Applied Mathematics & Information Sciences **4**(2) (2010), 237–251
– An International Journal

Description of Weak Periodic Ground States of Ising Model with Competing Interactions on Cayley Tree

M. M. Rahmatullaev

Namangan State University, 316, Uychi str., Namangan, Uzbekistan

Email Address: mrahmatullaev@rambler.ru

Received August 3, 2008; Revised January 12, 2009

Recently Rozikov studied an Ising model with competing interactions and spin values ± 1, on a Cayley tree of order $k \geq 1$ and described the ground states of the model. In this paper we describe some weak periodic ground states of the model.

Keywords: Cayley tree, configuration, Ising model, ground state.

1 Introduction

This paper gives a class of weak periodic ground states for an Ising model with competing interactions and spin values ± 1, on a Cayley tree of order $k \geq 1$. One of the key problems related to the spin models is the description of the set of Gibbs measures. This problem has a good connection with the problem of the description the set of ground states. Because the phase diagram of Gibbs measures (see [6, 12] for details) is close to the phase diagram of the ground states for sufficiently small temperatures. Usually, more simple and interesting ground states are periodic ones. But for some set of parameters such a ground state does not exist. In such a case it would be necessary to find some a weak periodic ground states.

The Ising model, with two values of spin ± 1 was considered in [9, 13] and became actively researched in the 1990's and afterwards (see for example [1–5, 7, 8, 10, 11]).

The *Cayley tree* Γ^k (See [1]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y \rangle$.

The distance $d(x,y), x,y \in V$ on the Cayley tree is defined by the formula

$$d(x,y)=\min\Big\{d:\exists x=x_0,x_1,\dots,x_{d-1},x_d=y\in V \text{ such that } \langle x_0,x_1\rangle,\dots,\langle x_{d-1},x_d\rangle\Big\}.$$

For the fixed $x^0 \in V$ we set

$$\begin{aligned} W_n &= \{x\in V: \quad \partial(x,x^0)=n\},\\ V_n &= \{x\in V: \quad d(x,x^0)\leq n\},\\ L_n &= \{l=\langle x,y\rangle \in L: \quad x,y\in V_n\}, \end{aligned} \tag{1.1}$$

and we denote $|x|=d(x,x^0)$, $x\in V$.

A collection of the pairs $\langle x,x_1\rangle,\dots,\langle x_{d-1},y\rangle$ is called a *path* from x to y and we write $\pi(x,y)$. We write $x<y$ if the path from x^0 to y goes through x.

It is known (see [5]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k\geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e,a_i\}$, $i=1,\dots,k+1$ of the second order (i.e. $a_i^2=e$, $a_i^{-1}=a_i$) with generators $a_1,a_2,\dots,a_{k+1}$.

Denote $S(x)$ the set of "direct successors" of $x\in G_k$. Let $S_1(x)$ be denotes the set of all nearest neighboring vertices of $x\in G_k$, i.e. $S_1(x)=\{y\in G_k:\langle x,y\rangle\}$ and $x_\downarrow=S_1(x)\setminus S(x)$. (see Figure 1.1).

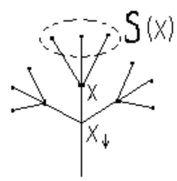

Figure 1.1

2 The Model

Here we shall give main definitions and facts about the model which we are going to study (see [11] for details). Consider models where the spin takes values in the set $\Phi=\{-1,1\}$. For $A\subseteq V$ a spin *configuration* σ_A on A is defined as a function $x\in A\to\sigma_A(x)\in\Phi$; the set of all configurations coincides with $\Omega_A=\Phi^A$. Denote $\Omega=\Omega_V$ and $\sigma=\sigma_V$. Also put $-\sigma_A=\{-\sigma_A(x), x\in A\}$. Define a *periodic configuration* as a configuration $\sigma\in\Omega$ which is invariant under a subgroup of shifts $G_k^*\subset G_k$ of finite index. More precisely, a configuration $\sigma\in\Omega$ is called G_k^* -periodic if $\sigma(yx)=\sigma(x)$ for any $x\in G_k$ and $y\in G_k^*$.

For a given periodic configuration the index of the subgroup is called the *period of the configuration*. A configuration that is invariant with respect to all shifts is called *translational-invariant*. Let $G_k/G_k^* = \{H_1, \dots, H_r\}$ factor group, where G_k^* is a normal subgroup of index $r \geq 1$. Configuration $\sigma(x), x \in V$ is called G_k^* *weak periodic*, if $\sigma(x) = \sigma_{ij}$ for $x \in H_i, x_\downarrow \in H_j, \forall x \in G_k$.

The Hamiltonian of the Ising model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle} \sigma(x)\sigma(y) + J_2 \sum_{x,y \in V:\ d(x,y)=2} \sigma(x)\sigma(y), \tag{2.1}$$

where $J_1, J_2 \in R$ are coupling constants and $\sigma \in \Omega$.

For a pair of configurations σ and φ that coincide almost everywhere, i.e. everywhere except for a finite number of positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, the difference between the energies of the configurations σ, φ of the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y\rangle} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{x,y \in V: d(x,y)=2} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)), \tag{2.2}$$

where $J = (J_1, J_2) \in R^2$ is an arbitrary fixed parameter.

Let M be the set of unit balls with vertices in V. We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b.

