligned} \tag{4.2}$$

A simple analysis of the concurrence measures (4.2) shows that if the ratio g_2/g_1 is not an integer number or a fraction of an integer number, no complete transfer of the initial entanglement from the atoms is possible to any of the qubit pairs. The complete transfer is possible only if the ratio is an integer number or a fraction of an integer number. However, the destination to where the initial entanglement can be completely transfered depends on whether the ratio is an even or an odd integer number.

If the ratio g_2/g_1 is an even number, the initial maximal entanglement between the atoms can be completely transferred only to the atom-field qubit pair $C_{\beta a}$. On the other hand, if the ratio is an odd integer number, the initial entanglement between the atoms can be completely transferred only to the field-field qubit pair C_{ab}. We illustrate this situation in Fig. 4.1, where we plot the concurrence of the different qubit pairs as a function of time for exact resonances but unequal coupling strengths. It is evident from the figure

that at particular discrete times, the initial entanglement between the atoms is completely transferred to the qubit pair C_{ab} or $C_{\beta a}$.

The reason for this feature of the entanglement transfer can be understood intuitively by noting that, for example, for $g_2/g_1 = 2$ the Rabi frequency g_2 of the population oscillation in the cavity system 2 is twice that of the Rabi frequency g_1 for the population oscillation in the system 1. This means that over a complete Rabi cycle $g_2 t = \pi$, the initial population in the system 2 returns to the atom, but at the same time the population makes a half Rabi cycle in the system 1, i.e. the excitation in system 1 will be in the cavity mode. Thus, $C_{\beta a} = 1$ at that time, with the concurrence in the other qubit pairs equal to zero.

5 Spontaneously Triggered Evolution

An another issue is creation of entanglement by spontaneous emission where a system prepared initially in an arbitrary state will start to evolve spontaneously due to the interaction of the atoms with an external environment [20–23].

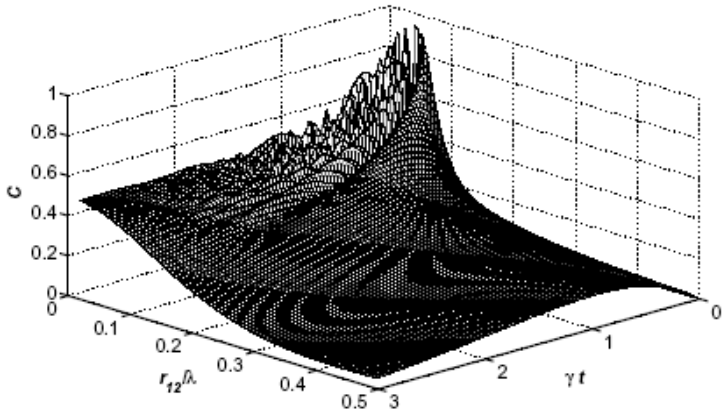

Figure 5.1: Concurrence as a function of the dimensionless time γt and the distance r_{12}/λ between the atoms for $\hat{\vec{r}}_{12} \perp \hat{\mu}$. The system was initially in the separable state $|e_1, g_2\rangle$.

We consider first the evolution of the concurrence from the initial separable one-photon state $|e_1, g_2\rangle$. In this case, the initial values for the density matrix elements are

$$\rho_{ee}(0) = 0, \quad \rho_{ss}(0) = \rho_{aa}(0) = \rho_{as}(0) = \rho_{sa}(0) = \frac{1}{2}. \tag{5.1}$$

We find Eqs. (2.26) and (2.39) that with the initial values (5.1), the time evolution of the concurrence is of the form

$$\tilde{C}(t) = \frac{1}{2}\left\{\left[\mathrm{e}^{-(\gamma+\gamma_{12})t} - \mathrm{e}^{-(\gamma-\gamma_{12})t}\right]^2 + 4\mathrm{e}^{-2\gamma t}\sin^2(2\Omega_{12}t)\right\}^{1/2}. \tag{5.2}$$

To visualize the behavior of the concurrence, we plot the time evolution of $\mathcal{C}(t)$ in Fig. 5.1 for different separations between the atoms. It is evident from the figure, that the atoms become entangled immediately after $t = 0$ and remain entangled for all times. The degree of the entanglement depends on the distance between the atoms and is large for short distances. It is not difficult to find from Eq.(2.26) that the long living entanglement created by spontaneous emission is due to a slow decay of the population of the antisymmetric state [23].

