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Estimating the Mean of an AR(1) Process with Infinite Variance

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Abstract: Peng [17] proposed an asymptotically normal estimator of the mean of a heavy tailed distribution with tail index $\alpha > 1$ based on an i.i.d. observations. The goal in this paper is to propose an extension of this estimator which is also asymptotically normal for a sequence $X_1, X_2, \ldots, X_n, \ldots$ resulting from an AR(1) stationary process with common heavy tailed distribution of innovations.

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1 Introduction

The mean is often the parameter of greatest interest during a given statistical analysis for a certain phenomenon. In the classical case when $Z_1, Z_2, \ldots,$ are i.i.d.(independent identically distributed) heavy tailed random variables with tail index satisfying $\alpha > 1$, this location parameter was estimated by many authors. For example, the Peng estimator under tail balance and regularly varying tails conditions is consistent and under a second order regularly varying condition it has a normal limiting distribution (see [17].)

In this paper, we provide an estimator for the mean of an AR(1) stationary time series which have an i.i.d.heavy tailed distribution of innovations and whose the unknown autoregressive coefficient verifies the condition $|\lambda| < 1$. Moreover, one imposes that innovations have to verify the same conditions above and have an non necessarily zero mean. In this situation, the AR(1) process which is a particular case of linear processes, is strictly stationary of causal type; what encourages us to propose an estimator resembling that one of Peng [17] which is also consistent and has a normal limit distribution.

Consider the moving average process of order infinity, written $MA(\infty)$ of the form

$$X_t = \sum_{j=-\infty}^{\infty} c_j \varepsilon_{t-j}, \quad t = 1, 2, \ldots \tag{1}$$

where the filter coefficients c_j and the ε_t satisfy the following assumptions:

Assumption 1*If for* $1 < \alpha < 2$, *we denote by* $G_{|\varepsilon_t|}(x) := P(|\varepsilon_t| \le x) = F_{\varepsilon_t}(x) - F_{\varepsilon_t}(-x), x \in \boldsymbol{R}$, *and let* $T(s) := \inf\{x > 0 : G_{\varepsilon_t}(x) \ge s\}, 0 < s < 1$, *be the quantile function:*

•*the survival function* $1 - G_{\varepsilon_t}$ *is regularly varying at infinity with index* $-\alpha$, *i.e.*

$$\lim_{v \to \infty} \frac{1 - G_{\varepsilon_t}(vx)}{1 - G_{\varepsilon_t}(v)} = x^{-\alpha}, x > 0, \tag{2}$$

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and

$$\lim_{x\to\infty}\frac{1-F_{\varepsilon_t}(x)}{1-G_{\varepsilon_t}(x)}=p,\ p\in[0,1]\ ,\ \lim_{x\to\infty}\frac{F_{\varepsilon_t}(-x)}{1-G_{\varepsilon_t}(x)}=(1-p), \tag{3}$$

•*There exists a function A, not changing sign near zero, such that*

$$\lim_{s\downarrow 0}(A(s))^{-1}\left(\frac{T(1-xs)}{T(1-s)}-x^{-1/\alpha}\right)=x^{-1/\alpha}\frac{x^{\eta}-1}{\eta},\ for\ \ any\ \ x>0, \tag{4}$$

where $\eta\leq 0$ is the second-order parameter. If $\eta=0$, interpret $(x^{\eta}-1)/\eta$ as $\log(x)$.

Assumption 2 $\sum_{j=-\infty}^{\infty}|c_j|^{\delta}<\infty$ *for some $\delta\in]0,\alpha[\cap[0,1]$ to insure that the sum converges almost surly (see [1]).*

*Remark.*Properties (2) and (3) of assumption (1) means that $\{\varepsilon_t\}$ are i.i.d. heavy-tailed random variables with tail index α and which satisfying the second-order condition with η parameter (of second order).

*Remark.*Property (3) of assumption (1) is equivalent to the statement that F_{ε_t} is in the domain of attraction of a stable law and thus which will be known in this work still applies to the α-stable distributions with $\alpha\in]1,2[$.

*Remark.*Assumption (2) insure the existence of process (1) with probability one and its strictly stationarity.

The causal α-stable and stationary processes AR(**p**), MA(**q**) and ARMA(**p**,**q**) are examples for this situation and where $\alpha>1$ is the characteristic exponent.

It's well known that the moving average (1) has the same tail behavior as the innovations ε_t for $t=1,2,\ldots$ Indeed, Mikosch and Samorodnistky [12] proved that

$$\lim_{x\to\infty}\frac{P(X_t>x)}{P(\varepsilon_t>x)}=\frac{1}{p}\sum_{j=-\infty}^{\infty}\left(pc_j^{\alpha}\mathbf{1}_{\{c_j>0\}}+(1-p)|c_j|^{\alpha}\mathbf{1}_{\{c_j<0\}}\right) \tag{5}$$

and

$$\lim_{x\to\infty}\frac{P(X_t<-x)}{P(\varepsilon_t<-x)}=\frac{1}{1-p}\sum_{j=-\infty}^{\infty}\left((1-p)c_j^{\alpha}\mathbf{1}_{\{c_j>0\}}+p|c_j|^{\alpha}\mathbf{1}_{\{c_j<0\}}\right) \tag{6}$$

The rest of this paper is organized as follows. Section 2 contains background material and our theoretical results. Section 3 contains the proofs, in some detail and Section 4 contains some finite-sample simulation studies.