Define the energy of a ball b for configuration σ by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y\rangle,\ x,y \in b} \sigma(x)\sigma(y) + J_2 \sum_{x,y \in b:\ d(x,y)=2} \sigma(x)\sigma(y), \tag{2.3}$$

where $J = (J_1, J_2) \in R^2$.

We shall say that two bounded configurations σ_b and $\sigma'_{b'}$ belong to the same class if $U(\sigma_b) = U(\sigma'_{b'})$ and we write $\sigma'_{b'} \sim \sigma_b$.

For any set A we denote by $|A|$ the number of elements in A.

Lemma 2.1 ([11]). 1) *For any configuration σ_b we have*

$$U(\sigma_b) \in \{U_0, U_1, \dots, U_{k+1}\},$$

$$U_i = \left(\frac{k+1}{2} - i\right) J_1 + \left(\frac{k(k+1)}{2} + 2i(i-k-1)\right) J_2, \quad i = 0, 1, \dots, k+1. \tag{2.4}$$

2) *Let $\mathcal{C}_i = \Omega_i \cup \Omega_i^-$, $i = 0, \dots, k+1$, where*

$$\Omega_i = \big\{\sigma_b : \sigma_b(c_b) = +1,\ \ |\{x \in b \setminus \{c_b\} : \sigma_b(x) = -1\}| = i\big\},$$
$$\Omega_i^- = \big\{ -\sigma_b = \{-\sigma_b(x), x \in b\} : \sigma_b \in \Omega_i\big\},$$

and let c_b be the center of the ball b. Then for $\sigma_b \in \mathcal{C}_i$ we have $U(\sigma_b) = U_i$.

3) *The class $\mathcal{C}_i$ contains*

$$\frac{2(k+1)!}{i!(k-i+1)!}$$

configurations.

Definition 2.1. A configuration φ is called a *ground state* for the relative Hamiltonian H if

$$U(\varphi_b) = \min\{U_0, U_1, \ldots, U_{k+1}\}, \quad \text{for any } \ b \in M. \tag{2.5}$$

We set

$$U_i(J) = U(\sigma_b, J), \ \text{ if } \ \sigma_b \in \mathcal{C}_i, \ \ i = 0, 1, \ldots, k+1.$$

The quantity $U_i(J)$ is a linear function of the parameter $J \in R^2$. For every fixed $m = 0, 1, \ldots, k+1$ we denote

$$A_m = \{J \in R^2 : U_m(J) = \min\{U_0(J), U_1(J), \ldots, U_{k+1}(J)\}\}. \tag{2.6}$$

It is easy to check that

$$A_0 = \{J \in R^2 : J_1 \le 0; \ \ J_1 + 2kJ_2 \le 0\},$$

$$A_m = \{J \in R^2 : J_2 \ge 0; \ \ 2(2m-k-2)J_2 \le J_1 \le 2(2m-k)J_2\}, \ \ m = 1, 2, \ldots, k,$$

$$A_{k+1} = \{J \in R^2 : J_1 \ge 0; \ \ J_1 - 2kJ_2 \ge 0\},$$

and $R^2 = \cup_{i=0}^{k+1} A_i$.

For any A_i and A_j with $i \ne j$, we have

$$A_i \cap A_j = \begin{cases} \{J : J_1 = 2(2i-k)J_2, \ \ J_2 \ge 0\} & \text{if } \ j = i+1, \ \ i = 0, 1, \ldots, k, \\ (0,0) & \text{if } \ 1 < |i-j| < k+1, \\ \{J : J_1 = 0, J_2 \le 0\} & \text{if } \ |i-j| = k+1. \end{cases} \tag{2.7}$$

We denote

$$B = A_0 \cap A_{k+1}, \ \ B_i = A_i \cap A_{i+1}, \ \ i = 0, \ldots, k,$$
$$\tilde{A}_0 = A_0 \setminus (B \cup B_0), \ \ \tilde{A}_{k+1} = A_{k+1} \setminus (B \cup B_k),$$
$$\tilde{A}_i = A_i \setminus (B_{i-1} \cup B_i), \ \ i = 1, \ldots, k,$$

and for fixed $J \in R^2$ we denote

$$N_J(\sigma_b) = |\{j : \sigma_b \in \mathcal{C}_j\}|.$$

We let $GS(H)$ denote the set of all ground states of the relative Hamiltonian H (see (2.3)). For any $\sigma = \{\sigma(x), \ \ x \in V\} \in \Omega$ we denote $\overline{\sigma} = -\sigma = \{-\sigma(x), \ \ x \in V\}$.

In [11] the set of periodic ground states for the model (1.1) is described, i.e., the following is proved:

Theorem 2.1. (i) *If* $J = (0,0)$ *then* $GS(H) = \Omega$.

