

Composita and its Properties

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Abstract: In this paper we study the coefficients of the powers of an ordinary generating function and their properties. A new class of functions based on compositions of an integer n is introduced and is termed composita. We present theorems about compositae and operations with compositae. We obtain the compositae of polynomials, trigonometric and hyperbolic functions.

Keywords: Generating function, composition of generating functions, composita

1 Introduction

The computations based on combinatorial objects are an important direction of research in enumerative combinatorics and related fields of mathematics. For example, ordered partitions of a finite set is used to derive the formula for a composition of exponential generating functions [1]. Computations that use compositions of an integer n are found in various problems: derivation of a convolution of convolutions [2], composition of ordinary generating functions [3] that allow us to obtain many properties of polynomials [4,5], calculation of the n -th order derivatives of a composite function [6], generation of ordered root trees [7], etc. However, there is no unified approach to solving problems based on compositions.

In this work, we consider a unified approach to the above problems, using a special function termed a *composita*. The notion of the composita is close to that of a Riordan array [8,9], but the composita characterizes only one function, and potential polynomials for exponential generating functions [10].

Most of all papers and books related to combinatorial problems and generating functions use coefficients of the powers of an ordinary generating function [1,3,10,11]. However, as an independent object of study this has not considered. So investigation of the coefficients of the powers of an ordinary generating function is very important.

2 Composita

Now we introduce the definition of composita.

Definition 1. The composita of the generating function $F(x) = \sum_{n>0} f(n)x^n$ is the function of two variables

$$F^\Delta(n, k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \dots f(\lambda_k), \quad (1)$$

where C_n is a set of all compositions of an integer n , π_k is the composition n into k parts such that $\sum_{i=1}^k \lambda_i = n$.

It follows from the definition of a composita that it is defined for a generating function $F(x)$ for which $f(0) = 0$. Let us consider a generating function $F(x) = \frac{x}{1-x} = \sum_{n>0} x^n$. On the strength of formula (1), the composita of this function is

$$F^\Delta(n, k) = \binom{n-1}{k-1}.$$

For all $n > 0$ we have $f(n) = 1$; therefore, the formula (1) counts the number of compositions of n into k parts.

Next we obtain a recurrent formula for the composita of a generating function.

Theorem 1. For the composita $F^\Delta(n, k)$ of the generating function $F(x) = \sum_{n>0} f(n)x^n$ the following recurrent relation holds true

$$F^\Delta(n, k) = \begin{cases} f(n), & \text{if } k = 1; \\ \sum_{i=1}^{n-k+1} f(i) F^\Delta(n-i, k-1), & \text{if } k \leq n. \end{cases} \quad (2)$$

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*Proof.*Raising the generating function $A(x)$ to the power of k and applying the binomial theorem, we obtain

$$A(x)^k = [B(x) - b(0)]^k = \sum_{j=0}^k \binom{k}{j} B(x)^j (-1)^{k-j} b(0)^{k-j}.$$

From

$$[B(x)]^k = \sum_{n \geq 0} B(n, k) x^n,$$

and $B(x)^0 = 1$, we have

$$B(n, 0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases}$$

Since $A(x) = \sum_{n > 0} a(n)x^n$, we get

$$A^\Delta(n, k) = \sum_{j=1}^k \binom{k}{j} B(n, j) (-1)^{k-j} b(0)^{k-j}.$$

Theorem 4. Suppose $F(x) = \sum_{n > 0} f(n)x^n$ is the generating function, $F^\Delta(n, k)$ is the composita of this generating function, and α is constant. Then for the generating function $A(x) = \alpha F(x)$ the composita is equal to

$$A^\Delta(n, k) = \alpha^k F^\Delta(n, k). \tag{8}$$

Proof. Using (3), we get

$$\begin{aligned} [A(x)]^k &= [\alpha F(x)]^k = \alpha^k [F(x)]^k = \\ &= \sum_{n \geq k} \alpha^k F^\Delta(n, k) x^n = \sum_{n \geq k} A^\Delta(n, k) x^n. \end{aligned}$$