2 Estimating the mean of a stationary AR(1)

In the sequel, we consider the particular case of an AR(1) in the form

$$X_n=\lambda X_{n-1}+\varepsilon_n \tag{7}$$

where $-1<\lambda<1$ is the autoregression coefficient and $\{\varepsilon_n\}$ the sequence of innovations satisfy assumption (1).

In these conditions, the AR(1) is equivalent to process (1) with filter coefficients $c_j=\lambda^j$ for $j=0,1,\ldots$ and $c_j=0$ for $j<0$ and verify assumption (2) for $\delta=1$ and $1<\alpha<2$ and consequently, it is strictly stationary (see [1])).

2.1 Estimating the tail index

There are two possible ways to estimate α:

1.Apply the Hill estimator [6] directly to $|X_1|,\ldots,|X_n|$,i.e

$$1/\widehat{\alpha}_X=\frac{1}{k}\sum_{i=1}^{k}\log^{+}|X_{n-i+1,n}|-\log^{+}|X_{n-k,n}|$$

[1] $\log^{+}(x)=\log(max(x,1))$

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2.Estimate autoregressive coefficient λ with the consistent estimator (see Davis and Resnik [4,5]) :

$$\widehat{\lambda} = \frac{\sum_{i=1}^{n-1}(X_{i+1}-\overline{X}_n)(X_i-\overline{X}_n)}{\sum_{i=1}^{n-1}(X_i-\overline{X}_n)^2} \tag{8}$$

then estimate the residuals

$$\widehat{\varepsilon}_t = X_t - \widehat{\lambda}X_{t-1},\ \ t=2,3,\ldots,n$$

and apply Hill's estimator to the absolute residuals, we get:

$$1/\widehat{\alpha}_{|\widehat{\varepsilon}|} = \frac{1}{k}\sum_{i=1}^{k}\log(|\widehat{\varepsilon}|_{n-i+1,n}) - \log(|\widehat{\varepsilon}|_{n-k,n}),$$

where $\widehat{\varepsilon}_{j,n-1}$ is the jth largest order statistics of the residuals $\widehat{\varepsilon}_t = X_t - \widehat{\lambda}X_{t-1}, 2 \leq t \leq n$ witch are consistent estimators for ε_t.

In general, for an ARMA(**p**) time series, Resnick and Stărică [19] demonstrated that the Hill estimator performs better in the second approach. A similar result was proved by Ling and Peng [9].

2.2 Estimating extreme quantile

We assume that $X_t,\ t=1,2,\ldots$ is the stationary AR(1) defined in (7). For estimate the right extreme quantile $F_{X_t}^{-1}(1-u), 0<u<1$ relation (5) reads as

$$\lim_{x\to\infty}\frac{\overline{F}_{X_t}(x)}{F_{\varepsilon_t}(x)} = \begin{cases}(1+|\lambda|^{\alpha}(1-p)/p)/(1-|\lambda|^{2\alpha}) & ,\ \lambda\in(-1,0),\\ 1/(1-\lambda^{\alpha}) & ,\ \lambda\in[0,1),\end{cases} \tag{9}$$

For simplicity we assume that λ is known if not we can estimate it by relationship (8) and we suppose $0<\lambda<1$, then (9) simplifies to

$$\lim_{x\to\infty}\frac{\overline{F}_{X_t}(x)}{F_{\varepsilon_t}(x)} = 1/(1-\lambda^{\alpha}),$$

which, by the regular variation of $\overline{F}_{\varepsilon_t}$, we obtain the following relationship between the corresponding right quantile functions:

$$\lim_{u\downarrow 0}\frac{F_{X_t}^{-}(1-u)}{F_{\varepsilon_t}^{-}(1-(1-\lambda^{\alpha})u)} = 1.$$

From (6) we find the following relationship between the corresponding left quantile functions as

$$\lim_{u\downarrow 0}\frac{F_{X_t}^{-}(u)}{F_{\varepsilon_t}^{-}((1-\lambda^{\alpha})u)} = 1$$

Then we estimate $F_{X_t}^{-}(1-u)$, and $F_{X_t}^{-}(u)$ as follows:

– Approximate $F_{X_t}^{-}(1-u)$ by $F_{\varepsilon_t}^{-}(1-(1-\lambda^{\alpha})u) \sim F_{\widehat{\varepsilon}_t}^{-}(1-k/n)\left(\frac{n(1-\lambda^{\alpha})u}{k}\right)^{-1/\alpha}$ and estimate the latter by the Weissman estimator [22]

$$\widehat{\varepsilon}_{n-k,n-1}\left(\frac{n(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}})u}{k}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}$$

where

$$1/\widehat{\alpha}_{\widehat{\varepsilon},R} = \frac{1}{k}\sum_{i=1}^{k}\log^{+}(\widehat{\varepsilon}_{n-i,n-1}) - \log^{+}(\widehat{\varepsilon}_{n-k-1,n-1})$$

is the corresponding Hill estimator.