(ii) *If* $J \in \tilde{A}_i, \ \ i = 0, \ldots, k+1$ *then*

$$GS(H) = \{\sigma^{(i)}, \overline{\sigma}^{(i)}\}.$$

(iii) *If* $J \in B_i \setminus \{(0,0)\}$, $i = 0,\dots,k$ *then*

$$GS(H) = \{\sigma^{(i)}, \overline{\sigma}^{(i)}, \sigma^{(i+1)}, \overline{\sigma}^{(i+1)}\} \cup S_i,$$

where S_i *contains at least a countable subset of non periodic ground states.*

(iv) *If* $J \in B \setminus \{(0,0)\}$, *then*

$$GS(H) = \{\sigma^{(0)}, \overline{\sigma}^{(0)}, \sigma^{(k+1)}, \overline{\sigma}^{(k+1)}\}.$$

Here $\sigma^{(i)}$, $\overline{\sigma}^{(i)}$, $i = 0,\dots,k+1$ *are periodic ground states such that on any* $b \in M$ *the bounded configurations* $\sigma_b^{(i)}, \overline{\sigma}_b^{(i)} \in \mathcal{C}_i$, *i.e.* $\sigma^{(0)}, \overline{\sigma}^{(0)}$ *are translational - invariant and* $\sigma^{(i)}, \overline{\sigma}^{(i)}$, $i = 1,\dots,k+1$ *are periodic with period 2.*

Remark 2.1. We note, that weak periodic (non periodic) ground states belong to the set S_i i.e. for parameters $J_1 = 2(2i-k)J_2, J_2 \neq 0$.

In this paper we explicitly describe the weak periodic (with respect to normal subgroups of index 2 and 4) ground states.

3 Weak Periodic Ground States

Case 1: index 2.

Let $A \subset \{1,2,\dots,k+1\}$, $H_A = \{x \in G_k : \sum_{j\in A} w_j(x) - \text{even}\}$, where $w_j(x)$-is the number of letters a_j in the word x. It is obvious that H_A is a normal subgroup of index two [5].

Let $G_k/H_A = \{H_A, G_k \setminus H_A\}$ be the quotient group. We set $H_0 = H_A, H_1 = G_k \setminus H_A$.

The H_A - weak periodic configurations are of the form

$$(1)\ \varphi_1(x) = \pm \begin{cases} +1, & x_\downarrow \in H_0\ x \in H_0 \\ +1, & x_\downarrow \in H_0\ x \in H_1 \\ +1, & x_\downarrow \in H_1\ x \in H_0 \\ +1, & x_\downarrow \in H_1\ x \in H_1, \end{cases} \qquad (2)\ \varphi_2(x) = \pm \begin{cases} -1, & x_\downarrow \in H_0\ x \in H_0 \\ +1, & x_\downarrow \in H_0\ x \in H_1 \\ +1, & x_\downarrow \in H_1\ x \in H_0 \\ +1, & x_\downarrow \in H_1\ x \in H_1, \end{cases}$$

$$(3)\ \varphi_3(x) = \pm \begin{cases} +1, & x_\downarrow \in H_0\ x \in H_0 \\ -1, & x_\downarrow \in H_0\ x \in H_1 \\ +1, & x_\downarrow \in H_1\ x \in H_0 \\ +1, & x_\downarrow \in H_1\ x \in H_1, \end{cases} \qquad (4)\ \varphi_4(x) = \pm \begin{cases} +1, & x_\downarrow \in H_0\ x \in H_0 \\ +1, & x_\downarrow \in H_0\ x \in H_1 \\ -1, & x_\downarrow \in H_1\ x \in H_0 \\ +1, & x_\downarrow \in H_1\ x \in H_1, \end{cases}$$

$$(5)\ \varphi_5(x) = \pm \begin{cases} +1, & x_\downarrow \in H_0\ x \in H_0 \\ +1, & x_\downarrow \in H_0\ x \in H_1 \\ +1, & x_\downarrow \in H_1\ x \in H_0 \\ -1, & x_\downarrow \in H_1\ x \in H_1, \end{cases} \qquad (6)\ \varphi_6(x) = \pm \begin{cases} -1, & x_\downarrow \in H_0\ x \in H_0 \\ -1, & x_\downarrow \in H_0\ x \in H_1 \\ +1, & x_\downarrow \in H_1\ x \in H_0 \\ +1, & x_\downarrow \in H_1\ x \in H_1, \end{cases}$$

(7) $\varphi_7(x)=\pm\begin{cases} -1, & x_\downarrow\in H_0\ x\in H_0\\ +1, & x_\downarrow\in H_0\ x\in H_1\\ -1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1,\end{cases}$ (8) $\varphi_8(x)=\pm\begin{cases} +1, & x_\downarrow\in H_0\ x\in H_0\\ -1, & x_\downarrow\in H_0\ x\in H_1\\ -1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1.\end{cases}$

Hence, we must choose weak periodic ground states among these 16 configurations. The following theorem gives the result.

Theorem 3.1. *Let* $|A|=i, i\in\{1,2,\dots,k+1\}$

1) *If* $|A|\neq(k+1)/2$ *then each* H_A*-weak periodic ground state is a* H_A*- periodic or translational-invariant i.e. belongs to the set* $\{\pm\varphi_1(x),\pm\varphi_7(x)\}$.

2) *If* $|A|=(k+1)/2$ *then there are at least two two* H_A *- weak periodic (non-periodic) ground states which are of the form* $\pm\varphi_8(x)$.

Proof. By (2.7) one can see that a configuration ϕ is a ground state if and only if there is $j\in\{0,\dots,k\}$ such that $\phi_b\in\mathcal{C}_j\cup\mathcal{C}_{j+1}$ for any $b\in M$. Thus we must check this property for above mentioned configurations.

1) Let

$$\varphi_1(x)=\begin{cases} +1, & x_\downarrow\in H_0\ x\in H_0\\ +1, & x_\downarrow\in H_0\ x\in H_1\\ +1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1.\end{cases}$$

It is obvious, that H_A weak periodic ground states are translational-invariant.

