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A Decay Estimate for Constrained Semilinear Systems

Azzeddine Tsouli[1,*], *Mohamed Ouzahra*[2] *and Ali Boutoulout*[1]

[1] TSI Team, MACS Laboratory, Department of Mathematics abd computer, Faculty of Sciences Meknes, Moulay Ismail University, Morocco
[2] AMES Team (Applied Mathematics and Engineering Sciences), Department of Mathematics & Informatics, ENS. Univesity of Sidi Mohamed Ben Abdellah, Fes, Morocco

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Abstract: The paper proposes a constrained feedback control that guarantee weak and strong stabilizability for distributed semilinear systems of the form :

$$\frac{dy(t)}{dt} = Ay(t) + p(t)Ny(t),$$

where A is the infinitesimal generator of a linear C_0-semigroup of contractions on a Hilbert space H and N is a (nonlinear) operator from H into its self. A decay rate of the state is estimated. Also the robustness of the considered control is discussed. Applications and simulations are provided.

Keywords: Distributed semilinear systems, constrained, stabilization, decay estimate, robustness.

1 Introduction

Semilinear systems can be used to represent a wide range of physical, chemical, biological and social systems as well as manufacturing processes. Semilinear structures are derived in a natural manner to approximate the description of nuclear fission and heat transfer. The semilinear nature of nuclear fission follows from the fact that the state (neutron level or power) is multiplied by the control function (reactivity or neutron). A multiplication of coolant flow rate (a control variable) and temperature (a state variable) is produced in heat transfer between a solid wall, such as a reactor core, and moving coolant fluid. Even the generation of poison products in nuclear reactors may be described by a bilinear model with thermal neutron flux (the control) multiplying xenon concentration (see [1,2,3]). Here we consider infinite-dimensional semilinear systems of the form

$$\frac{dy(t)}{dt} = Ay(t) + p(t)Ny(t), \; y(0) = y_0, \tag{1}$$

on a Hilbert space H with inner product $\langle \cdot,\cdot \rangle$ and corresponding norm $\|.\|$, where A generates a semigroup of contractions $S(t)$ on H and N is a nonlinear operator from H to H such that $N(0) = 0$. While the scalar valued function $p(.)$ is a control. The conventional control for stabilization problem of (1) is given by

$$p(t) = -\langle Ny(t), y(t) \rangle \tag{2}$$

(see [4,5,6,7]). The problem of stabilizing the system (1) was considered in [4], where N is sequentially continuous from H_w (H endowed with the weak topology) to H. Then it has been shown that under the condition :

$$\langle NS(t)y, S(t)y \rangle = 0, \; \forall t \geq 0 \Longrightarrow y = 0, \tag{3}$$

the quadratic feedback (2) weakly stabilizes the system (1).
Under the assumption

$$\int_0^T |\langle NS(t)y, S(t)y \rangle| dt \geq \delta \|y\|^2, \; \forall y \in H, \; (T, \delta > 0), \tag{4}$$

a strong stabilization result has been obtained using the control (2) (see [5,7]). However, in this way the convergence of the resulting closed loop state is not better than $\|y(t)\| = O(\frac{1}{\sqrt{t}})$.

* Corresponding author e-mail: azzeddine.tsouli@gmail.com

Control systems are often subject to constraints on their manipulated inputs. Input constraints arise as a manifestation of the physical limitations inherent in the capacity of control actuators. Stabilization question of constrained bilinear and semilinear systems have been considered in many works (see [8,6,9,10,11,12]). In this paper, we study weak and strong stabilizability of the system (1) using a control constraint of the form (eventually, after re-escaling) $|p(t)| \leq 1$. Among saturating feedbacks, the following law

$$p(t) = \begin{cases} -\dfrac{< Ny(t), y(t) >}{\|y(t)\|^2}, & y(t) \neq 0 \\ 0, & y(t) = 0, \end{cases} \tag{5}$$

has been considered in [11,12]. In [13] the rational decay rates are established i.e.,

$$\|y(t)\| = O(t^{\frac{-1}{2-r}}),$$

using the following feedback control

$$p_r(t) = -\frac{< Ny(t), y(t) >}{\|y(t)\|^r}, r \in (-\infty, 2). \tag{6}$$

Here, we consider the following continuous control

$$p(t) = -\frac{\langle Ny(t), y(t)\rangle}{1 + |\langle Ny(t), y(t)\rangle|}. \tag{7}$$

This type of feedback has been treated in [6], where it has been shown that if the resolvent of A is compact, N is a bounded linear self-adjoint and monotone operator then under the assumption (3), the feedback (7) strongly stabilizes (1), but no estimate has been given. Here, we will establish an explicit decay estimate of the stabilized state for a large class of semilinear systems. The paper is organized as follows : In the second section, we establish an existence and uniqueness result for the mild solution and we show that the feedback (7) guarantees the weak and strong stability of (1) with a decay estimate. Also we analyze the robustness of the stabilizing control. In the third section, we give some applications.