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–Approximate $F_{X_t}^{-}(u)$ by $F_{\varepsilon_t}^{-}((1-\lambda^{\alpha})u) \sim F_{\varepsilon_t}^{-}(k/n)\left(\frac{n(1-\lambda^{\alpha})u}{k}\right)^{-1/\alpha}$ and estimate the latter by the Weissman estimator

$$\widehat{\varepsilon}_{k,n-1}\left(\frac{n(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}})u}{k}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}$$

Where

$$1/\widehat{\alpha}_{\widehat{\varepsilon},L} = \frac{1}{k}\sum_{i=1}^{k} \log^{+}(-\widehat{\varepsilon}_{i,n-1}) - \log^{+}(-\widehat{\varepsilon}_{k,n-1})$$

is the Hill estimator and $\log^{+} x := \max(0, \log(x))$.

2.3 Defining the estimator and main results

To estimate $\mathfrak{m}_X$ the mean of X_t, let $k = k_n$ be sequence of integers satisfying $1 < k < n, k \to \infty, k/n \to 0$. We present the mean of X_t as sum of three integrals as follows

$$\begin{aligned}\mathfrak{m}_X &= \int_0^{k/n} F_{X_t}^{-}(s)ds + \int_{k/n}^{1-k/n} F_{X_t}^{-}(s)ds + \int_{1-k/n}^{1} F_{X_t}^{-}(s)ds \\ &=: I_{L,n} + I_{M,n} + I_{R,n}.\end{aligned} \tag{10}$$

By substituting $\widehat{F}_{X_t}^{-}(s)$ and $\widehat{F}_{X_t}^{-}(1-s)$ for $F_{X_t}^{-}(s)$ and $F_{X_t}^{-}(1-s)$ in $I_{L,n}$ and $I_{R,n}$ respectively, we have for all large n

$$\begin{aligned}\int_0^{k/n} \widehat{F}_{X_t}^{-}(s)ds &= (k/n)^{1/\widehat{\alpha}_{\widehat{\varepsilon},L}}\widehat{\varepsilon}_{k,n-1}\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} \int_0^{k/n} s^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}ds \\ &= (1+o(1))\frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}\widehat{\varepsilon}_{k,n-1}.\end{aligned}$$

and

$$\begin{aligned}\int_0^{k/n} \widehat{F}_{X_t}^{-}(1-s)ds &= (k/n)^{1/\widehat{\alpha}_{\widehat{\varepsilon},R}}\widehat{\varepsilon}_{n_k,n-1}\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}} \int_0^{k/n} s^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}ds \\ &= (1+o(1))\frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}\widehat{\varepsilon}_{n-k,n-1}.\end{aligned}$$

Thus, we can estimate $I_{L,n}$ and $I_{R,n}$ by

$$\widehat{I}_{L,n} = \frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}\widehat{\varepsilon}_{k,n-1}, \tag{11}$$

and

$$\widehat{I}_{R,n} = \frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}\widehat{\varepsilon}_{n-k,n-1}, \tag{12}$$

respectively. We take the sample one to estimate the $I_{M,n}$, that is

$$\begin{aligned}\widehat{I}_{M,n} &= \int_{k/n}^{1-k/n} \widehat{F}_{X_t}^{-}(s)ds \\ &= n^{-1}\sum_{i=k+1}^{n-k} X_{i,n}\end{aligned} \tag{13}$$

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our estimator becomes

$$\begin{aligned}\widehat{\mathfrak{m}}_X &= \frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},L}}\widehat{\varepsilon}_{k,n-1} \\ &+n^{-1}\sum_{i=k+1}^{n-k} X_{i,n} \\ &+\frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}}{1-1/\widehat{\alpha}_{\widehat{\varepsilon},R}}\widehat{\varepsilon}_{n-k,n-1}\end{aligned} \tag{14}$$

Theorem 1. *Assume that F_{ε_t} have the α-stable marginal distribution with $\alpha \in]1,2[$, and for any sequence of integer k such that $1<k<n, k\to\infty, k/n\to 0$, as $n\to\infty$, there exists a probability space $(\Omega,\mathscr{A},P)$ carrying the sequence $X_1,X_2,\ldots$ and a sequence of Brownian bridges $(B_n(s), 0\le s\le 1, n=1,2,\ldots)$ such that we have for all large n*

$$\frac{\sqrt{n}\left(\widehat{I}_{M,n}-I_{M,n}\right)}{\sigma_n} = -\frac{\int_{k/n}^{1-k/n} B_n(s)dF_{\varepsilon_t}^{-}(s)}{(1-\lambda)\sigma_n} + o_P(1),$$

and therefore

$$\frac{\sqrt{n}\left(\widehat{I}_{M,n}-I_{M,n}\right)}{\sigma_n} \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \frac{1}{(1-\lambda)^2}\right) \text{ as } n\to\infty,$$

with

$$\sigma_n^2 = \int_{k/n}^{1-k/n}\int_{k/n}^{1-k/n}(\min(s,t)-st)dF_{\varepsilon_t}^{-}(s)dF_{\varepsilon_t}^{-}(t),$$

The asymptotic normality of our estimator is established in the following theorem.