2) Let

$$\varphi_2(x)=\begin{cases} -1, & x_\downarrow\in H_0\ x\in H_0\\ +1, & x_\downarrow\in H_0\ x\in H_1\\ +1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1.\end{cases}$$

$\forall b\in M$ we have $|\{x\in S_1(c_b):x\in H_0\}|=i$, $|\{x\in S_1(c_b):x\in H_1\}|=k+1-i$.

Denote $A_-=\{x\in S_1(x):\varphi_b(x)=-1\}$, $A_+=\{x\in S_1(x):\varphi_b(x)=+1\}$, and $\varphi_{i,b}=(\varphi_i)_b, i=1,2,\dots,8$.

Assume $c_b\in H_0$. The possible cases are:

a) $c_{b\downarrow}\in H_0$ and $\varphi_{2,b}(c_{b\downarrow})=+1$, then

$$\varphi_{2,b}(c_b)=-1\quad\text{and}\quad |A_-|=i-1,\quad |A_+|=k+2-i,\ \ \varphi_{2,b}\in C_{k+2-i}.$$

b) $c_{b\downarrow}\in H_0$ and $\varphi_{2,b}(c_{b\downarrow})=-1$, then

$$\varphi_{2,b}(c_b)=-1\quad\text{and}\quad |A_-|=i,\quad |A_+|=k+1-i,\ \ \varphi_{2,b}\in C_{k+1-i}.$$

c) $c_{b\downarrow}\in H_1$ and $\varphi_{2,b}(c_{b\downarrow})=+1$, then

$$\varphi_{2,b}(c_b)=+1\quad\text{and}\quad |A_-|=i,\quad |A_+|=k+1-i,\ \ \varphi_{2,b}\in C_i.$$

If $c_b \in H_1$ then we have

d) if $c_{b\downarrow} \in H_0$ and $\varphi_{2,b}(c_{b\downarrow}) = -1$, then

$$\varphi_{2,b}(c_b) = +1 \quad \text{and} \quad |A_-| = 1, \quad |A_+| = k, \;\; \varphi_{2,b} \in C_1.$$

e) If $c_{b\downarrow} \in H_0 ? \varphi_{2,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{2,b}(c_b) = +1 \quad \text{and} \quad A_-| = 0, \quad |A_+| = k+1, \;\; \varphi_{2,b} \in C_0.$$

By (2.7) we find that $C_{k+2-i} \cap C_{k+1-i} \cap C_i = \emptyset$ if $i \neq (k+1)/2, (k+2)/2$. If $i = (k+1)/2$, from d) and e) we get $i = 0$ and $k = -1$, which is impossible. If $i = (k+2)/2$, then $i = 1$ and $k = 0$, which also is impossible. Thus, $\varphi_2(x)$ is not a ground state.

3) Let

$$\varphi_3(x) = \begin{cases} +1, & x_\downarrow \in H_0 \; x \in H_0 \\ -1, & x_\downarrow \in H_0 \; x \in H_1 \\ +1, & x_\downarrow \in H_1 \; x \in H_0 \\ +1, & x_\downarrow \in H_1 \; x \in H_1. \end{cases}$$

Let $c_b \in H_0$. We consider several cases:

a) $c_{b\downarrow} \in H_0$ and $\varphi_{3,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{3,b}(c_b) = +1 \quad \text{and} \quad |A_-| = k+1-i, \quad |A_+| = i, \;\; \varphi_{3,b} \in C_{k+1-i}.$$

b) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{3,b}(c_b) = +1 \quad \text{and} \quad |A_-| = k-i, \quad |A_+| = i+1, \;\; \varphi_{3,b} \in C_{k-i}.$$

c) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = -1$, then

$$\varphi_{3,b}(c_b) = +1 \quad \text{and} \quad |A_-| = k+1-i, \quad |A_+| = i, \;\; \varphi_{3,b} \in C_{k+1-i}.$$

Let $c_b \in H_1$. We have

d) $c_{b\downarrow} \in H_0$ and $\varphi_{3,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{3,b}(c_b) = -1 \quad \text{and} \quad |A_-| = 0, |A_+| = k+1, \quad \varphi_{3,b} \;\; \in C_{k+1}.$$

e) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = -1$, then

$$\varphi_{3,b}(c_b) = +1 \quad \text{and} \quad |A_-| = 1, \quad |A_+| = k, \;\; \varphi_{3,b} \in C_1,$$

f) $c_{b\downarrow} \in H_1$ and $\varphi_{3,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{3,b}(c_b) = +1 \quad \text{and} \quad |A_-| = 0, \quad |A_+| = k+1, \;\; \varphi_{3,b} \in C_0.$$

By (2.7) $C_0 \cap C_1 \cap C_{k+1} = \emptyset$ if $k \neq 0$. Thus, $\varphi_3(x)$ is not a ground state.

4) For $\varphi_j(x),\ \ j=4,5,6$ one similarly can prove that they are not ground states.

5) Consider now

$$\varphi_7(x)=\begin{cases}-1, & x_\downarrow\in H_0\ x\in H_0\\ +1, & x_\downarrow\in H_0\ x\in H_1\\ -1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1\end{cases}=\begin{cases}-1, & x\in H_0\\ +1, & x\in H_1.\end{cases}$$

Consequently $\varphi_7(x)$ is a periodic ground state which is not interesting for us.

