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Spectrum and Fine Spectrum Generalized Difference Operator Over The Sequence Space ℓ_1

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Abstract: In this paper, we examined the fine spectrum of upper triangular double-band matrices over the sequence spaces ℓ_1. Also, we determined the point spectrum, the residual spectrum and the continuous spectrum of the operator $A(\tilde{r},\tilde{s})$ on ℓ_1. Further, we derived the approximate point spectrum, defect spectrum and compression spectrum of the matrix operator $A(\tilde{r},\tilde{s})$ over the space ℓ_1.

Keywords: Spectrum of an operator, double sequential band matrix, spectral mapping theorem, the sequence space ℓ_1, Goldberg's classification.

1 Introduction, notations and known results

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of these three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

Several authors studied the spectrum and fine spectrum of linear operators defined by some triangle matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. Cesàro operator of order one on the sequence space ℓ_p studied by Gonzàlez [16], where $1 < p < \infty$. Also, weighted mean matrices of operators on ℓ_p have been investigated by Cartlidge [12]. The spectrum of the Cesàro operator of order one on the sequence spaces bv_0 and bv investigated by Okutoyi [21, 22]. The spectrum and fine spectrum of the Rhally operators on the sequence spaces ℓ_p, examined by Yıldırım [24]. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c studied by Altay and Başar [4]. The same authors also worked the fine spectrum of the generalized difference operator $B(r,s)$ over c_0 and c, in [5]. Recently, the fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p studied by Akhmedov and Başar [1,2], where bv_p is the space consisting of the sequences $x=(x_k)$ such that $x=(x_k-x_{k-1})\in \ell_p$ and introduced by Başar and Altay [9] with $1\le p\le\infty$. In the recent paper, Furkan [13] has studied fine spectrum of $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p with $1<p<\infty$, where $B(r,s,t)$ is a lower triangular triple-band matrix. Later, Karakaya and Altun have determined the fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c, in [19]. Quite recently, Karaisa [6] have determined the fine spectrum of the generalized difference operator $A(\tilde{r},\tilde{s})$, defined as a upper triangular double-band matrix with the convergent sequences $\tilde{r}=(r_k)$ and $\tilde{s}=(s_k)$ having certain properties, over the sequence space ℓ_p, where $1<p<\infty$. Finally, Karaisa and Başar [17,18] have determined the fine spectrum of the upper triangular triple-band matrix $A(r,s,t)$ over the sequence space ℓ_p, where $0<p<\infty$. Further informations on the spectrum and fine spectra of different operators over some sequence spaces can be found in the list of references [3,7,10,11, 14,23]

In this paper, we study the spectrum and fine spectrum of the generalized difference operator $A(\tilde{r},\tilde{s})$ defined by a double sequential band matrix acting on the sequence space ℓ_1 with respect to the Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum.

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By ω, we denote the space of all complex valued sequences. Any vector subspace of ω is called a sequence space. We write ℓ_∞, c_0, c and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively, which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty}|x_k - x_{k+1}|$, respectively, where $\mathbb{N}=\{0,1,2,\ldots\}$. Also by ℓ_1 and ℓ_p, we denote the spaces of all absolutely summable and p-absolutely summable sequences, which are the Banach spaces with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}$, respectively, where $1 \le p < \infty$.

Let X and Y is a Banach space and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator $T_\alpha = T - \alpha I$, where α is a complex number and I is the identity operator on $D(T)$. If T_α has an inverse that is linear, we denote it by T_α^{-1}, that is

$$T_\alpha^{-1} = (T - \alpha I)^{-1}$$

and call it the resolvent operator of T.

Many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α in the complex plane such that T_α^{-1} exists. The boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α the domain of T_α^{-1} is dense in X, to name just a few aspects For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows (see [20, pp. 370-371]):

Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value α of T is a complex number such that

(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set $\rho(T)$ *of* T is the set of all regular values α of T. Its complement $\mathbb{C}\setminus\rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T . Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows. *The point spectrum* $\sigma_p(T)$ is the set such that T_α^{-1} does not exist. $\alpha \in \sigma_p(T)$ is called an eigenvalue of T. *The continuous spectrum* $\sigma_c(T)$ is the set such that T_α^{-1} exists and satisfies $(R3)$ but not $(R2)$. *The residual spectrum* $\sigma_r(T)$ is the set such that T_α^{-1} exists but not satisfy $(R3)$.

Table 1: Subdivisions of spectrum of a linear operator.