Theorem 2. *Assume that F_{ε_t} have the α-stable marginal distribution with $\alpha \in]1,2[$, and for any sequence of integer k such that $1<k<n, k\to\infty, k/n\to 0$, and $\sqrt{k}A(k/n)\to 0$ as $n\to\infty$, we have*

$$\frac{\sqrt{n}(\widehat{\mathfrak{m}}_X-\mathfrak{m}_X)}{\sigma_n} \xrightarrow{\mathscr{D}} \mathscr{N}(0,\sigma_0^2), \text{ as } n\to\infty,$$

where

$$\begin{aligned}\sigma_0^2 = (2-\alpha)\Bigg[&(1-\lambda^{\alpha})^{-2/\alpha}\left(\frac{2\alpha^2-2\alpha+1}{2(\alpha-1)^4}\right) \\ &+\frac{(1-\lambda^{\alpha})^{-1/\alpha}}{(1-\lambda)(\alpha-1)}\Bigg]+\frac{1}{(1-\lambda)^2}\end{aligned}$$

3 Proofs

Lemma 1. *For any integer k such that $1<k<n$ and for any j with $0\le j\le m$ one has*

$$n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i-j,n} = o_P(1) + n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n} \tag{15}$$

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3.0.1 Proof of lemma (1)

Setting $r = i - k$ and $l = n - 2k$ then by Davis and Resnick, 1985, p.190 we have :

$$\begin{aligned} n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i-j,n} &= n^{-1}\sum_{r=1}^{l}\varepsilon_{r+k-j,n} \\ &= o_P(1) + n^{-1}\sum_{r=1}^{l}\varepsilon_{r+k,n} \\ &= o_P(1) + n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n} \end{aligned}$$

Lemma 2. *For any integer k such that $1 < k < n$ one has*

$$n^{-1}\sum_{i=k+1}^{n-k}\sum_{j=0}^{\infty}\lambda^j\varepsilon_{i-j,n} = o_P(1) + \frac{1}{1-\lambda}\left(n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n}\right)$$

3.0.2 Proof of lemma (2)

From lemma 1 if we take the dot product between the vectors $(n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i-j,n})_{j=0}^{m}$ and $(\lambda^j)_{j=0}^{m}$ (asymptotic equivalence is preserved because dot product is continuous mapping), we find

$$n^{-1}\sum_{i=k+1}^{n-k}\sum_{j=0}^{m}\lambda^j\varepsilon_{i-j,n} = o_P(1) + n^{-1}\left(\sum_{j=0}^{m}\lambda^j\right)\sum_{i=k+1}^{n-k}\varepsilon_{i,n}$$

We let $m \to \infty$ on both sides to have the statement of lemma 2.

3.0.3 Proof of Theorem (1)

Now, from lemma 2

$$n^{-1}\sum_{i=k+1}^{n-k}X_{i,n} = n^{-1}\sum_{i=k+1}^{n-k}\sum_{j=0}^{\infty}\lambda^j\varepsilon_{i-j,n} = \frac{1}{1-\lambda}\left(n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n}\right) + o_p(1)$$

using expectations we have

$$\int_{k/n}^{1-k/n}F_{X_t}^{-}(u)du = \frac{1}{1-\lambda}\int_{k/n}^{1-k/n}F_{\varepsilon_t}^{-}(u)du$$

hence :

$$n^{-1}\sum_{i=k+1}^{n-k}X_{i,n} - \int_{k/n}^{1-k/n}F_{X_t}^{-}(u)du = \frac{1}{1-\lambda}\left[n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n} - \int_{k/n}^{1-k/n}F_{\varepsilon_t}^{-}(u)du\right] + o_p(1)$$

by Csörgő & al. [3] in theorem(5), we have :

$$\left\{\frac{\sqrt{n}}{\sigma_n}\left(n^{-1}\sum_{i=k+1}^{n-k}\varepsilon_{i,n} - \int_{k/n}^{1-k/n}F_{\varepsilon_t}^{-}(u)du\right)\right\} = \left\{-\frac{\int_{k/n}^{1-k/n}B_n(s)dF_{\varepsilon_t}^{-}(s)}{\sigma_n} + o_P(1)\right\}$$

from where the result :

$$\frac{\sqrt{n}\left(\widehat{I}_{M,n} - I_{M,n}\right)}{\sigma_n} = -\frac{\int_{k/n}^{1-k/n}B_n(s)dF_{\varepsilon_t}^{-}(s)}{(1-\lambda)\sigma_n} + o_P(1).$$

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3.1 Proof of Theorem (2)

Recall (10), (11), (12) and (13) and write

$$\widehat{\mathfrak{m}}_X - \mathfrak{m}_X = \left(\widehat{I}_{L,n} - I_{L,n}\right) + \left(\widehat{I}_{M,n} - I_{M,n}\right) + \left(\widehat{I}_{R,n} - I_{R,n}\right).$$

We have

$$\begin{aligned}\widehat{I}_{L,n} - I_{L,n} &= \frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} \widehat{\varepsilon}_{k,n-1}\widehat{\alpha}_{\widehat{\varepsilon},L}}{\widehat{\alpha}_{\widehat{\varepsilon},L}-1} - \int_0^{k/n} F_{\varepsilon_t}^{-}((1-\lambda^{\alpha})s)ds \\ &= K_1^L + K_2^L + K_3^L,\end{aligned}$$

where

$$K_1^L = (k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} \widehat{\varepsilon}_{k,n-1}\left\{\frac{\widehat{\alpha}_{\widehat{\varepsilon},L}}{\widehat{\alpha}_{\widehat{\varepsilon},L}-1} - \frac{\alpha}{\alpha-1}\right\}$$

$$K_2^L = \frac{\alpha(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} F_{\varepsilon_t}^{-}(k/n)}{\alpha-1}\left\{\frac{\widehat{\varepsilon}_{k,n-1}}{F_{\varepsilon_t}^{-}(k/n)} - 1\right\},$$

$$K_3^L = \frac{\alpha(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} F_{\varepsilon_t}^{-1}(k/n)}{\alpha-1} - \int_0^{k/n} F_{X_t}^{-}(s)ds.$$