6) Consider

$$\varphi_8(x)=\begin{cases}+1, & x_\downarrow\in H_0\ x\in H_0\\ -1, & x_\downarrow\in H_0\ x\in H_1\\ -1, & x_\downarrow\in H_1\ x\in H_0\\ +1, & x_\downarrow\in H_1\ x\in H_1.\end{cases}$$

Let $c_b\in H_0$. The possible cases are:

a) $c_{b\downarrow}\in H_0$ and $\varphi_{8,b}(c_{b\downarrow})=+1$, then

$$\varphi_{8,b}(c_b)=+1 \quad \text{and} \quad |A_-|=k+1-i,\ |A_+|=i, \varphi_{8,b}\in C_{k+1-i}.$$

b) $c_{b\downarrow}\in H_0$ and $\varphi_{8,b}(c_{b\downarrow})=-1$, then

$$\varphi_{8,b}(c_b)=+1 \quad \text{and} \quad |A_-|=k+2-i,\ |A_+|=i-1, \varphi_{8,b}\in C_{k+2-i}.$$

c) $c_{b\downarrow}\in H_1$ and $\varphi_{8,b}(c_{b\downarrow})=-1$, then

$$\varphi_{8,b}(c_b)=-1 \quad \text{and} \quad |A_-|=k+1-i,\ |A_+|=i, \varphi_{8,b}\in C_i.$$

d) $c_{b\downarrow}\in H_1$ and $\varphi_{8,b}(c_{b\downarrow})=+1$, then

$$\varphi_{8,b}(c_b)=-1 \quad \text{and} \quad |A_-|=k-i,\ |A_+|=i+1, \varphi_{8,b}\in C_{i+1}.$$

For $c_b\in H_1$ we have

e) $c_{b\downarrow}\in H_0$ and $\varphi_{8,b}(c_{b\downarrow})=+1$, then

$$\varphi_{8,b}(c_b)=-1 \quad \text{and} \quad |A_-|=k-i,\ |A_+|=i+1, \varphi_{8,b}\in C_{i+1}.$$

f) $c_{b\downarrow}\in H_0$ and $\varphi_{8,b}(c_{b\downarrow})=-1$, then

$$\varphi_{8,b}(c_b)=-1 \quad \text{and} \quad |A_-|=k+1-i,\ |A_+|=i, \varphi_{8,b}\in C_i.$$

g) $c_{b\downarrow}\in H_1$ and $\varphi_{8,b}(c_{b\downarrow})=-1$, then

$$\varphi_{8,b}(c_b)=+1 \quad \text{and} \quad |A_-|=k+2-i,\ |A_+|=i-1, \varphi_{8,b}\in C_{k+2-i}.$$

h) $c_{b\downarrow} \in H_1$ and $\varphi_{8,b}(c_{b\downarrow}) = +1$, then

$$\varphi_{8,b}(c_b) = +1 \quad \text{and} \quad |A_-| = k+1-i,\ |A_+| = i, \varphi_{8,b} \in C_{k+1-i}.$$

If

$$\begin{cases} i \neq k+1-i, \\ i+1 \neq k+2-i, \end{cases} \quad \text{or} \quad \begin{cases} i \neq k+2-i, \\ i+1 \neq k+1-i, \end{cases}$$

We obtain $C_i \cap C_{i+1} \cap C_{k+1-i} \cap C_{k+2-i} = \emptyset$. Only following system has solution

$$\begin{cases} i = k+1-i \\ i+1 = k+2-i \end{cases} \Rightarrow i = \frac{k+1}{2}.$$

Thus the configuration φ_8 is a ground state iff $i = (k+1)/2$. The theorem is proved. □

Corollary 3.1. *If k is a even number then each weak periodic ground state is a periodic one.*

Case 2: index 4.

We take

$$\begin{aligned} A \subset \{1,2,\ldots,k+1\}, \quad & H_A = \Big\{x \in G_k : \sum_{j\in A} w_j(x) - \text{even}\Big\}, \\ G_k^{(2)} &= \{x \in G_k : |x| - \text{even}\}, \quad \text{where } w_j(x) \text{ is the number of } a_j \text{ in word} x; \\ G_k^{(4)} &= H_A \cap G_k^{(2)} - \text{normal subgroup of index 4 [5]}, \\ G_k/G_k^{(4)} &= \{H_0, H_1, H_2, H_3\}, \end{aligned}$$

where

$$\begin{aligned} H_0 &= \Big\{x \in G_k : \sum_{j\in A} w_j(x) - \text{even}, |x| - \text{even}\Big\}, \\ H_1 &= \Big\{x \in G_k : \sum_{j\in A} w_j(x) - \text{odd}, |x| - \text{even}\Big\}, \\ H_2 &= \Big\{x \in G_k : \sum_{j\in A} w_j(x) - \text{even}, |x| - \text{odd}\Big\}, \\ H_3 &= \Big\{x \in G_k : \sum_{j\in A} w_j(x) - \text{odd}, |x| - \text{odd}\Big\}. \end{aligned}$$

The $G_k^{(4)}-$weak periodic configuration is of the form

$$\varphi(x) = \begin{cases} a_{13}, & x_\downarrow \in H_1 \; x \in H_3, \\ a_{31}, & x_\downarrow \in H_3 \; x \in H_1, \\ a_{03}, & x_\downarrow \in H_0 \; x \in H_3, \\ a_{30}, & x_\downarrow \in H_3 \; x \in H_0, \\ a_{21}, & x_\downarrow \in H_2 \; x \in H_1, \\ a_{12}, & x_\downarrow \in H_1 \; x \in H_2, \\ a_{02}, & x_\downarrow \in H_0 \; x \in H_2, \\ a_{20}, & x_\downarrow \in H_2 \; x \in H_0, \end{cases} \tag{3.1}$$

where $a_{pq} = \pm 1$ and $p, q \in \{0, 1, 2, 3\}$.