		1	2	3
		T_α^{-1} exists and is bounded	T_α^{-1} exists and is unbounded	T_α^{-1} does not exist
A	$R(\alpha I - T) = X$	$\alpha \in \rho(T,X)$	–	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$
B	$\overline{R(\alpha I - T)} = X$	$\alpha \in \rho(T,X)$	$\alpha \in \sigma_c(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$
C	$\overline{R(\alpha I - T)} \neq X$	$\alpha \in \sigma_r(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$	$\alpha \in \sigma_r(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$	$\alpha \in \sigma_p(T,X)$ $\alpha \in \sigma_{ap}(T,X)$ $\alpha \in \sigma_\delta(T,X)$ $\alpha \in \sigma_{co}(T,X)$

In this section, following Appell et al. [8], we define the three more subdivisions of the spectrum called as the *approximate point spectrum*, *defect spectrum* and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(T,X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - T\} \quad (1)$$

the *approximate point spectrum of* T. Moreover, the subspectrum

$$\sigma_\delta(T,X) := \{\alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective}\} \quad (2)$$

is called *defect spectrum* of T.

The two subspectra given by (1) and (2) form a (not necessarily disjoint) subdivisions

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\alpha \in \mathbb{C} : \overline{R(\alpha I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature.

By the definitions given above, we can illustrate the subdivisions spectrum in the following table:

From Goldberg [15] if $T \in B(X)$, X a Banach space, then there are three possibilities for $R(T)$ the range of T :

(A) $R(T) = X$.
(B) $R(T) \neq \overline{R(T)} = X$.
(C) $\overline{R(T)} \neq X$.

and and three possibilities for T^{-1}

(1) T^{-1} exists and is continuous.
(2) T^{-1} exists but is discontinuous.
(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$ then α is in the resolvent set $\rho(X,T)$ of T. The further classification gives rise to the fine spectrum of T. If an operator is in state B_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\alpha \in B_2\sigma(X,T)$.

Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k \in \mathbb{N} = \{0,1,2,\ldots\}$. Then, we say that A defines a matrix mapping from μ into γ and we denote it by writing $A : \mu \to \gamma$ if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x is in γ; where

$$(Ax)_n = \sum_k a_{nk}x_k \quad \text{for each} \quad n \in \mathbb{N}. \tag{3}$$

By $(\mu : \gamma)$, we denote the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right side of (3) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$.

Proposition 1.1. [8, Proposition 1.3, p. 28] Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:

(a)$\sigma(T^*,X^*) = \sigma(T,X)$.
(b)$\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.
(c)$\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$.
(d)$\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$.
(e)$\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.
(f)$\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.
(g)$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that $\sigma(T,X) = \sigma_{ap}(T,X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see [8]).

Lemma 1.1.[15, p. 60] The adjoint operator T^* of T is onto if and only if T is a bounded operator.

Let $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be sequences whose entries either constants or distinct none-zero real numbers satisfying the following conditions:

$$\lim_{k\to\infty} r_k = r,$$
$$\lim_{k\to\infty} s_k = s \neq 0,$$
$$|r_k - r| \neq |s|.$$

Then, we define the sequential generalized difference matrix $A(\widetilde{r},\widetilde{s})$ by

$$A(\widetilde{r},\widetilde{s}) = \begin{bmatrix} r_0 & s_0 & 0 & 0 & \ldots \\ 0 & r_1 & s_1 & 0 & \ldots \\ 0 & 0 & r_2 & s_2 & \ldots \\ 0 & 0 & 0 & r_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, we introduce the operator $A(\widetilde{r},\widetilde{s})$ from ℓ_1 to itself by

$$A(\widetilde{r},\widetilde{s})x = (r_kx_k + s_kx_{k+1})_{k=0}^{\infty} \quad \text{where} \quad x = (x_k) \in \ell_1.$$

2 The fine spectrum of the operator $A(\widetilde{r},\widetilde{s})$ over the sequence space ℓ_1

Theorem 2.1. The operator $A(\widetilde{r},\widetilde{s}) : \ell_1 \to \ell_1$ is a bounded linear operator and

$$\|A(\widetilde{r},\widetilde{s})\|_{\ell_1} = \sup_{k\in\mathbb{N}} |r_k| + \sup_{k\in\mathbb{N}} |s_k|.$$

Proof. The proof is simple. So we omit detail.

Throughout the paper, by $\mathscr{C}$ and $\mathscr{SD}$ we denote the set of constant sequences and the set of sequences of distinct none-zero real numbers, respectively.