Likewise we have

$$\begin{aligned}\widehat{I}_{R,n} - I_{R,n} &= \frac{(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}} \widehat{\varepsilon}_{n-k,n-1}\widehat{\alpha}_{\widehat{\varepsilon},R}}{\widehat{\alpha}_{\widehat{\varepsilon},R}-1} \\ &\quad - \int_0^{k/n} F_{\varepsilon_t}^{-}(1-(1-\lambda^{\alpha})s)ds \\ &= H_1^R + H_2^R + H_3^R,\end{aligned}$$

where

$$H_1^R = (k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}} \widehat{\varepsilon}_{n-k,n-1}\left\{\frac{\widehat{\alpha}_{\widehat{\varepsilon},R}}{\widehat{\alpha}_{\widehat{\varepsilon},R}-1} - \frac{\alpha}{\alpha-1}\right\}$$

$$H_2^R = \frac{\alpha(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}} F_{\varepsilon_t}^{-}(1-k/n)}{\alpha-1}\left\{\frac{\widehat{\varepsilon}_{n-k,n-1}}{F_{\varepsilon_t}^{-}(1-k/n)} - 1\right\},$$

$$H_3^R = \frac{\alpha(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},R}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},R}} F_{\varepsilon_t}^{-}(1-k/n)}{\alpha-1} - \int_{1-k/n}^{1} F_{X_t}^{-}(s)ds.$$

We can write K_1^L as

$$K_1^L = \frac{\widehat{\alpha}_{\widehat{\varepsilon},L}\,\alpha\,(k/n)\left(1-\lambda^{\widehat{\alpha}_{\widehat{\varepsilon},L}}\right)^{-1/\widehat{\alpha}_{\widehat{\varepsilon},L}} \widehat{\varepsilon}_{k,n-1}}{(\widehat{\alpha}_{\widehat{\varepsilon},L}-1)(\alpha-1)}\left(\frac{1}{\widehat{\alpha}_{\widehat{\varepsilon},L}} - \frac{1}{\alpha}\right).$$

Since $\widehat{\alpha}_{\widehat{\varepsilon},L}$ and $\widehat{\varepsilon}_{k,n-1}$ are consistent estimators of α and $\varepsilon_{k,n-1}$ respectively, then for all large n

$$K_1^L = (1+o_P(1))\frac{\alpha^2\,(k/n)(1-\lambda^{\alpha})^{-1/\alpha}\;\varepsilon_{k,n-1}}{(\alpha-1)^2}\left(\frac{1}{\widehat{\alpha}_{\widehat{\varepsilon},L}} - \frac{1}{\alpha}\right).$$

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and

$$K_2^L = (1+o_P(1))\frac{\alpha(k/n)(1-\lambda^\alpha)^{-1/\alpha}F_{\varepsilon_t}^-(k/n)}{\alpha-1}\left\{\frac{\varepsilon_{k,n-1}}{F_{\varepsilon_t}^-(k/n)}-1\right\}$$
$$= (1+o_P(1))\frac{\alpha(k/n)(1-\lambda^\alpha)^{-1/\alpha}F_{\varepsilon_t}^-(k/n)}{\alpha-1}\left\{\frac{\varepsilon_{k,n-1}}{F_{\varepsilon_t}^-(k/n)}-1\right\}$$

and

$$K_3^L = (1+o_P(1))\frac{\alpha(k/n)(1-\lambda^\alpha)^{-1/\alpha}F_{\varepsilon_t}^-(k/n)}{\alpha-1} - \int_0^{k/n} F_{X_t}^-(s)ds.$$

In view of Theorems 2.3 and 2.4 of Csörgő and Mason [2], Peng [18], and Necir& al. [13] it has been shown that under second-order condition (4) and for all large n,

$$\sqrt{k}\alpha\left(\frac{1}{\widehat{\alpha}_{\widehat{\varepsilon},L}}-\frac{1}{\alpha}\right) = -\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds+o_P(1),$$

$$\sqrt{k}\left(\frac{\varepsilon_{k,n-1}}{F_{\varepsilon_t}^-(k/n)}-1\right) = -\alpha^{-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_P(1),$$

and

$$\frac{\varepsilon_{k,n-1}}{F_{\varepsilon_t}^-(k/n)} = 1+o_P(1)$$

where $\{B_n(s), 0 \le s \le 1, n = 1,2,\ldots\}$ is the sequence of Brownian bridges defined in Theorem (1) This implies that for all large n

$$K_1^L = (1+o_P(1))\frac{\alpha\ (k^{1/2}/n)(1-\lambda^\alpha)^{-1/\alpha}F_{\varepsilon_t}^-(k/n)}{(\alpha-1)^2}$$
$$\times\left(-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds+o_P(1)\right),$$

$$K_2^L = (1+o_P(1))\frac{\alpha\ (k^{1/2}/n)(1-\lambda^\alpha)^{-1/\alpha}F_{\varepsilon_t}^-(k/n)}{\alpha-1}\left(-\alpha^{-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_P(1)\right).$$

