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Periodic and Solitary Wave Solutions of the Davey-Stewartson Equation

Hassan A. Zedan and Ahmed Al Saedi
Mathematics Deptartment, Faculty of Science, King Abdul Aziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia
Email Addresses: hassanzedan2003@yahoo.com; aalsaedi@hotmail.com

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In this paper, we establish exact solutions for the Davey-Stewartson (DS) equation. The rational expansion method is used to construct periodic and solitary wave solutions of the Davey-Stewartson equation. Moreover, we extend the analysis of this method to solve different types of nonlinear system of Davey-Stewartson.

Keywords: Solitary wave solutions, periodic solutions, Davey-Stewartson equation.

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1 Introduction

Searching and constructing exact solutions for nonlinear partial differential equations (NPDEs) is an ongoing research. These exact solutions when they exist can help to understand the mechanism of the complicated physical phenomena and dynamically processes modelled by these nonlinear partial differential equations (NPDEs). During the past decades, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear DS equations have given rise to a variety of powerful methods, such as homogeneous balance method [4], the tanh-sech method [3, 6–8], inverse scattering method [1], bilinear transformation [5], etc.

The motivation of this paper is to extend the analysis of a new compound Riccati equations rational expansion method to solve different types of nonlinear system of partial differential equations (NPDEs). The higher-dimensional nonlinear wave fields have richer phenomena than that of one-dimensional case since various localized solitons may be considered in higher-dimensional space. The DS equation has four kinds of soliton solutions: the conventional line, algebraic, periodic and lattice solitons. The conventional line soliton has an essentially one-dimensional structure. On the other hand, the algebraic, periodic and lattice solitons have a two-dimensional localized structure.

The Davey–Stewartson equation may be written as

$$
\begin{aligned}
iq_t + \frac{1}{2}\sigma^2\left(q_{xx} + \sigma^2 q_{yy}\right) + \lambda \quad | \quad & q\,|^2\, q - \phi_x q = 0, \\
\phi_{xx} - \sigma^2 \phi_{yy} - 2\lambda\left(|\, q\,|^2\right)_x \quad = \quad & 0, \\
\lambda = \pm 1, \qquad\qquad & \sigma^2 = \pm 1.
\end{aligned}
\tag{1.1}
$$

When $\sigma = 1$ (1.1) is called the DS I equation, while when $\sigma = i$ (1.1) is the DS II equation. The parameter λ characterises the focusing or defocusing case.

Davey and Stewartson first derived their model in the context of water wave, with purely physical considerations. In this context,$q(t, x, y)$ is the amplitude of a surface wave packet, while $\phi(t, x, y)$ is the velocity potential of the mean flow interacting with the surface wave [2].

The Davey-Stewartson I and II equations (Denoted by DSI and DSII) are two well-know examples of integrable equation in two dimensions space, which arise as higher dimensional generalizations of the nonlinear Schrodinger (NLS) equation, as well as from physical considerations [2]. Indeed, they appear in many applications, for example in the description of gravity-capillarity surface wave packets in the limit of shallow water.

2 New Compound Riccati Equations Rational Expansion Method

The Known Riccati equation is used as sub-equation to set up many algorithms to construct particular travelling solutions for a large number of NPDEs. Generally speaking, the various extensions and improvement of the Riccati sub-equation methods focus mainly on presenting more general answer. Recently, Wang [9, 10] claimed that solutions of two different Riccati equations with different parameters are used as two variables in the components of finite rational expansion. In this work,we further develop the method Wang [9,10], named the compound Riccati equations rational expansion (CRERE) method, to construct new styles solutions of NPDEs. For illustration, we apply the new method to the (2+1)-dimensional Davey-Stewartson system and successfully construct new solutions.