Theorem 2.2.
(i) If $\widetilde{r},\widetilde{s} \in \mathscr{C}$,
$\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1) = \{\alpha \in \mathbb{C} : |r-\alpha| < |s|\}$.
(ii) If $\widetilde{r},\widetilde{s} \in \mathscr{SD}$,
$\left\{\alpha \in \mathbb{C} : \sup_{n\in\mathbb{N}} \left|\frac{\alpha-r_n}{s_n}\right| < 1\right\} \subseteq \sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$.
(iii) If $\widetilde{r},\widetilde{s} \in \mathscr{SD}$,
$\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1) \subseteq \left\{\alpha \in \mathbb{C} : \inf_{n\in\mathbb{N}} \left|\frac{\alpha-r_n}{s_n}\right| < 1\right\}$.
iv) If $\widetilde{r},\widetilde{s} \in \mathscr{SD}$,
$\{r_k : k \in \mathbb{N}\} \subseteq \sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$.
(v)If $\widetilde{r},\widetilde{s} \in \mathscr{SD}$,
$\{\alpha \in \mathbb{C} : |r-\alpha| < |s|\} \subseteq \sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$.

Proof. Let $A(\widetilde{r},\widetilde{s})x = \alpha x$ for $\theta \neq x \in \ell_1$. Then, by solving linear equation

$$\begin{aligned} r_0x_0 + s_0x_1 &= \alpha x_0 \\ r_1x_1 + s_1x_2 &= \alpha x_1 \\ r_2x_2 + s_2x_3 &= \alpha x_2 \\ &\vdots \\ r_{k-1}x_{k-1} + s_{k-1}x_k &= \alpha x_k \\ &\vdots \end{aligned}$$

$x_k = \left(\frac{\alpha-r_{k-1}}{s_{k-1}}\right)x_{k-1}$ for all $k \geq 1$ and

$$x_k = \left[\frac{(\alpha-r_{k-1})(\alpha-r_{k-2})\cdots(\alpha-r_1)(\alpha-r_0)}{s_{k-1}s_{k-2}\ldots s_1s_0}\right]x_0.$$

(i) Assume that $\widetilde{r},\widetilde{s} \in \mathscr{C}$. Let $r_k = r$ and $s_k = s$ for all $k \in \mathbb{N}$. We observe that $x_k = \left(\frac{\alpha-r}{s}\right)^k x_0$. This shows that $x \in \ell_1$ if and only if $|\alpha - r| < |s|$, as asserted.
(ii) Let $\widetilde{r},\widetilde{s} \in \mathscr{SD}$ and for $\alpha \in \mathbb{C}$, $\sup_{n\in\mathbb{N}} \left|\frac{\alpha-r_n}{s_n}\right| < 1$. So we have

$$\begin{aligned} &\sum_{k=0}^{\infty} |x_k| \\ &= |x_0| + \sum_{k=1}^{\infty} \left|\frac{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_0-\alpha)}{s_{k-1}s_{k-2}\ldots s_0}\right| |x_0| \\ &\leq |x_0| + \sum_{k=1}^{\infty} \left[\sup_{n\in\mathbb{N}} \left|\frac{\alpha-r_n}{s_n}\right|\right]^k |x_0|. \end{aligned}$$

Hence, $x=(x_k)\in\ell_1$.
(iii) Let $\widetilde{r},\widetilde{s}\in\mathscr{S}\mathscr{D}$ and $x=(x_k)\in\ell_1$. Thus,

$$\begin{aligned}&\sum_{k=0}^{\infty}|x_k|\\&=|x_0|+\sum_{k=1}^{\infty}\left|\frac{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_0-\alpha)}{s_{k-1}s_{k-2}\ldots s_0}\right||x_0|\\&\geq|x_0|+\sum_{k=1}^{\infty}\left[\inf_{n\in\mathbb{N}}\left|\frac{\alpha-r_n}{s_n}\right|\right]^k|x_0|.\end{aligned}\tag{4}$$

If we use inequality of (4) and we consider $x=(x_k)\in\ell_1$, $\inf_{n\in\mathbb{N}}\left|\frac{\alpha-r_n}{s_n}\right|<1$.

(iv) Let $\widetilde{r},\widetilde{s}\in\mathscr{S}\mathscr{D}$. It is clear that, for all $k\in\mathbb{N}$, the vector $x=(x_0,x_1,\ldots,x_k,0,0,\ldots)$ is an eigenvector of the operator $A(\widetilde{r},\widetilde{s})$ corresponding to the eigenvalue $\alpha=r_k$, where $x_0\neq0$ and $x_n=\left(\frac{\alpha-r_n}{s_{n-1}}\right)x_{n-1}$, for $1\leq n\leq k$. Thus $\{r_k:k\in\mathbb{N}\}\subseteq\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$.
(v) Let $\widetilde{r},\widetilde{s}\in\mathscr{S}\mathscr{D}$ and $|\alpha-r|<|s|$. Since $\lim_{k\to\infty}\left|\frac{x_k}{x_{k-1}}\right|=\lim_{k\to\infty}\left|\frac{r_{k-1}-\alpha}{s_{k-1}}\right|=\left|\frac{r-\alpha}{s}\right|<1$, $x\in\ell_1$. This completes the proof.