Then by the Lemma 3.5 and Lemma 3.6 of Necir and Meraghni [16], we get for all large n

$$\frac{\sqrt{n}(K_1^L+K_2^L)}{\sigma_n} = -\frac{(1-\lambda^\alpha)^{-1/\alpha}\ \alpha\ \omega}{(\alpha-1)^2}\times\left(-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right)$$
$$-\frac{(1-\lambda^\alpha)^{-1/\alpha}\ \omega}{\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_P(1), \tag{16}$$

where

$$\omega^2 = \lim_{n\to\infty}\frac{(k/n)(F_{\varepsilon_t}^-(k/n))^2}{\sigma_n^2}$$
$$= \frac{(2-\alpha)}{4}$$

By the same arguments, we show that for all large n

$$\frac{\sqrt{n}(H_1^R+H_2^R)}{\sigma_n} = \frac{(1-\lambda^\alpha)^{-1/\alpha}\ \alpha\ \omega}{(\alpha-1)^2}\times\left(\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)-\sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds\right)$$
$$+\frac{(1-\lambda^\alpha)^{-1/\alpha}\ \omega}{\alpha-1}\left(-\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)\right)+o_P(1), \tag{17}$$

Similar arguments as those used in the proof of Theorem 1 by Necir et al. [13], yield that

$$\frac{\sqrt{n}K_3^L}{\sigma_n} = \frac{\sqrt{n}H_3^R}{\sigma_n} = o(1)\ \ as\ \ n\to\infty. \tag{18}$$

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Then, by (16), (17) and (18) we get

$$\frac{\sqrt{n}(\widehat{\mathfrak{m}_X}-\mathfrak{m}_X)}{\sigma_n} = o_P(1) - \frac{(1-\lambda^{\alpha})^{-1/\alpha}\,\alpha\,\omega}{(\alpha-1)^2} \times \left(-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right)$$

$$-\frac{(1-\lambda^{\alpha})^{-1/\alpha}\,\omega}{\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_P(1) - \frac{\int_{k/n}^{1-k/n}B_n(s)ds}{(1-\lambda)\sigma_n}$$

$$+\frac{(1-\lambda^{\alpha})^{-1/\alpha}\,\alpha\,\omega}{(\alpha-1)^2} \times \left(\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right) - \sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds\right)$$

$$+\frac{(1-\lambda^{\alpha})^{-1/\alpha}\,\omega}{\alpha-1}\left(-\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)\right)$$

Table 1: For $\lambda = 0.2$ as autoregressive parameter and $\mathfrak{m}_X = 5$ theoretical mean of the AR(1).

n	$\widehat{\mathfrak{m}}_X$	l_b	u_b	length	Cov Prob	$\overline{X}$
1000	4.939398	3.413290	6.465505	3.052215	0.91	4.986124
2000	4.874174	3.836672	5.911676	2.075004	0.86	4.955199
3000	4.984762	4.072243	5.897282	1.825039	0.85	5.351354
5000	5.071300	4.351357	5.791244	1.439887	0.84	5.017171
10000	5.040796	4.496334	5.585258	1.088924	0.74	5.068563

Table 2: For $\lambda = 0.5$ as autoregressive parameter and $\mathfrak{m}_X = 8$ theoretical mean of the AR(1).

n	$\widehat{\mathfrak{m}}_X$	l_b	u_b	length	Cov Prob	$\overline{X}$
1000	8.328174	6.273375	10.382970	4.109599	0.96	7.689167
2000	7.810230	6.412216	9.208243	2.796028	0.92	8.130348
3000	7.977200	6.784955	9.169445	2.384491	0.83	8.174059
5000	8.260606	7.273002	9.248210	1.975208	0.79	8.082490
10000	8.032830	7.299913	8.765747	1.465835	0.79	8.452839

Table 3: For $\lambda = 0.7$ as autoregressive parameter and $\mathfrak{m}_X = 13.3333$ theoretical mean of the AR(1).

n	$\widehat{\mathfrak{m}}_X$	l_b	u_b	length	Cov Prob	$\overline{X}$
1000	13.42359	10.57943	16.26775	5.688324	0.91	13.66918
2000	12.98009	10.89539	15.06480	4.169401	0.78	13.07235
3000	13.16939	11.43796	14.90082	3.462863	0.76	13.27977
5000	14.02810	12.63599	15.42021	2.784219	0.88	13.39029
10000	13.26368	12.23761	14.28976	2.052147	0.76	13.31790