Thus we have to determine which of them are ground states.

Theorem 3.2. *Let* $|A| = i, \; i \in \{1, 2, \dots, k+1\}$.

i) *If* $i \neq (k+1)/2$ *then each* $G_k^{(4)}$*-weak periodic ground state is a periodic.*

ii) *If* $i = (k+1)/2$*, then there are periodic and four* $G_k^{(4)}$*-weak periodic ground states:*

$$\pm\varphi'(x) = \begin{cases} +1, & x_\downarrow \in H_1 \; x \in H_3 \\ +1, & x_\downarrow \in H_3 \; x \in H_1 \\ -1, & x_\downarrow \in H_0 \; x \in H_3 \\ -1, & x_\downarrow \in H_3 \; x \in H_0 \\ -1, & x_\downarrow \in H_2 \; x \in H_1 \\ -1, & x_\downarrow \in H_1 \; x \in H_2 \\ +1, & x_\downarrow \in H_0 \; x \in H_2 \\ +1, & x_\downarrow \in H_2 \; x \in H_0, \end{cases} \quad \textit{and} \quad \pm\varphi''(x) = \begin{cases} -1, & x_\downarrow \in H_1 \; x \in H_3 \\ +1, & x_\downarrow \in H_3 \; x \in H_1 \\ +1, & x_\downarrow \in H_0 \; x \in H_3 \\ -1, & x_\downarrow \in H_3 \; x \in H_0 \\ -1, & x_\downarrow \in H_2 \; x \in H_1 \\ +1, & x_\downarrow \in H_1 \; x \in H_2 \\ -1, & x_\downarrow \in H_0 \; x \in H_2 \\ +1, & x_\downarrow \in H_2 \; x \in H_0. \end{cases}$$

Remark 3.1.

a) Using Theorems 1-3 we can give the phase diagram of the ground states for any k. For $k = 3$ it is shown in Figure 3.1.

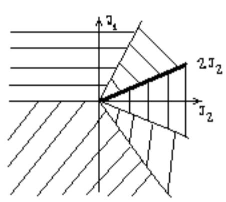

Figure 3.1: Only on the line $J_1 = 2J_2$ there are weak periodic ground states.

b) By Remark 2.1 and our theorems we get $J_1 = 2J_2, J_2 > 0$. Consequently, for ordinary Ising model (i.e. $J_2 = 0$) there is no weak periodic ground state. Since for $J_1 = 2J_2 = 0$ the Hamiltonian equals to zero.

c) For cases of index other than 2 and 4 the description of weak periodic ground states becames a technically difficult problem.
d) We note that any normal subgroup of index two has a form H_A for a suitable $A \subset \{1, 2, \dots, k+1\}$. But there are many normal subgroups of index four which do not coincide with $G_k^{(4)}$, for example, $H_A \cap H_B$ for $A, B \subsetneqq \{1, 2, \dots, k+1\}$ with $A \neq B$ is a normal subgroup of index four but it does not coincide with $G_k^{(4)}$.

Proof of Theorem 3.2. Consider several cases of configurations (3.1).

1) Let $a_{pq} = +1 (a_{pq} = -1), \forall p, q \in \{0, 1, 2, 3\}$. Obviously, $G_k^{(4)}$-weak periodic ground states are translation invariant.

2) $\forall b \in M$ we have

$$|\{x \in S_1(c_b) : c_b \in H_0, x \in H_3\}| = i, \quad |\{x \in S_1(c_b) : c_b \in H_0, x \in H_2\}| = k + 1 - i,$$

$$|\{x \in S_1(c_b) : c_b \in H_1, x \in H_2\}| = i, \quad |\{x \in S_1(c_b) : c_b \in H_1, x \in H_3\}| = k + 1 - i,$$

$$|\{x \in S_1(c_b) : c_b \in H_2, x \in H_1\}| = i, \quad |\{x \in S_1(c_b) : c_b \in H_2, x \in H_0\}| = k + 1 - i,$$

$$|\{x \in S_1(c_b) : c_b \in H_3, x \in H_0\}| = i, \quad |\{x \in S_1(c_b) : c_b \in H_3, x \in H_1\}| = k + 1 - i.$$

We denote $A_- = \{x \in S_1(x) : \varphi_b(x) = -1\}, \quad A_+ = \{x \in S_1(x) : \varphi_b(x) = +1\}$. and we put $a_{13} = -1$, and $a_{pq} = +1$ for others. Assume $c_b \in H_0$, then the following cases are possible:

a) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{13} = -1$, then

$$\varphi_b(c_b) = a_{30} = +1, \quad |A_-| = 1, \quad |A_+| = k.$$

Consequently $\varphi_b \in C_1$.
b) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{03} = +1$, then

$$\varphi_b(c_b) = a_{30} = +1, \quad |A_-| = 0, ;\ |A_+| = k + 1.$$

Thus $\varphi_b \in C_0$.