Theorem 2.3. $\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_1^*)=\begin{cases}\emptyset & ,\ \widetilde{s},\widetilde{r}\in\mathscr{C},\\ \mathscr{B} & ,\ \widetilde{s},\widetilde{r}\in\mathscr{S}\mathscr{D}\end{cases}$ where, $\mathscr{B}=\{r_k:k\in\mathbb{N},|r-r_k|>|s|\}$.

Proof. We prove the theorem by dividing into two parts.

Part 1. Assume that $\widetilde{s},\widetilde{r}\in\mathscr{C}$. Consider $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in $\ell_1^*=\ell_\infty$. Then, by solving the system of linear equations

$$\begin{aligned}r_0f_0&=\alpha f_0\\ s_0f_0+r_1f_1&=\alpha f_1\\ s_1f_1+r_2f_2&=\alpha f_2\\&\vdots\\ s_{k-1}f_{k-1}+r_kf_k&=\alpha f_k\\&\vdots\end{aligned}$$

we find that $f_0=0$ if $\alpha\neq r=r_k$ and $f_1=f_2=\cdots=0$ if $f_0=0$ which contradicts $f\neq\theta$. If f_{n_0} is the first non zero entry of the sequence $f=(f_n)$ and $\alpha=r$, then we get $sf_{n_0}+rf_{n_0+1}=\alpha f_{n_0+1}$ which implies $f_{n_0}=0$ which contradicts the assumption $f_{n_0}\neq0$. Hence, the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ has no solution $f\neq\theta$.

Part 2. Assume that $\widetilde{r},\widetilde{s}\in\mathscr{S}\mathscr{D}$. Then, by solving the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in ℓ_∞ we obtain $(r_0-\alpha)f_0=0$ and $(r_{k+1}-\alpha)f_{k+1}+s_kf_k=0$ for all $k\in\mathbb{N}$. Hence, for all $\alpha\notin\{r_k:k\in\mathbb{N}\}$, we have $f_k=0$ for all $k\in\mathbb{N}$, which contradicts our assumption. So, $\alpha\notin\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_\infty)$. This shows that $\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_\infty)\subseteq\{r_k:k\in\mathbb{N}\}\backslash\{r\}$. Now, we prove that

$\alpha\in\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_\infty)$ if and only if $\alpha\in\mathscr{B}$.

Let $\alpha\in\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_\infty)$. Then, by solving the equation $A(\widetilde{r},\widetilde{s})^*f=\alpha f$ for $f\neq\theta=(0,0,0,\ldots)$ in ℓ_1 with $\alpha=r_0$

$$f_k=\frac{s_0s_1s_2\ldots s_{k-1}}{(r_0-r_k)(r_0-r_{k-1})(r_0-r_{k-2})\cdots(r_0-r_1)}f_0$$

for all $k\geq1$. Since $\ell_1\subseteq\ell_\infty$, we can applying ratio test and we have

$$\lim_{k\to\infty}\left|\frac{f_k}{f_{k-1}}\right|=\lim_{k\to\infty}\left|\frac{s_{k-1}}{r_k-r_0}\right|=\left|\frac{s}{r-r_0}\right|\leq1.$$

But our assumption $\left|\frac{s}{r-r_0}\right|\neq1$. Hence,
$\alpha=r_0\in\{r_k:k\in\mathbb{N},|r_k-r|>|s|\}=\mathscr{B}$.
Similarly we can prove that $\alpha=r_k\in\{r_k:k\in\mathbb{N},|r_k-r|>|s|\}=\mathscr{B}$, for $\alpha=r_k\neq r$ for all $k\in\mathbb{N}_1$. Conversely, let $\alpha\in\mathscr{B}$. Then, exists $k\in\mathbb{N}$, $\alpha=r_k\neq r$ and

$$\lim_{n\to\infty}\left|\frac{f_n}{f_{n-1}}\right|=\lim_{n\to\infty}\left|\frac{s_n}{r_{n+1}-r_k}\right|=\left|\frac{s}{r-r_k}\right|<1.$$

That is $f\in\ell_1$. Since $\ell_1\subseteq\ell_\infty$, $f\in\ell_\infty$. So we have $\mathscr{B}\subseteq\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_\infty)$. This completes the proof.

Theorem 2.4
$\sigma_r(A(\widetilde{r},\widetilde{s}),\ell_1)=\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_1^*)\backslash\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$.

Proof. The proof is obvious so is omitted.

Theorem 2.5. Let $(r_k),(s_k)$ in $\mathscr{S}\mathscr{D}$ and $\mathscr{C}$. $\sigma_r(A(\widetilde{r},\widetilde{s}),\ell_1)=\emptyset$.