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The asymptotic variance of $\dfrac{\sqrt{n}(\widehat{\mathfrak{m}}_X - \mathfrak{m}_X)}{\sigma_n}$ will be computed by

$$
\begin{aligned}
\sigma_0^2 = \lim_{n\to\infty} \Bigg\{ & (1-\lambda^\alpha)^{-2/\alpha} \frac{\omega^2\alpha^2}{(\alpha-1)^4} \frac{n}{k} \int_0^{k/n} ds \int_0^{k/n} \frac{\min(s,t)-st}{st} dt \\
& + (1-\lambda^\alpha)^{-2/\alpha} \frac{\omega^2}{(\alpha-1)^4} \frac{n}{k} \frac{k}{n} (1-k/n) \\
& + \frac{\int_{k/n}^{1-k/n} dF_\varepsilon^-(s) \int_{k/n}^{1-k/n} (\min(s,t)-st) dF_{\varepsilon_t}^-(s)}{(1-\lambda)^2 \sigma_n} \\
& + (1-\lambda^\alpha)^{-2/\alpha} \frac{\omega^2}{(\alpha-1)^4} \frac{n}{k} \frac{k}{n} (1-k/n) \\
& + (1-\lambda^\alpha)^{-2/\alpha} \frac{\omega^2\alpha^2}{(\alpha-1)^4} \frac{n}{k} \int_{1-k/n}^{1} ds \int_{1-k/n}^{1} \frac{\min(s,t)-st}{st} dt \\
& - (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2\alpha}{(\alpha-1)^4} \frac{n}{k} \int_0^{k/n} \frac{t-(k/n)t}{t} dt \\
& + \frac{(1-\lambda^\alpha)^{-1/\alpha}}{(1-\lambda)} \frac{2\omega\alpha}{(\alpha-1)^2} \sqrt{\frac{n}{k}} \int_0^{k/n} ds \int_{k/n}^{1-k/n} \frac{s-st}{s} dF_{\varepsilon_t}^-(t)/\sigma_n \\
& - (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2\alpha}{(\alpha-1)^4} \sqrt{\frac{n}{k}} \sqrt{\frac{n}{k}} \int_0^{k/n} \frac{s-(1-k/n)s}{s} ds \\
& + (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2\alpha^2}{(\alpha-1)^4} \sqrt{\frac{n}{k}} \sqrt{\frac{n}{k}} \int_0^{k/n} ds \int_{1-k/n}^{1} \frac{\min(s,t)-st}{st} dt \\
& - \frac{(1-\lambda^\alpha)^{-1/\alpha}}{(1-\lambda)} \frac{2\omega}{(\alpha-1)^2} \sqrt{\frac{n}{k}} \int_{k/n}^{1-k/n} \left(\frac{k}{n} - s\frac{k}{n}\right) dF_{\varepsilon_t}^-(t)/\sigma_n \\
& + (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2}{(\alpha-1)^4} \sqrt{\frac{n}{k}} \sqrt{\frac{n}{k}} \left(\frac{k}{n} - \frac{k}{n}\left(1-\frac{k}{n}\right)\right) \\
& - (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2\alpha}{(\alpha-1)^4} \int_{1-k/n}^{1} \frac{k/n-(k/n)s}{1-s} ds \\
& - \frac{(1-\lambda^\alpha)^{-1/\alpha}}{(1-\lambda)} \frac{2\omega}{(\alpha-1)^2} \sqrt{\frac{n}{k}} \int_{k/n}^{1-k/n} \left(s - s\left(1-\frac{k}{n}\right)\right) dF_{\varepsilon_t}^-(t)/\sigma_n \\
& + \frac{(1-\lambda^\alpha)^{-1/\alpha}}{(1-\lambda)} \frac{2\omega\alpha}{(\alpha-1)^2} \sqrt{\frac{n}{k}} \int_{1-k/n}^{1} ds \int_{k/n}^{1-k/n} \frac{t-st}{1-s} dF_{\varepsilon_t}^-(t)/\sigma_n \\
& - (1-\lambda^\alpha)^{-2/\alpha} \frac{2\omega^2\alpha}{(\alpha-1)^4} \frac{n}{k} \int_{1-k/n}^{1} \frac{(1-k/n)-(1-k/n)s)}{1-s} ds \Bigg\},
\end{aligned}
$$

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After calculation we get

$$\begin{aligned}
\sigma_0^2 &= (1-\lambda^{\alpha})^{-2/\alpha}\frac{2\omega^2\alpha^2}{(\alpha-1)^4} + (1-\lambda^{\alpha})^{-2/\alpha}\frac{\omega^2}{(\alpha-1)^4} + \frac{1}{(1-\lambda)^2} \\
&\quad + (1-\lambda^{\alpha})^{-2/\alpha}\frac{\omega^2}{(\alpha-1)^4} + (1-\lambda^{\alpha})^{-2/\alpha}\frac{2\omega^2\alpha^2}{(\alpha-1)^4} \\
&\quad - (1-\lambda^{\alpha})^{-2/\alpha}\frac{2\omega^2\alpha}{(\alpha-1)^4} + \frac{(1-\lambda^{\alpha})^{-1/\alpha}\,2\omega^2\alpha}{(1-\lambda)(\alpha-1)^2} \\
&\quad - \frac{(1-\lambda^{\alpha})^{-1/\alpha}\,2\omega^2}{(1-\lambda)(\alpha-1)^2} - \frac{(1-\lambda^{\alpha})^{-1/\alpha}\,2\omega^2}{(1-\lambda)(\alpha-1)^2} \\
&\quad + \frac{(1-\lambda^{\alpha})^{-1/\alpha}\,2\omega^2\alpha}{(1-\lambda)(\alpha-1)^2} - (1-\lambda^{\alpha})^{-2/\alpha}\frac{2\omega^2\alpha}{(\alpha-1)^4} \\
&= (2\omega^2)\left[(1-\lambda^{\alpha})^{-2/\alpha}\left(\frac{2\alpha^2+1-2\alpha}{(\alpha-1)^4}\right)\right. \\
&\qquad \left. + \frac{2\,(1-\lambda^{\alpha})^{-1/\alpha}}{(1-\lambda)(\alpha-1)}\right] + \frac{1}{(1-\lambda)^2} \\
&= \left(\frac{2-\alpha}{2}\right)\left[(1-\lambda^{\alpha})^{-2/\alpha}\left(\frac{2\alpha^2+1-2\alpha}{(\alpha-1)^4}\right)\right. \\
&\qquad \left. + \frac{2\,(1-\lambda^{\alpha})^{-1/\alpha}}{(1-\lambda)(\alpha-1)}\right] + \frac{1}{(1-\lambda)^2}.
\end{aligned}$$