Therefore,

$$A^\Delta(n, k) = \alpha^k F^\Delta(n, k).$$

Theorem 5. Suppose $F(x) = \sum_{n > 0} f(n)x^n$ is the generating function, $F^\Delta(n, k)$ is the composita of this generating function, and α is constant. Then for the generating function $A(x) = F(\alpha x)$ the composita is equal to

$$A^\Delta(n, k) = \alpha^n F^\Delta(n, k). \tag{9}$$

Proof. Using (3), we get

$$\begin{aligned} [A(x)]^k &= [F(\alpha x)]^k = \sum_{n \geq k} F^\Delta(n, k) (\alpha x)^n = \\ &= \sum_{n \geq k} \alpha^n F^\Delta(n, k) x^n = \sum_{n \geq k} A^\Delta(n, k) x^n. \end{aligned}$$

Therefore,

$$A^\Delta(n, k) = \alpha^n F^\Delta(n, k).$$

Theorem 6. Suppose we have the generating function $F(x) = \sum_{n > 0} f(n)x^n$, the composita of this generating function $F^\Delta(n, k)$; the following generating functions $B(x) = \sum_{n \geq 0} b(n)x^n$ and $[B(x)]^k = \sum_{n \geq 0} B(n, k)x^n$. Then for the generating function $A(x) = F(x)B(x)$ the composita is equal to

$$A^\Delta(n, k) = \sum_{i=k}^n F^\Delta(i, k) B(n-i, k). \tag{10}$$

Proof. Since $a(0) = f(0)b(0) = 0$, the function $A(x)$ has the composita $A^\Delta(n, k)$.

Using (3), we get

$$[A(x)]^k = [F(x)]^k [B(x)]^k.$$

Then, from the rule of product of generating functions, we have

$$A^\Delta(n, k) = \sum_{i=k}^n F^\Delta(i, k) B(n-i, k).$$

Corollary 3. If for the generating function $B(x)$ we have $b(0) = 0$, then the formula (10) takes the form

$$A^\Delta(n, k) = \sum_{i=k}^{n-k} F^\Delta(i, k) B^\Delta(n-i, k). \tag{11}$$

Theorem 7. Suppose we have the generating functions $F(x) = \sum_{n > 0} f(n)x^n$, $G(x) = \sum_{n > 0} g(n)x^n$, and their compositae $F^\Delta(n, k)$, $G^\Delta(n, k)$ respectively. Then for the generating function $A(x) = F(x) + G(x)$ the composita is equal to

$$\begin{aligned} A^\Delta(n, k) &= F^\Delta(n, k) + G^\Delta(n, k) + \\ &+ \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^\Delta(i, j) G^\Delta(n-i, k-j). \end{aligned} \tag{12}$$

Proof. Using (3) and the binomial theorem, we get

$$[A(x)]^k = \sum_{j=0}^k \binom{k}{j} [F(x)]^j [G(x)]^{k-j}.$$

Note that

$$[F(x)]^j = \sum_{n \geq j} F^\Delta(n, j) x^n,$$

and

$$[G(x)]^{k-j} = \sum_{n \geq k-j} G^\Delta(n, k-j) x^n.$$

Then, from $F(x)^0 = 1$, $G(x)^0 = 1$ and the rule of product of generating functions, we have

$$\begin{aligned} A^\Delta(n, k) &= F^\Delta(n, k) + G^\Delta(n, k) + \\ &+ \sum_{j=1}^{k-1} \binom{k}{j} \sum_{i=j}^{n-k+j} F^\Delta(i, j) G^\Delta(n-i, k-j). \end{aligned}$$

Remark. For the case $k = 0$, we have $F(x)^0 = 1$. It is mean that

$$F^\Delta(n, 0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases} \tag{13}$$

4 Compositae of generating functions

In this section we consider several examples of computation of compositae.

For derivation of a composita of the generating function $F(x) = \sum_{n>0} f(n)x^n$, we have to find coefficients of the generating function $F(x)^k$. As an example, in Table 1 we present compositae of several known generating functions [1, 2, 10].