In the following we outline the main steps of our method:

Step 1. Consider a given NPDEs system with some physical fields u_i in three variables x, y, t, where i numbers the physical field and the subscripts x or t indicate differentiation,

$$
P_i\left(u_{\ i}, u_{i\ t}, u_{i\ x}, u_{i\ y}, u_{i\ tt}, u_{i\ xx}, u_{i\ tx}, u_{i\ ty,} u_{i\ xy}, u_{i\ yy}, \ldots\right) = 0. \tag{2.1}
$$

By using the wave transformation

$$
u_i(x, y, t) \ = U(\xi), \qquad \xi = k(x + ly - \lambda t\ ), \tag{2.2}
$$

where k, l and λ are constants to be determined latter, the nonlinear partial differential equation (2.1) is reduced to a nonlinear ordinary differential equations (ODEs)

$$Q_i(U_i, U_i', U_i'', U_i''', \ldots) = 0. \tag{2.3}$$

Step 2. We introduce a new solutions in terms of finite rational formal expansion in the following form

$$U_i(\xi) = a_0 + \sum_{j=1}^{m_i} \frac{\sum_{r_1+r_2=j} a^{ij}_{r_{j1} r_{j2}} \left(\phi''\right)^{r_{j1}} \left(\psi'\right)^{r_{j2}}}{\left(\mu_1\, \phi'' + \mu_2\, \psi'\right)^j}, \tag{2.4}$$

where $a^{ij}_{r_{j1} r_{j2}}, \mu_1$ and μ_2 $(r_{jn} = 1, 2, 3, \ldots, j;\ n = 1, 2)$ are constants to be determined and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy the Riccati equation, i.e.,

$$\frac{d\,\phi}{d\,\xi} = h_1 + h_2 \phi^2, \qquad \frac{d\,\psi}{d\,\xi} = h_3 + h_4 \psi^2, \tag{2.5}$$

where h_1, h_2, h_3 and h_4 are arbitrary constants.

Step 3. The parameter m_i in Eq. (2.4) can be found by balancing the highest nonlinear terms and the highest-order partial derivative term in (2.1) or (2.3).

Step 4. Substituting Eq. (2.4) into Eq. (2.3) along with Eq. (2.5) and then seting all coefficients of ϕ^i and $\psi^j, (i = 1, 2, \ldots; j = 1, 2, \ldots)$ in the resulting system's numerator zero to get an over-determined system of nonlinear algebraic equations with respect to λ and $a^{ij}_{r_{j1} r_{j2}}$ $(r_{jn} = 1, 2, 3, \ldots, j; n = 1, 2)$.

Step 5. Solve the over-determined system of nonlinear algebraic equations by using Maple program and we would end up with the explicit expressions for λ and $a^{ij}_{r_{j1} r_{j2}}$ $(r_{jn} = 1, 2, 3, \ldots, j;\ n = 1, 2)$.

Step 6. It is well known that the general solutions of the Riccati equation

$$\frac{d\,F}{d\,\xi} = r_1 + r_2\, F^{\,2}\,(\xi),$$

are

1). if $r_1 = \frac{1}{2},\ r_2 = -\frac{1}{2}$

$$F(\xi) = \tanh(\xi) \pm i\,\text{sech}(\xi), \qquad F(\xi) = \coth(\xi) \pm \text{csch}(\xi);$$

2). if $r_1 = r_2 = \pm\frac{1}{2}$, $\quad F(\xi) = \csc(\xi) \pm \cot(\xi)$;

3). if $r_1 = 1,\ r_2 = -1$, $\quad F(\xi) = \tanh(\xi)$, $\quad F(\xi) = \coth(\xi)$;

4). if $r_1 = r_2 = 1$, $\quad F(\xi) = \tan(\xi)$;

5). if $r_1 = r_2 = -1$, $\quad F(\xi) = \cot(\xi)$;

6). if $r_1 = 0, r_2 \neq 0$, $\quad F(\xi) = \dfrac{-1}{r_2 \xi + r_0}$,

where $\xi = k\,(x + ly - \lambda t)\,,\ i = \sqrt{-1}$ and r_0 is arbitrary constant.