Proof.
By Theorem 2.2-2.4, we get $\sigma_r(A(\widetilde{r},\widetilde{s}),\ell_1)=\emptyset$.

Theorem 2.6. $\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)=\mathscr{A}\cup\mathscr{B}$, where $\mathscr{A}=\{\alpha\in\mathbb{C}:|r-\alpha|\leq|s|\}$.

Proof. We prove the theorem by dividing into two parts.
Part 1. Assume that $\widetilde{r},\widetilde{s}\in\mathscr{C}$ and $y=(y_k)\in\ell_\infty$. Then, by solving the equation $A((\widetilde{r},\widetilde{s})-\alpha I)^*x=y$ for $x=(x_k)$ in terms of y, we obtain

$$x_k=\frac{s^{k-1}y_0}{(r-\alpha)^k}+\cdots-\frac{sy_{k-1}}{(r-\alpha)^2}+\frac{y_k}{r-\alpha}.$$

We get,

$$x_k=\frac{1}{r-\alpha}\sum_{i=0}^{k}\left(\frac{s}{r-\alpha}\right)^{k-i}y_i$$

for all $k\in\mathbb{N}$. Hence,

$$|x_k|\leq\frac{1}{|r-\alpha|}\sum_{i=0}^{\infty}\left|\frac{s}{r-\alpha}\right|^i\|y\|_\infty.$$

For $|s|<|r-\alpha|$, we can observe that

$$\|x\|_\infty\leq\frac{1}{|r-\alpha|-|s|}\|y\|_\infty.$$

Thus for $|s|<|r-\alpha|$, $A(\widetilde{r},\widetilde{s})^*-\alpha I$ is onto and by Lemma 1.1, $A(\widetilde{r},\widetilde{s})-\alpha I$ bounded inverse. This means that

$$\sigma_c(A(\widetilde{r},\widetilde{s}),\ell_1)\subseteq\{\alpha\in\mathbb{C}:|r-\alpha|\leq|s|\}.$$

Combining this with Theorem 2.2 and Theorem 2.5, we get

$$\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}\subseteq\sigma(A(\widetilde{r},\widetilde{s}),\ell_1).$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)=\{\alpha\in\mathbb{C}:|r-\alpha|\leq|s|\}.$$

Part 2. Assume that $\widetilde{r},\widetilde{s}\in\mathscr{SD}$ and $y=(y_k)\in\ell_\infty$. Then, by solving the equation $A((\widetilde{r},\widetilde{s})-\alpha I)^*x=y$ terms of y, we obtain

$$x_k=\frac{(-1)^k s_0s_1s_2\cdots s_{k-1}y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\cdots(r_k-\alpha)}+\cdots-\frac{s_{k-1}y_{k-1}}{(r_k-\alpha)(r_{k-1}-\alpha)}+\frac{y_k}{r_k-\alpha}.$$

Then, $|x_k|\le S_k\|y\|_\infty$, where

$$S_k=\left|\frac{1}{r_k-\alpha}\right|+\left|\frac{s_{k-1}}{(r_{k-1}-\alpha)(r_k-\alpha)}\right|+\left|\frac{s_{k-1}s_{k-2}}{(r_{k-2}-\alpha)(r_{k-1}-\alpha)(r_k-\alpha)}\right|+\cdots+\left|\frac{s_0s_1\ldots s_{k-1}}{(r_0-\alpha)(r_1-\alpha)\cdots(r_k-\alpha)}\right|.$$

Now, we prove that $(S_k)\in\ell_\infty$. Since $\lim_{k\to\infty}|s_k/(r_k-\alpha)|=|s/(r-\alpha)|=p<1$, then there exists $k_0\in\mathbb{N}$ such that $|s_k/(r_k-\alpha)|<p_0$ with $p_0<1$, for all $k\ge k_0+1$,

$$\begin{aligned}S_k&=\frac{1}{|r_k-\alpha|}\Bigg[1+\left|\frac{s_{k-1}}{r_{k-1}-\alpha}\right|+\left|\frac{s_{k-1}s_{k-2}}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)}\right|+\cdots+\left|\frac{s_{k-1}s_{k-2}\ldots s_{k_0+1}s_{k_0}\ldots s_0}{(r_{k-1}-\alpha)(r_{k-2}-\alpha)\cdots(r_{k_0+1}-\alpha)(r_{k_0}-\alpha)\cdots(r_0-\alpha)}\right|\Bigg]\\&\le\frac{1}{|r_k-\alpha|}\Bigg[1+p_0+p_0^2+\cdots+p_0^{k-k_0}+p_0^{k-k_0}\frac{|s_{k_0-1}|}{|r_{k_0-1}-\alpha|}+\cdots+p_0^{k-k_0}\left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|\Bigg].\end{aligned}$$