Table 1: Examples of generating functions and their compositae

Generating function $F(x)$	Composita $F^\Delta(n, k)$
x^m	$\delta_{n,mk}, m > 0$
$\frac{bx}{1-ax}$	$\binom{n-1}{k-1} a^{n-k} b^k$
xe^x	$\frac{k^{n-k}}{(n-k)!}$
$\ln(1+x)$	$\frac{k!}{n!} s(n, k)$
$e^x - 1$	$\frac{k!}{n!} S(n, k)$

Here $\delta_{n,k}$ is the Kronecker delta, $s(n, k)$ and $S(n, k)$ stand for the Stirling numbers of the first kind and of the second kind, respectively (see [10, 2]).

The Stirling numbers of the first kind $s(n, k)$ count the number of permutations of n elements with k disjoint cycles. The Stirling numbers of the first kind are defined by the following generating function

$$\psi_k(x) = \sum_{n \geq k} s(n, k) \frac{x^n}{n!} = \frac{1}{k!} \ln^k(1+x).$$

The Stirling numbers of the second kind $S(n, k)$ count the number of ways to partition a set of n elements into k nonempty subsets. A general formula for the Stirling numbers of the second kind is given as follows

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

The Stirling numbers of the second kind are defined by the following generating function

$$\Phi_k(x) = \sum_{n \geq k} S(n, k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k.$$

Compositae of polynomials

Let us obtain compositae for polynomials. First we obtain the composita of the generating function $F(x) = ax + bx^2$. Raising this generating function to the power of k and applying the binomial theorem, we get

$$[F(x, a, b)]^k = x^k (a + bx)^k = x^k \sum_{m=0}^k \binom{k}{m} a^{k-m} b^m x^m.$$

Substituting n for $m + k$, we get the following expression

$$[F(x, a, b)]^k = \sum_{n=k}^{2k} \binom{k}{n-k} a^{2k-n} b^{n-k} x^n = \sum_{n=k}^{2k} F^\Delta(n, k, a, b) x^n.$$

Therefore, the composita is

$$F^\Delta(n, k, a, b) = \binom{k}{n-k} a^{2k-n} b^{n-k}. \tag{14}$$

Next we obtain the composita of the generating function $F(x) = ax + bx^2 + cx^3$. For this purpose, we write the generating function as the sum of the functions $F_1(x) = ax$ and $F_2(x) = x(bx + cx^2)$.

The composita of the generating function $F_1(x) = ax$, according to Theorem 4, is equal to

$$F_1^\Delta(n, k, a) = a^k \delta_{n,k}.$$

Using Theorem 2 and the formula (14), the composita of the generating function $F_2(x)$ is equal to

$$F_2^\Delta(n, k, b, c) = \binom{k}{n-2k} b^{3k-n} c^{n-2k}.$$

Using Theorem 7, we obtain

$$F^\Delta(n, k, a, b, c) = \sum_{j=0}^k \binom{k}{j} \sum_{i=j}^{n-k+j} F_1^\Delta(i, j, b, c) \delta_{n-i, k-j} a^{k-j}.$$

Since

$$\delta_{n-i, k-j} = \begin{cases} 1, & \text{if } n-i = k-j; \\ 0, & \text{otherwise,} \end{cases}$$

the composita of $F(x) = ax + bx^2 + cx^3$ is

$$F^\Delta(n, k, a, b, c) = \sum_{j=0}^k \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}. \tag{15}$$

With the above theorems (Section 3), we can obtain compositae for different polynomials. Some examples are presented in Table 2.

Compositae of trigonometric functions

For computation of compositae of trigonometric functions, we use the Euler identity $e^{ix} = \cos(x) + i \sin(x)$.

Let us obtain the composita of the generating function $F(x) = \sin(x)$.

Using the expression

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i},$$

Table 2: Compositae of polynomials

GF $F(x)$	Composita $F^\Delta(n, k)$
$ax + bx^2$	$\binom{k}{n-k} a^{2k-n} b^{n-k}$
$ax + bx^2 + cx^3$	$\sum_{j=0}^k \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}$
$ax + cx^3$	$\frac{1+(-1)^{n-k}}{2} \binom{k}{\frac{n-k}{2}} a^{\frac{3k-n}{2}} c^{\frac{n-k}{2}}$
$ax + bx^2 + dx^4$	$\sum_{j=\lfloor \frac{4k-n}{3} \rfloor}^k a^{4k-n-2j} b^{n-4k+3j} d^{k-j} \binom{j}{n-4k+3j} \binom{k}{j}$