3 Application to the (2+1)-Dimensional Davey-Stewartson System

Let us consider the (2+1)-dimensional system DS (1.1). With the transformations

$$q(x,y,t) = u(\xi)\, e^{i\,\theta}, \qquad \phi(x,y,t) = V(\xi), \tag{3.1}$$

where

$$\xi = k\,(x + ly - \lambda t)\,, \qquad \theta = k_1 x + k_2 y + k_3 t,$$

the system (1.1) is converted to the system

$$\Big\{ - k_3 u - i\,\lambda k u' + \frac{1}{2}\sigma^2 \Big[- k_1^2 u\, + 2i\, k k_1 u' + k^2 u''(-k^2 u + 2ikk_2 l\, u' + k^2 l^{\,2} u'') \Big] + \lambda\, u^3 - k\,\mathrm{V}' \Big\} e^{i\,\theta} = 0, \tag{3.2}$$

$$k(1 - \sigma^2 l^2\,)V'' - 2\lambda (u^2)' = 0. \tag{3.3}$$

With straightforward calculations, Eq. (3.2) and Eq.(3.3) give

$$\big[\sigma^2(k_1 + \sigma^2 k_2 l\,) - \lambda\big]\, k = 0, \tag{3.4}$$

$$\sigma^2 k^2 (1+\sigma^2) u'' + 2\lambda u^3 - \sigma^2(k_1^2 + \sigma^2 k_2^2 + 2k_3)u - 2k\,\mathrm{V}' = 0, \tag{3.5}$$

$$k(1 - \sigma^2 l^2\,)V' - 2\lambda u^2 = 0. \tag{3.6}$$

By balancing the highest nonlinear terms and the highest-order partial derivative terms in (3.5) and (3.6) we have $n_1 = n_2 = 1$.

Thus, we obtain the following formulas as solutions of (3.5) and (3.6), respectively,

$$u(\xi) = a_0 + \frac{a_1 \phi'' + b_1 \psi'}{\mu_1 \phi'' + \mu_2 \psi'}, \qquad V(\xi) = b_0 + \frac{a_2 \phi'' + b_2 \psi'}{\mu_1 \phi'' + \mu_2 \psi'}, \tag{3.7}$$

where ϕ and ψ satisfy Eq. (2.5).

Substitution of (3.7) and (2.5) into equations (3.5) and (3.6) gives a set of algebraic equations in ϕ^i and $\psi^j (i = 1, 2, \ldots; j = 1, 2, \ldots)$. Setting the coefficients of these terms with ϕ^i and ψ^j to zero yields a set of over-determined algebraic equations with respect to $a_0, a_1, a_2, b_0, b_1, b_2$, and l. Now, solving the over-determined algebraic equations, we get

$$a_0 = \frac{-\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1\mu_2}}{2\mu_1\mu_2}, \qquad a_1 = \frac{-2\mu_1}{-\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1\mu_2}} + 1, \tag{3.8}$$

$$b_0 = b_0, \quad a_2 = a_2, \quad b_2 = b_2, \quad l = \pm\frac{1}{\sigma}. \tag{3.9}$$

Thus briefly ,we can write (3.8) and (3.9) in the following form

$$a_0 = \alpha, \quad a_1 = \frac{-1}{\alpha\mu_2} + 1, \quad b_1 = -\alpha\mu_2, \tag{3.10}$$

$$b_0 = b_0, \quad a_2 = a_2, \quad b_2 = b_2, \quad l = \pm\frac{1}{\sigma}, \tag{3.11}$$

where

$$\alpha = \frac{-\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1\mu_2}}{2\mu_1\mu_2}.$$

From (3.1), (3.7), (3.10) and (3.11), we obtain a new type of solutions for the system (1.1) as the following:

Family 1.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] - \alpha\mu_2\left[\text{csch}^2\xi \pm \text{csch}\xi\coth\xi\right]}{\mu_1\left[2\ \tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] + \mu_2\left[\text{csch}^2\xi \pm \text{csch}\xi\coth\xi\right]} \tag{3.12}$$

$$\therefore\ q = u\, e^{i\ (k_1 x\ + k_2 y\ + k_3 t\ )}$$

$$\begin{aligned}\phi(x,y,t) &= V(\xi)\\ &= b_0 + \frac{a_2\left[2\ \tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] + b_2\left[\text{csch}^2\xi \pm \text{csch}\xi\coth\xi\right]}{\mu_1\left[2\tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] + \mu_2\left[\text{csch}^2\xi \pm \text{csch}\xi\coth\xi\right]};\end{aligned}$$