Therefore;

$$S_k\le\frac{1}{|r_k-\alpha|}\left(1+p_0+p_0^2+\cdots p_0^{k-k_0}+p_0^{k-k_0}Mk_0\right),$$

where

$$Mk_0=1+\left|\frac{s_{k_0-1}}{r_{k_0-1}-\alpha}\right|+\left|\frac{s_{k_0-1}s_{k_0-2}}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)}\right|+\cdots+\left|\frac{s_{k_0-1}s_{k_0-2}\ldots s_0}{(r_{k_0-1}-\alpha)(r_{k_0-2}-\alpha)\cdots(r_0-\alpha)}\right|.$$

Then, $Mk_0\ge1$ and so

$$S_k\le\frac{Mk_0}{|r_k-\alpha|}\left(1+p_0+p_0^2+\cdots+p_0^{k-k_0}\right).$$

But there exists $k_1\in\mathbb{N}$ and a real number p_1 such that $\frac{1}{|r_k-\alpha|}<p_1$ for all $k\ge k_1$. Then, $S_k\le(Mp_1k_0)/(1-p_0)$ for all $k>max\{k_0,k_1\}$. Hence, $\sup_{k\in\mathbb{N}}S_k<\infty$. This shows that $\|x\|_\infty\le\|(S_k)\|_\infty\|y\|_\infty<\infty$, since $(y_k)\in\ell_\infty$. Thus for $|s|<|r-\alpha|$, $A(\widetilde{r},\widetilde{s})^*-\alpha I$ is onto and by Lemma 1.1 $A(\widetilde{r},\widetilde{s})-\alpha I$ bounded inverse. This means that

$$\sigma_c(A(\widetilde{r},\widetilde{s}),\ell_1)\subseteq\{\alpha\in\mathbb{C}:|r-\alpha|\le|s|\}.$$

Combining this with Theorem 2.2 and Theorem 2.5, we get

$$\mathscr{B}\cup\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}\subseteq\sigma(A(\widetilde{r},\widetilde{s}),\ell_1).$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)=\mathscr{A}\cup\mathscr{B}.$$

This completes the proof.

Theorem 2.7. Let $(r_k),(s_k)\in\mathscr{SD}$, $\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)=\{\alpha\in\mathbb{C}:|r-\alpha|<|s|\}\cup\mathscr{B}\cup\mathscr{H}$. Where;

$$\mathscr{H}=\left\{\alpha\in\mathbb{C}:|\alpha-r|=|s|,\sum_{k=1}^{\infty}\prod_{i=0}^{k}\left|\frac{\alpha-r_{i-1}}{s_{i-1}}\right|<\infty\right\}.$$

Proof. The proof is obvious.

Theorem 2.8.

$$\sigma_c(A(\widetilde{r},\widetilde{s}),\ell_1)=\begin{cases}\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}&,\widetilde{r},\widetilde{s}\in\mathscr{C},\\\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}\setminus\mathscr{H}&,\widetilde{r},\widetilde{s}\in\mathscr{SD}.\end{cases}$$

Proof. The proof follows of immediately from Theorem 2.2, Theorem 2.5, Theorem 2.6 and Theorem 2.7 because the parts $\sigma_c(A(\widetilde{r},\widetilde{s}),\ell_1)$, $\sigma_r(A(\widetilde{r},\widetilde{s}),\ell_1)$ and $\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$ are pairwise disjoint sets and union of these sets is $\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)$.

Theorem 2.9. Let $(r_k),(s_k)\in\mathscr{SD}$ and $\mathscr{C}$. If $|\alpha-r|<|s|$, $\alpha\in\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)A_3$.

Proof. From Theorem 2.2, $\alpha\in\sigma_p(A(\widetilde{r},\widetilde{s}),\ell_1)$. Thus, $(A(\widetilde{r},\widetilde{s})-\alpha I)^{-1}$ does not exist. It is sufficient to show that the operator $(A(\widetilde{r},\widetilde{s})-\alpha I)$ is onto, i.e., for given $y=(y_k)\in\ell_1$, we have to find $x=(x_k)\in\ell_1$ such that $(A(\widetilde{r},\widetilde{s})-\alpha I)x=y$. Solving the linear equation $(A(\widetilde{r},\widetilde{s})-\alpha I)x=y$,

$$[A(\widetilde{r},\widetilde{s})-\alpha I]x=\begin{bmatrix}r_0-\alpha&s_0&0&0&\ldots\\0&r_1-\alpha&s_1&0&\ldots\\0&0&r_2-\alpha&s_2&\ldots\\0&0&0&r_3-\alpha&\ldots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}\begin{bmatrix}x_0\\x_1\\x_2\\\vdots\end{bmatrix}=\begin{bmatrix}y_0\\y_1\\y_2\\\vdots\end{bmatrix}$$