we obtain $\sin(x)^k$

$$\begin{aligned} \sin(x)^k &= \frac{1}{2^k i^k} \sum_{m=0}^k \binom{k}{m} e^{imx} e^{-i(k-m)x} (-1)^{k-m} = \\ &= \frac{1}{2^k i^k} \sum_{m=0}^k \binom{k}{m} e^{i(2m-k)x} (-1)^{k-m} = \\ &= \sum_{n \geq k} \frac{1}{2^k} i^{n-k} \sum_{m=0}^k \binom{k}{m} \frac{(2m-k)^n}{n!} (-1)^{k-m} x^n. \end{aligned}$$

Then the composita is equal to

$$\frac{1}{2^k} i^{n-k} \sum_{m=0}^k \binom{k}{m} \frac{(2m-k)^n}{n!} (-1)^{k-m}.$$

Since $n - k$ is an even number and the function is symmetric with respect to k , we obtain the composita of the generating function $F(x) = \sin(x)$:

if $n - k$ is even, we have

$$F^\Delta(n, k) = \frac{1}{2^{k-1} n!} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m}, \quad (16)$$

if $n - k$ is odd, we have

$$F^\Delta(n, k) = 0.$$

With the above theorems (Section 3), we can obtain compositae for different trigonometric and hyperbolic functions. Some examples are presented in Table 3.

5 Composition of generating functions and its composita

Let us consider the application of compositae for computation of compositions of ordinary generating functions. For this purpose, we prove the following theorem.

Theorem 8. Suppose we have the generating function $F(x) = \sum_{n>0} f(n)x^n$, the composita of this generating

Table 3: Compositae of trigonometric and hyperbolic functions

GF $F(x)$	Composita $F^\Delta(n, k)$
$\sin(x)$	$\frac{1+(-1)^{n-k}}{2^k n!} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m}$
$x \cos(x)$	$\frac{1+(-1)^{n-k}}{2^k (n-k)!} (-1)^{\frac{n-k}{2}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} (k-2j)^{n-k}$
$\tan(x)$	$\frac{1+(-1)^{n-k}}{n!} \sum_{j=k}^n 2^{n-j-1} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} j! (-1)^{\frac{n+k}{2}+j} \binom{j-1}{k-1}$
$\arctan(x)$	$\frac{((-1)^{\frac{3n+k}{2}} + (-1)^{\frac{n+k}{2}}) k!}{2^{k+1}} \sum_{j=k}^n \frac{2^j}{j!} \binom{n-1}{j-1} s(j, k)$
$\sinh(x)$	$\frac{1}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(k-2j)^n}{n!}$
$x \cosh(x)$	$\frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{(k-2j)^{n-k}}{(n-k)!}$

function $F^\Delta(n, k)$, and the generating function $R(x) = \sum_{n \geq 0} r(n)x^n$. Then for the composition of generating functions $A(x) = R(F(x))$ the following condition holds

$$a(n) = \begin{cases} r(0), & \text{if } n = 0; \\ \sum_{k=1}^n F^\Delta(n, k)r(k), & \text{if } n > 0, \end{cases} \quad (17)$$

where $A(x) = \sum_{n \geq 0} a(n)x^n$.

Proof. For computation $A(x) = R(F(x))$ we can write

$$A(x) = R(F(x)) = \sum_{k \geq 0} r(k)F(x)^k.$$

Replacing $F(x)^k$ by $\sum_{n \geq k} F^\Delta(n, k)x^n$ and considering that $F(x)^0 = 1$, we get

$$\begin{aligned} A(x) &= r(0) + \\ &+ r(1)F(1, 1)x + r(1)F(2, 1)x^2 + \dots + r(1)F(n, 1)x^n + \dots \\ &+ r(2)F(2, 2)x^2 + \dots + r(2)F(n, 2)x^n + \dots \\ &\dots \\ &+ r(n)F(n, n)x^n + \dots \\ &+ \dots \end{aligned}$$

Summing the coefficients of equal powers of x^n , we obtain the desired formula

$$a(0) = r(0), \quad n = 0;$$

$$a(n) = \sum_{k=1}^n F^\Delta(n, k)r(k), \quad n > 0.$$

Further, for the composition $A(x) = R(F(x))$ the condition $a(0) = r(0)$ is implied.