If $c_b \in H_3$, then we have
c) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = a_{31} = +1$, then

$$\varphi_b(c_b) = a_{13} = -1, \quad |A_-| = 0, \quad |A_+| = k + 1.$$

Consequently $\varphi_b \in C_{k+1}$.

By (2.7) one can see that a configuration ϕ is a ground state if and only if there is $j \in \{0, \dots, k\}$ such that $\phi_b \in \mathcal{C}_j \cup \mathcal{C}_{j+1}$ for any $b \in M$. Thus it is enough to check this property for above mentioned configurations. We have $C_0 \cap C_1 \cap C_{k+1} \neq \emptyset$ for any $k \geq 1$. Thus φ is not a ground state. Similarly one can prove that if $p_0, q_0, a_{p_0 q_0} = -1$ and others $a_{pq} = +1$, then the configuration is not a ground state.

3) Let $a_{13} = a_{31} = -1$ and others $a_{pq} = +1$.

If $c_b \in H_0$, we have

a) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{13} = -1$, then

$$\varphi_b(c_b) = a_{30} = +1, \quad |A_-| = 1, \quad |A_+| = k.$$

Thus $\varphi_b \in C_1$.

b) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{03} = +1$, then

$$\varphi_b(c_b) = a_{30} = +1, \quad |A_-| = 0, \quad |A_+| = k+1.$$

Consequently $\varphi_b \in C_0$.

If $c_b \in H_1$ then

c) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{03} = +1$, then

$$\varphi_b(c_b) = a_{31} = -1, \quad |A_-| = k-i, \quad |A_+| = i+1.$$

Thus $\varphi_b \in C_{i+1}$.

d) $c_{b\downarrow} \in H_3$ and $\varphi_b(c_{b\downarrow}) = a_{13} = -1$, then

$$\varphi_b(c_b) = a_{31} = -1, \quad |A_-| = k-i+1, \quad |A_+| = i.$$

Consequently $\varphi_b \in C_i$.

e) $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = a_{12} = +1$, then

$$\varphi_b(c_b) = a_{21} = +1, \quad |A_-| = k-i+1, \quad |A_+| = i.$$

Thus $\varphi_b \in C_{k-i+1}$.

By (2.7) $C_0 \cap C_1 \cap C_i \cap C_{i+1} \cap C_{k+1} \neq \emptyset$ for all $k \geq 1$. Thus φ is not a ground state. All other cases can be checked similarly.

Now we shall prove (ii) for configurations φ'. Let $a_{13} = a_{31} = a_{02} = a_{20} = -1$, and $a_{pq} = +1$ for others. If $c_b \in H_0$, we consider the cases:

a) $c_{b\downarrow} \in H_3$ and $\varphi'_b(c_{b\downarrow}) = a_{13} = -1$, then

$$\varphi'_b(c_b) = a_{30} = +1, \quad |A_-| = k+2-i, \quad |A_+| = i-1.$$

Thus $\varphi'_b \in C_{k+2-i}$.

b) $c_{b\downarrow} \in H_3$ and $\varphi'_b(c_{b\downarrow}) = a_{03} = +1$, thus

$$\varphi'_b(c_b) = a_{30} = +1, \quad |A_-| = k+1-i, \quad |A_+| = i.$$

Thus $\varphi'_b \in C_{k+1-i}$.

c) $c_{b\downarrow} \in H_2$ and $\varphi'_b(c_{b\downarrow}) = a_{02} = -1$, then

$$\varphi'_b(c_b) = a_{20} = -1, \quad |A_-| = k+1-i, \quad |A_+| = i.$$

Consequently $\varphi'_b \in C_i$.
d) $c_{b\downarrow} \in H_2$ and $\varphi'_b(c_{b\downarrow}) = a_{12} = +1$, then

$$\varphi'_b(c_b) = a_{20} = -1, \quad |A_-| = k - i, \quad |A_+| = i + 1.$$

Consequently $\varphi'_b \in C_{i+1}$.

Assume $c_b \in H_1$, then
a1) $c_{b\downarrow} \in H_3$ and $\varphi'_b(c_{b\downarrow}) = a_{03} = +1$, then

$$\varphi'_b(c_b) = a_{31} = -1, \quad |A_-| = k - i, \quad |A_+| = i + 1.$$

Consequently $\varphi'_b \in C_{i+1}$,
b1) $c_{b\downarrow} \in H_3$ and $\varphi'_b(c_{b\downarrow}) = a_{13} = -1$, then

$$\varphi'_b(c_b) = a_{31} = -1, \quad |A_-| = k + 1 - i, \quad |A_+| = i.$$

Consequently $\varphi'_b \in C_i$.
c1) $c_{b\downarrow} \in H_2$ and $\varphi'_b(c_{b\downarrow}) = a_{12} = +1$, then

$$\varphi'_b(c_b) = a_{21} = +1, \quad |A_-| = k + 1 - i, \quad |A_+| = i.$$

Thus $\varphi'_b \in C_{k+1-i}$.
d1) $c_{b\downarrow} \in H_2$ and $\varphi'_b(c_{b\downarrow}) = a_{02} = -1$, then

$$\varphi'_b(c_b) = a_{21} = +1, \quad |A_-| = k + 2 - i, \quad |A_+| = i - 1.$$

Thus $\varphi'_b \in C_{k+2-i}$.