$$\begin{aligned}\text{let }x_0&=0.\\x_1&=\frac{y_0}{s_0},\\x_2&=\frac{(\alpha-r_1)y_0}{s_1s_0}+\frac{y_1}{s_1},\\&\vdots\\x_k&=\frac{(\alpha-r_1)(\alpha-r_2)\cdots(\alpha-r_{k-1})y_0}{s_0s_1\cdots s_{k-1}}+\cdots+\frac{(r_{k-2}-\alpha)y_{k-2}}{s_{k-1}s_{k-2}}+\frac{y_{k-1}}{s_{k-1}}.\end{aligned}$$

Then, $\sum_k|x_k|\le\sup_k(T_k)\sum_k|y_k|$, where

$$T_k=\left|\frac{1}{s_k}\right|+\left|\frac{(r_{k+1}-\alpha)}{s_ks_{k+1}}\right|+\left|\frac{(r_{k+1}-\alpha)(r_{k+2}-\alpha)}{s_ks_{k+1}s_{k+2}}\right|+\cdots$$

for all $k\in\mathbb{N}$. Since $|(r_{k+1}-\alpha)/s_{k+1}|\longrightarrow|s/(r-\alpha)|<1$, as $k\longrightarrow\infty$, then there exists $k_0\in\mathbb{N}$ and a real number z_0 such that $|s_{k+1}/(r_{k+1}-\alpha)|<z_0$ for all $k\geq k_0$. Then, for all $k\geq k_0+1$,

$$T^k\leq\frac{1}{|s_k|}\left(1+z_0+z_0^2+\cdots\right).$$

But, there exists $k_1\in\mathbb{N}$ and a real number z_1 such that $|1/s_k|<z_1$ for all $k\geq k_1$. Then, $T^k\leq z_1/(1-z_0)$, for all $k>\max\{k_0,k_1\}$. Thus, $\sup_{k\in\mathbb{N}}T^k<\infty$. Therefore,

$$\sum_k|x_k|\leq\sup_{k\in\mathbb{N}}(T_k)\sum_k|y_k|<\infty.$$

This shows that $x=(x_k)\in\ell_1$. Thus $A(\widetilde{r},\widetilde{s})-\alpha I)$ is onto. So we have $\alpha\in\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)A_3$.

Theorem 2.10. Let $(r_k),(s_k)\in\mathscr{C}$ with $r_k=r$, $s_k=s$ for all $k\in\mathbb{N}$. Then, the following statements hold:

(i)$\sigma_{ap}(A(\widetilde{r},\widetilde{s}),\ell_1)=\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)$,
(ii)$\sigma_{\delta}(A(\widetilde{r},\widetilde{s}),\ell_1)=\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}$,
(iii)$\sigma_{co}(A(\widetilde{r},\widetilde{s}),\ell_1)=\emptyset$.

Proof. (i) From Table 1, we obtain

$$\sigma_{ap}(A(\widetilde{r},\widetilde{s}),\ell_1)=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)\setminus\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_1.$$

We have by Theorem 2.5

$$\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_1=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_2=\emptyset.$$

Hence;

$$\sigma_{ap}(A(\widetilde{r},\widetilde{s}),\ell_1)=\mathscr{A}.$$

(ii) Since the following equality

$$\sigma_{\delta}(A(\widetilde{r},\widetilde{s}),\ell_1)=\sigma(A(\widetilde{r},\widetilde{s}),\ell_1)\setminus\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)A_3$$

holds from Table 1, we derive by Theorem 2.6 and Theorem 2.9 that $\sigma_{\delta}(A(\widetilde{r},\widetilde{s}),\ell_1)=\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}$.

(iii) From Table 1, we have

$$\begin{aligned}&\sigma_{co}(A(\widetilde{r},\widetilde{s}),\ell_1)\\=\;&\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_1\cup\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_2\cup\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_3\end{aligned}$$

by Theorem 2.3 it is immediate that $\sigma_{co}(A(\widetilde{r},\widetilde{s}),\ell_1)=\emptyset$.