Family 2.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] + \alpha\mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]}{\mu_1\left[2\ \tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] - \mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]} \tag{3.13}$$

$$\therefore\ q = u\, e^{i\ (k_1 x\ + k_2 y\ + k_3 t\ )}$$

$$\begin{aligned}\phi(x,y,t) &= V(\xi)\\ &= b_0 + \frac{a_2\left[2\tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] - b_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]}{\mu_1\left[2\ \tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] - \mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]};\end{aligned}$$

Family 3.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] - \alpha\mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]}{\mu_1\left[2\ \tanh\xi(\text{sech}^2\xi \mp i\text{sech}\xi\tanh\xi) \pm i\text{sech}\xi\right] + \mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]} \tag{3.14}$$

$$\therefore\ q = u\, e^{i\ (k_1 x\ + k_2 y\ + k_3 t\ )}$$

$$\phi(x,y,t) = V(\xi)$$

$$= b_0 + \frac{a_2\left[2\tanh\xi(\sec h^2\xi \mp i\mathrm{sech}\xi\tanh\xi) \pm i\mathrm{sech}\xi\right] + b_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]}{\mu_1\left[2\ \tanh\xi(\mathrm{sech}^2\xi \mp i\mathrm{sech}\xi\tanh\xi) \pm i\mathrm{sech}\xi\right] + \mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]};$$

Family 4.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] + \alpha\mu_2\left[\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - \mu_2\left[\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi\right]} \tag{3.15}$$

$\therefore\ q = u\, e^{i\,(k_1 x\,+k_2 y\,+k_3 t\,)}$

$$\phi(x,y,t) = V(\xi)$$
$$= b_0 + \frac{a_2\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - b_2\left[\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - \mu_2\left[\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi\right]};$$

Family 5.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - \alpha\mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] + \mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]}. \tag{3.16}$$

$\therefore\ q = u\, e^{i\,(k_1 x\,+k_2 y\,+k_3 t\,)}$

$$\phi(x,y,t) = V(\xi)$$
$$= b_0 + \frac{a_2\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] + b_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] + \mu_2\left[\sec\xi\tan\xi \pm \sec^2\xi\right]};$$

Family 6.

$$u = \alpha + \frac{\frac{\alpha\mu_2 - 1}{\alpha\mu_2}\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] + \alpha\mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - \mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]} \tag{3.17}$$

$\therefore\ q = u\, e^{i\,(k_1 x\,+k_2 y\,+k_3 t\,)}$

$$\phi(x,y,t) = V(\xi)$$
$$= b_0 + \frac{a_2\left[2\coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - b_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]}{\mu_1\left[2\ \coth\xi(\mathrm{csch}^2\xi \pm \mathrm{csch}\xi\coth\xi) \mp \mathrm{csch}\xi\right] - \mu_2\left[\csc\xi\cot\xi \pm \csc^2\xi\right]}.$$

Here $\xi = k\left[x \pm y/\sigma - \sigma^2 (k_1 + \sigma k_2) t\right]$, $k,\ \mu_{1,}\ \mu_2,\ b_0,\ b_2,\ a_2,\ k_1,\ k_2$ and k_3 are arbitrary constants,

$$\alpha = \frac{-\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1\mu_2}}{2\mu_1\mu_2}. \tag{3.18}$$

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Dr. Hassan A. Zedan is a Professor of Mathematics (Partial Differential Equations), Faculty of Science, Kafr El-Sheikh University, Egypt. Dr. Zedan earned a PhD in Mathematics in 1995 (from Italy).

Dr. Ahmed Al Saedi is a Professor of Differential Equations, Faculty of Science, King AbdulAziz University. Dr. Al Saedi earned a PhD in Differential Equations (2002) from University of Wales Swansea (UK).