If $c_b \in H_2$, we consider:
a2) $c_{b\downarrow} \in H_0$ and $\varphi'_b(c_{b\downarrow}) = a_{20} = -1$, then

$$\varphi'_b(c_b) = a_{02} = -1, \quad |A_-| = k + 1 - i, \quad |A_+| = i.$$

Thus $\varphi'_b \in C_i$.
b2) $c_{b\downarrow} \in H_0$ and $\varphi'_b(c_{b\downarrow}) = a_{30} = +1$, then

$$\varphi'_b(c_b) = a_{02} = -1, \quad |A_-| = k - i, \quad |A_+| = i + 1.$$

Hence $\varphi'_b \in C_{i+1}$.
c2) $c_{b\downarrow} \in H_1$ and $\varphi'_b(c_{b\downarrow}) = a_{21} = +1$, then

$$\varphi'_b(c_b) = a_{12} = +1, \quad |A_-| = k + 1 - i, \quad |A_+| = i.$$

Thus $\varphi'_b \in C_{k+1-i}$.
d2) $c_{b\downarrow} \in H_1$ and $\varphi'_b(c_{b\downarrow}) = a_{31} = -1$, then

$$\varphi'_b(c_b) = a_{12} = +1, \quad |A_-| = k + 2 - i, \quad |A_+| = i - 1.$$

Consequently $\varphi'_b \in C_{k+2-i}$.

Suppose $c_b \in H_3$, then

a3) $c_{b\downarrow} \in H_0$ and $\varphi'_b(c_{b\downarrow}) = a_{20} = -1$, then

$$\varphi'_b(c_b) = a_{03} = +1, \quad |A_-| = k+2-i, \quad |A_+| = i-1.$$

Consequently $\varphi'_b \in C_{k+2-i}$.

b3) $c_{b\downarrow} \in H_0$? $\varphi'_b(c_{b\downarrow}) = a_{30} = +1$, then

$$\varphi'_b(c_b) = a_{03} = +1, \quad |A_-| = k+1-i, \quad |A_+| = i.$$

Thus $\varphi'_b \in C_{k+1-i}$.

c3) $c_{b\downarrow} \in H_1$ and $\varphi'_b(c_{b\downarrow}) = a_{21} = +1$, then

$$\varphi'_b(c_b) = a_{13} = -1, \quad |A_-| = k-i, \quad |A_+| = i+1.$$

Thus $\varphi'_b \in C_{i+1}$.

d3) $c_{b\downarrow} \in H_1$ and $\varphi'_b(c_{b\downarrow}) = a_{31} = -1$, then

$$\varphi'_b(c_b) = a_{13} = -1, \quad |A_-| = k-i+1, \quad |A_+| = i.$$

Hence $\varphi'_b \in C_i$.

If

$$\begin{cases} i \neq k+1-i, \\ i+1 \neq k+2-i, \end{cases} \quad \text{or} \quad \begin{cases} i \neq k+2-i, \\ i+1 \neq k+1-i, \end{cases}$$

We have $C_i \cap C_{i+1} \cap C_{k+1-i} \cap C_{k+2-i} = \emptyset$. Thus it is easy to see that the configuration is a ground state iff $i = (k+1)/2$. Similarly one can prove that $\varphi''(x)$ is a ground state iff $i = (k+1)/2$. The theorem is proved. $\square$

References

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London/New York, 1982.

[2] P. M. Bleher and N. N. Ganikhodjaev, On pure phases of the Ising model on the Bethe lattice, *Theor. Probab. Appl.* **35** (1990), 216–227.

[3] P. M. Bleher, J. Ruiz, and V. A. Zagrebnov, On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, *Jour. Statist. Phys.* **79** (1995), 473–482.

[4] P. M. Bleher, J. Ruiz, R. H. Schonmann, S. Shlosman, and V. A. Zagrebnov, Rigidity of the critical phases on a Cayley tree, *Moscow Math. Journ*. **3** (2001), 345–362.

[5] N. N. Ganikhodjaev and U. A. Rozikov, A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* **111** (1997), 480–486.

[6] R. A. Minlos, *Introduction to Mathematical Statistical Physics*, University lecture series, v.19, 2000.

[7] F. M. Mukhamedov and U. A. Rozikov, On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras, *Jour. of Stat. Phys.* **114** (2004), 825–848.

[8] Kh. A. Nazarov and U. A. Rozikov, Periodic Gibbs measures for the Ising model with competing interactions, *Theor. Math. Phys.* **135** (2003), 881–888.

[9] C. Preston, *Gibbs States on Countable Sets*, Cambridge University Press, London, 1974.

[10] U. A. Rozikov, Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions, *Theor. Math. Phys.* **112** (1997), 929–933.

[11] U. A. Rozikov, A Constructive Description of Ground States and Gibbs Measures for Ising Model With Two-Step Interactions on Cayley Tree, *Jour. Statist. Phys.* **122** (2006), 217–235.

[12] Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results*, Pergamon, Oxford, 1982.

[13] S. Zachary, Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11** (1983), 894–903.

Rahmatullaev Muzaffar is a PhD student at the Namangan State University. His current research interest is in the description of Gibbs measures and ground states of some lattice models on trees. He has published several papers in verious refereed journals.