Theorem 2.11. Let $\widetilde{r},\widetilde{s}\in\mathscr{SD}$. Then

$$\begin{aligned}\sigma_{ap}(A(\widetilde{r},\widetilde{s}),\ell_1)&=\mathscr{A}\cup\mathscr{B},\\ \sigma_{\delta}(A(\widetilde{r},\widetilde{s}),\ell_1)&=\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}\cup\mathscr{B},\\ \sigma_{co}(A(\widetilde{r},\widetilde{s}),\ell_1)&=\mathscr{B}.\end{aligned}$$

Proof. We have by Theorem 2.3 and Part (e) of Proposition 1.1 that

$$\sigma_p(A(\widetilde{r},\widetilde{s})^*,\ell_1^*)=\sigma_{co}(A(\widetilde{r},\widetilde{s}),\ell_1)=\mathscr{B}.$$

By Theorem 2.5 and Theorem 2.3, we must have

$$\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_1=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_p\right)C_2=\emptyset.$$

Hence, $\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_3=\{r_k\}$. Therefore, we derive from Table 1, Theorem 2.6 and Theorem 2.9 that

$$\begin{aligned}\sigma_{ap}(A(\widetilde{r},\widetilde{s}),\ell_1)&=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)\setminus\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)C_1\\&=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right),\\ \sigma_{\delta}(A(\widetilde{r},\widetilde{s}),\ell_1)&=\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)\setminus\sigma\left(A(\widetilde{r},\widetilde{s}),\ell_1\right)A_3\\&=\{\alpha\in\mathbb{C}:|r-\alpha|=|s|\}\cup\mathscr{B}.\end{aligned}$$

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References

[1] A.M. Akhmedov, F. Başar, On the spectra of the difference operator Δ over the sequence space ℓ_p, Demonstratio Math. 39(3)(2006), 585–595.

[2] A.M. Akhmedov, F. Başar, On the fine spectra of the difference operator Δ over the sequence space bv_p, $(1\leq p<\infty)$, Acta Math. Sin. Eng. Ser. 23(10)(2007), 1757–1768.

[3] A.M. Akhmedov, S.R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c, Comput. Math. Appl. 61(10)(2011), 2994–3002.

[4] B. Altay, F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, Inform. Sci. 168(2004), 217–224.

[5] B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. 2005: 18(2005), 3005–3013.

[6] A. Karaisa, Fine spectra of upper triangular double-band matrices over the sequence space ℓ_p, $(1<p<\infty)$, Discrete Dyn. Nature Soc. 2012 Article ID 381069 (2012).

[7] M. Altun, On the fine spectra of triangular Toeplitz operators, Appl. Math. Comput.217(20)(2011), 8044–8051.

[8] J. Appell, E. Pascale, A. Vignoli, Nonlinear Spectral Theory, de Gruyter Series in Nonlinear Analysis and Applications 10, Walter de Gruyter · Berlin · New York, 2004.

[9] F. Başar, B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. 55(1)(2003), 136–147.

[10] H. Bilgiç, H. Furkan, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p $(1<p<\infty)$, Nonlinear Anal. 68(3)(2008), 499–506.

[11] H. Bilgiç, H. Furkan, On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv, Math. Comput. Modelling 45(2007), 883–891.

[12] P.J. Cartlidge, Weighted mean matrices as operators on ℓ^p, Ph.D. Dissertation, Indiana University, 1978.

[13] H. Furkan, H. Bilgiç, F. Başar, On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p, $(1<p<\infty)$, Comput. Math. Appl. 60(7)(2010), 2141–2152.

[14] H. Furkan, H. Bilgiç, K. Kayaduman, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv, Hokkaido Math. J. 35(2006), 897–908.

[15] S. Goldberg, Unbounded Linear Operators, Dover Publications, Inc. New York, 1985.

[16] M. Gonzàlez, The fine spectrum of the Cesàro operator in ℓ_p $(1<p<\infty)$, Arch. Math. 44(1985), 355–358.

[17] A. Karaisa, F. Başar, Fine spectra of upper triangular triple-band matrices over the sequence space ℓ_p, $(0<p<\infty)$, Abstr. Appl. Anal. 2013, Article ID 342682, (2013).

[18] A. Karaisa, F. Başar, Fine spectra of upper triangular triple-band matrix over the sequence space ℓ_p where $(0<p<1)$, AIP Conference Proceedings, 1470(2012), 134–137.

[19] V. Karakaya, M. Altun, Fine spectra of upper triangular double-band matrices, J. Comput. Appl. Math. 234 (2010) 1387-1394.
[20] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons Inc. New York · Chichester · Brisbane · Toronto, 1978.
[21] J.I. Okutoyi, On the spectrum of C_1 as an operator on bv_0, J. Austral. Math. Soc. Ser. A. 48(1990), 79–86.
[22] J.I. Okutoyi, On the spectrum of C_1 as an operator on bv, Commun. Fac. Sci. Univ. Ank. Ser. A_1. 41(1992), 197–207.
[23] P.D. Srivastava, S. Kumar, Fine spectrum of the generalized difference operator Δ_v on sequence space ℓ_1, Thai J. Math. 8(2)(2010), 7–19.
[24] M. Yıldırım, On the spectrum of the Rhaly operators on ℓ_p, Indian J. Pure Appl. Math. 32(2001), 191–198.

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