Example 1. Let us obtain an expression of coefficients of the generating function

$$A(x) = \frac{1}{1 - ax - bx^2 - cx^3},$$

where $a, b, c \neq 0$.

Represent $A(x)$ as the composition of generating functions $A(x) = R(F(x))$, where $F(x) = ax + bx^2 + cx^3$ and $R(x) = \frac{1}{1-x}$.

According to Table 2, the composita of $F(x) = ax + bx^2 + cx^3$ is

$$\sum_{j=0}^k \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.$$

Using Theorem 8, we obtain the expression of coefficients of $A(x)$

$$a(n) = \sum_{k=1}^n \sum_{j=0}^k \binom{k}{j} \binom{j}{n-k-j} a^{k-j} b^{2j+k-n} c^{n-k-j}.$$

Example 2. Let us consider the generating function $A(x) = e^{\sinh(x)}$.

Using the composita of $F(x) = \sinh(x)$ (see Table 3) and Theorem 8, we obtain the expression of coefficients of $A(x)$

$$a(n) = \sum_{k=1}^n \frac{1}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(k-2j)^n}{n!} \frac{1}{k!}.$$

Theorem 9. Suppose we have the generating functions $F(x) = \sum_{n>0} f(n)x^n$, $G(x) = \sum_{n>0} g(n)x^n$, and their compositae $F^\Delta(n, k)$, $G^\Delta(n, k)$ respectively. Then for the composition of generating functions $A(x) = G(F(x))$ the composita is equal to

$$A^\Delta(n, k) = \sum_{m=k}^n F^\Delta(n, m) G^\Delta(m, k). \tag{18}$$

Proof. Using the formula (3), we have

$$[A(x)]^k = [G(F(x))]^k = \sum_{n \geq k} A^\Delta(n, k) x^n.$$

The function of coefficients of the generating function $[G(x)]^k$ is the composita $G^\Delta(n, k)$

$$[G(x)]^k = \sum_{n \geq k} G^\Delta(n, k) x^n.$$

Then, according to Theorem 8, we get

$$[G(F(x))]^k = \sum_{n \geq k} \sum_{m=1}^n F^\Delta(n, m) G^\Delta(m, k).$$

Since

$$G^\Delta(m, k) = 0, \quad \text{if } m < k,$$

we obtain the composita of the composition of generating functions $A(x) = G(F(x))$

$$A^\Delta(n, k) = \sum_{m=k}^n F^\Delta(n, m) G^\Delta(m, k).$$

6 Composita of reciprocal generating function

First we consider the notion of *reciprocal generating functions* [12].

Definition 2. Reciprocal generating functions are functions that satisfy the condition

$$H(x)B(x) = 1.$$

Remark. If we have the reciprocal generating functions $H(x) = \sum_{n \geq 0} h(n)x^n$ and $B(x) = \sum_{n \geq 0} B(n)x^n$ such that $H(x)B(x) = 1$, then by the composita of the reciprocal generating function of $B(x)$ we mean the composita of $xH(x) = \frac{x}{B(x)}$.

In the following theorem we give the formula of the composita of a reciprocal generating function.

Theorem 10. Suppose $H(x) = \sum_{n \geq 0} h(n)x^n$ is a generating function, $B(x) = \sum_{n \geq 0} b(n)x^n$ is the reciprocal generating function of $H(x)$, and $B_x^\Delta(n, k)$ is the composita of $xH(x)$. Then the composita of the generating function $xH(x)$ is equal to

$$H_x^\Delta(n, k) = \begin{cases} \frac{1}{B_x^\Delta(1,1)^k}, & \text{if } n = k; \\ \sum_{m=1}^{n-k} \binom{m+k-1}{k-1} \sum_{j=1}^m \frac{(-1)^j \binom{m}{j}}{B_x^\Delta(1,1)^{j+k}} B_x^\Delta(n-k+j, j), & \text{if } n > k. \end{cases} \tag{19}$$