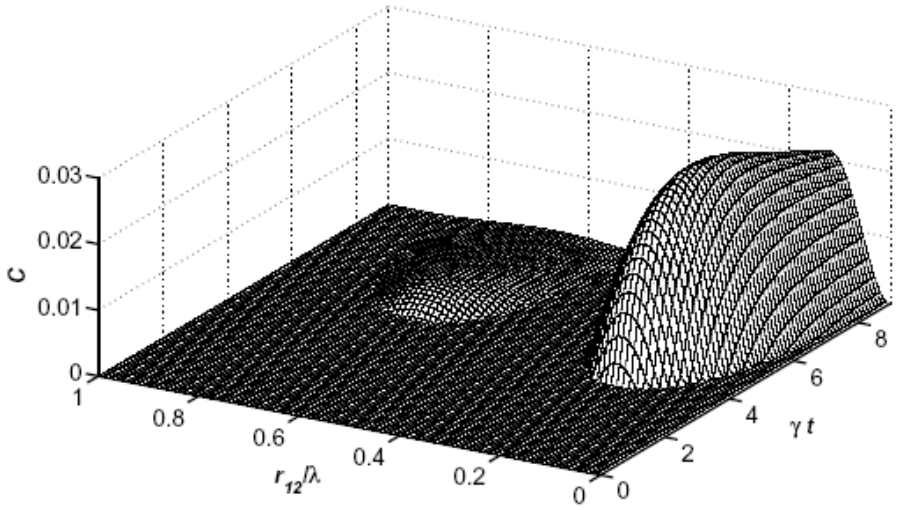

Figure 5.2: Concurrence as a function of the dimensionless time γt and the distance r_{12}/λ between the atoms for $\hat{r}_{12} \perp \hat{\mu}$.

The entangled properties of the atoms are completely different when the system is initially prepared in the upper state $|e\rangle$. If, at $t = 0$, both atoms were excited, the initial values of the density matrix elements are

$$\rho_{ee}(0) = 1, \quad \rho_{ss}(0) = \rho_{aa}(0) = \rho_{as}(0) = \rho_{sa}(0) = 0. \tag{5.3}$$

Then, the concurrence, evaluated from Eq. (2.26) and (2.39), shows completely different time behavior [24]. This is illustrated in Fig. 5.2, where we plot the concurrence as a function of time and the distance between the atoms. We see that in contrast to the previous case where entanglement was created immediately after $t = 0$, the creation of entanglement is now delayed, that it occurs after a finite time. The reason for the delayed creation of entanglement can be understood as follows. When the system is prepared in the state $|e\rangle$, the resulting spontaneous transitions are cascades: The system decays first to the intermediate states $|s\rangle$ and $|a\rangle$, from which then decays to the ground state $|g\rangle$. Since the transition rates to and from the states $|s\rangle$ and $|a\rangle$ are different when $\gamma_{12} \neq 0$, there appears unbalanced

population distribution between these states. According to Eq. (2.39), this may result in a transient entanglement between the atoms.

6 Conclusions

In this paper, we have investigated three schemes for triggered and controlled evolution of an initial stabile entanglement between two two-level atoms. It has been shown that the time evolution of an initially "frozen" entanglement can be triggered by varying one of the parameters of a given system. We have shown that by changing the coupling constants between the atoms and the field modes or detuning of the frequencies of the cavity modes from the atomic resonance frequencies one can trigger an evolution of an initial entanglement to a desired qubit pair. We have also shown that spontaneous emission which is always regarded as a source of decoherence, can create entanglement between two interacting atoms. The creation of entanglement depends on the initial state of the atoms and can be delayed if initially both atoms are excited. The delay can be controlled by changing the distance between the atoms.

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