This completes the proof of Theorem (2).

4 Simulation

In what follows we simulate samples of various sizes ($n = 1000; 2000, 3000, 5000$ and 10000 observations) and for one hundred replications ($r = 100$) of each one generated by an AR(1) α-stable with three values for autoregressive coefficient $\lambda = 0.2$, 0.5 and 0.7 driven by $S(\alpha, m, \beta, \sigma)$ innovations with characteristic exponent $\alpha = 1.3$, mean $m = 4$, skewness $\beta = 0$ and scale $\sigma = 0.5$.

The simulation results are presented in the table1, table2 and table3 where $\widehat{\mathfrak{m}}_X$, lb, ub are averages for estimated mean of X_t, lower bound and upper bound of 95% confidence intervals respectively. In addition, we indicate coverage probabilities and lengths of these intervals. Finally and for comparison, we indicate in each case the average of sample means $\overline{X}$.

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References

[1] Brockwell, P. and Davis, R., Time Series: Theory and Methods, 2nd edition, Springer-Verlag, New York, 1991.

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[2] Csörgő, M., Mason, D. M., (1985). Central limit theorems for sums of extreme values. Mathematical Proceedings of the Cambridge Philosophical Society. 98, 547-558.
[3] Csörgő. S, Haeusler. E, Mason. D. M, (1988). The Asymptotic Distribution of Trimmed Sums. Ann. Probab. 16, 672-699.
[4] Davis, R. Resnick, S., (1985). Limit theory for moving averages of random variables with regulary varying tai probabilities. Ann. Probab. 13,179-195.
[5] Davis, R. Resnick, S., (1986). Limit theory for the sample covariance and correlation functions of moving averages. Ann. Statist. 14,533-558.
[6] Hill, B. M. (1975). A simple approach to inference about the tail of a distribution. Ann. Statist. 3, 1136-1174.
[7] Jones, B.L., Zitikis, R., (2003). Empirical estimation of risk measures and related quantities. North American Actuarial Journal.7, 44-54.
[8] Loretan, M. and Phillips, P. Testing the covariance stationarity of heavy-tailed time series. Journal of Empirical Finance 1 (1994) 21 l-248.
[9] Ling, S. and Peng L., (2004): Hill's estimator for the tail index of an ARMA model. J. Stat. Plan. Inference. 123, 279-293.
[10] McCulloch, J.H. (1986) Simple Consistent Estimators of Stable Distribution Parameters. Commun. Statist.-Simula., **15**(4) ,1109-1136.
[11] Mason, D. (1982). Laws of the large numbers for sums of extreme values. Ann. Probab. 10, 754-764.
[12] Mikosch, T., Samorodnitsky, G., (2000). The supremum of a negative drift random walk with dependent heavy-tailed steps. Ann. Appl. Probab. 10, 1025-1064.
[13] Necir, A., Meraghni, D., and Meddi, F., (2007). Statistical estimate of the proportional hazard premium of loss. Scandinavian Actuarial Journal. 3, 147-161.
[14] Necir, A. and Meraghni, D., (2009). Empirical Estimation of the Proportional Hazard Premium for Heavy-Tailed Claim Amounts, Insurance Math. Econom. 45, 49-58.
[15] Necir, A., Rassoul, A., Zitikis, R. (2010). Estimating the conditional tail expectation in the case of heavy-tailed losses. Journal of Probability and Statistics 2010, ID 956838.
[16] Necir, A., Meraghni, D., (2010). Estimating L-functionals for heavy-tailed distributions and applications. Journal of Probability and Statistics 2010, ID 707146.
[17] Peng, L. (2001). Estimating the mean of a heavy tailed distribution. Statistics and Probability Letters 52, 255-264.
[18] Peng, L., (2004). Empirical-likelihood-based confidence interval for the mean with a heavy-tailed distribution. Ann. Statist. 32(3), 1192-1214.
[19] Resnick, S., Stărică, C., (1997): Asymptotic behavior of Hill's estimator for autoregressive data. Comm. Stat. Stoch. Models 13, 703721.
[20] Samorodnitsky, G. and Taqqu, M., (1994). Stable Non-Gaussian Random Processes. New York: Chapman and Hall.
[21] Stigler, S. M., (1974). Linear functions of order statistics with smooth weight functions. Ann. Statist. 2, 676-693.
[22] Weissman,I., (1978). Estimation of parameters and large quantiles based on the k largest observations. Journal of American Statistical Association. 73, 812-815.

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