Proof. By Definition 2, we get

$$xH(x) = \frac{x}{b(0) + B(x) - b(0)}.$$

Raising this generating function to the power of k , we obtain

$$[xH(x)]^k = \left[\frac{x}{b(0) + B(x) - b(0)} \right]^k = \left[\frac{1}{b(0)} \frac{x}{1 + \frac{1}{b(0)}(B(x) - b(0))} \right]^k.$$

Using Corollary 2, Theorem 3 and Theorem 4, we obtain the composita of $F(x) = \frac{1}{b_0}(B(x) - b_0)$

$$F^\Delta(n, k) = \sum_{j=1}^k b(0)^{-j} (-1)^{k-j} \binom{k}{j} B_x^\Delta(n+j, j).$$

The expression of coefficients of the generating function $R(x) = \left[\frac{1}{b(0)} \frac{1}{1+x} \right]^k$ is equal to

$$R(n, k) = \frac{1}{b(0)^k} \binom{n+k-1}{k-1} (-1)^n.$$

Then, according to Theorem 8, we get

$$H(n, k) = \begin{cases} \frac{1}{b(0)^k}, & \text{if } n = 0; \\ \sum_{m=1}^{n-k} \binom{m+k-1}{k-1} \sum_{j=1}^m \frac{(-1)^j \binom{m}{j}}{b(0)^{k+j}} B_x^\Delta(n+j, j), & \text{if } n > 0. \end{cases}$$

Therefore, from Corollary 1 and $b(0) = B_x^\Delta(1, 1)$, we obtain the composita of the reciprocal generating function

of $B(x)$

$$H_x^\Delta(n, k) = \begin{cases} \frac{1}{B_x^\Delta(1,1)^k}, & \text{if } n = k; \\ \sum_{m=1}^{n-k} \binom{m+k-1}{k-1} \sum_{j=1}^m \frac{(-1)^j \binom{m}{j}}{B_x^\Delta(1,1)^{j+k}} B_x^\Delta(n-k+j, j), & \text{if } n > k. \end{cases}$$

For applications of Theorem 10 we give some examples.

Example 3. Let us find a composita of the generating function $F(x) = x^2 \csc(x)$. For this purpose, we write

$$F(x) = x^2 \csc(x) = \frac{x}{\frac{\sin(x)}{x}},$$

or

$$\frac{F(x)}{x} \frac{\sin(x)}{x} = 1.$$

According to Table 3, the composita of $\sin(x)$ is

$$\frac{1 + (-1)^{n-k} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{m} (2m-k)^n (-1)^{\frac{n+k}{2}-m}}{2^k n!}$$

Then, using Theorem 10, we obtain the composita of $F(x)$: if $n = k$, we have

$$F^\Delta(n, k) = 1,$$

if $n > k$, we have

$$F^\Delta(n, k) = \sum_{m=1}^{n-k} \binom{m+k-1}{k-1} \sum_{j=1}^m \binom{m}{j} \frac{1 + (-1)^{n-k}}{2^j (n-k+j)!} \times \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{i} (2i-j)^{n-k+j} (-1)^{\frac{n-k}{2}-i}.$$

Example 4. Let us find a composita of the generating function $F(x) = xH(x)$, where $H(x) = \sum_{n \geq 0} h(n)x^n$ is the generating function for Bernoulli numbers

$$H(x) = \frac{x}{e^x - 1}.$$

According to Table 1, the composita of the generating function $e^x - 1$ is equal to

$$\frac{k!}{n!} S(n, k).$$

The generating function $H(x)$ is the reciprocal generating function of $\frac{e^x-1}{x}$. Then, using Theorem 10, we obtain the composita of the generating function $F(x) = \frac{x^2}{e^x-1}$

$$F^\Delta(n, k) = \sum_{m=0}^{n-k} \binom{m+k-1}{k-1} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{j! S(n-k+j, j)}{(n-k+j)!}.$$

7 Conclusion

In this paper we introduce the concept of composita for ordinary generating functions and provide a number of applications. The proposed apparatus of compositae is applicable to solve the following problems: calculation of the composition of ordinary generating functions; finding expressions of reciprocal generating functions; finding expressions of inverse generating functions; finding solutions of functional equations; obtaining expressions of polynomials and etc.

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