2 Stabilization results

Let us recall the following definition concerning the asymptotic behavior of the system (1).

Definition 2.1. The system (1) is weakly (resp. strongly) stabilizable if there exists a feedback control $p(t) = f(y(t))$, $f : H \to K := \mathbb{R}, \mathbb{C}$ such that the corresponding mild solution satisfies the properties :

1. for each y_0 there exists a unique mild solution $y(t)$, defined for all $t \in \mathbb{R}^+$ of (1),
2. $\{0\}$ is a an equilibrium of (1),
3. $y(t) \to 0$, weakly (resp. strongly), as $t \to +\infty$ for all $y_0 \in H$.

In the sequel the following result will be needed for our stabilization problem and constitutes an extension of the one given in [9].

Lemma 2.1. Let u be a positive and increasing function such that $u(0) = 0$, and let $v(x) = x - (I+u)^{-1}(\gamma x)$. Consider a sequence $(s_k)_{k\geq 0}$ of positive numbers which satisfies

$$s_{k+1} + u(s_{k+1}) \leq \gamma s_k, \text{ where } \forall k \geq 0, \tag{8}$$

for some $\gamma > 0$. Then $s_k \leq X(k)$, where $X(t)$ is a solution of the differential equation

$$\frac{dX(t)}{dt} + v(X(t)) = 0, \; X(0) = s_0. \tag{9}$$

Proof. The proof uses similar techniques as in [9]. It is done by induction on k. Assume that $s_k \leq X(k)$ (this is the induction hypothesis) and prove that $s_{k+1} \leq X(k+1)$. Since $(I+u)^{-1}$ is monotone increasing, then the inequality (8) is equivalent to

$$s_{k+1} \leq (I+u)^{-1}(\gamma s_k) = s_k - v(s_k). \tag{10}$$

Integrating the equation (9) from k to $k+1$; yields

$$X(k+1) - X(k) + \int_k^{k+1} v(X(\tau))d\tau = 0.$$

On the other hand, since v is an increasing function, the solution $X(t)$ of (9) is such that

$$X(t) \leq X(\tau), \; \forall t \geq \tau \geq 0. \tag{11}$$

Using (11), the induction assumption and the fact that v and $(I+u)^{-1}$ are increasing, we obtain

$$\begin{aligned} X(k+1) &\geq (I+u)^{-1}(\gamma X(k)) \geq (I+u)^{-1}(\gamma s_k) \\ &= s_k - v(s_k) \geq s_{k+1}. \end{aligned} \tag{12}$$

This yields the desired result ■

Remark. For $\gamma = 1$, we retrieve the result of [9].

2.1 Decay estimate

We begin with the following result concerning the existence of the mild solution and giving a useful estimate for our stabilization problem.

Theorem 2.1. Let A generate a semigroup $S(t)$ of contractions on H and let N be locally Lipschitz. Then the system (1) controlled with (7) possesses a unique mild solution $y(t)$ which verifies

$$\begin{aligned} &\left(\int_0^T |\langle NS(s)y(t), S(s)y(t)\rangle| ds \right)^2 = \\ &O\left(\int_t^{t+T} \frac{|\langle Ny(s), y(s)\rangle|^2}{1 + |\langle Ny(s), y(s)\rangle|} ds \right), \quad \text{as } t \to +\infty. \end{aligned} \tag{13}$$

Proof. Let us consider the closed loop-system :

$$\frac{dy(t)}{dt} = Ay(t) + f(y(t)),\ y(0) = y_0, \tag{14}$$

where

$$f(y) = -\frac{\langle Ny,y\rangle}{1+|\langle Ny,y\rangle|}Ny,\ \forall y \in H.$$

To establish the existence and uniqueness of the solution of (14), let us show that the function f is locally Lipschitz. For all $y,z \in H$, we have

$$\begin{aligned}\|f(y)-f(z)\| \le \| &\frac{\langle Ny,y-z\rangle Ny}{1+|\langle Ny,z\rangle|}\| \\ &+\|\frac{\langle Ny,z\rangle Ny}{1+|\langle Ny,y\rangle|} - \frac{\langle Nz,z\rangle Nz}{1+|\langle Nz,z\rangle|}\|\end{aligned}$$

Since N is locally Lipschitz, then for each $R>0$ there exists a positive constant L_R such that

$$\|Nz - Ny\| \le L_R\|z-y\|,\ \ \forall (z,y) \in H^2:\ \|z\| \le R,\ \|y\| \le R. \tag{15}$$

Using (15), we deduce that

$$\begin{aligned}\|f(y)-f(z)\| \le\ & C_1\|y-z\| + \\ & \|\frac{\langle Ny,z\rangle Ny}{1+|\langle Ny,y\rangle|} - \frac{\langle Nz,z\rangle Nz}{1+|\langle Nz,z\rangle|}\|,\ \ C_1 = R^2L_R^2 \\ & \le C_1\|y-z\| + \|\frac{\langle Nz-Ny,z\rangle Nz}{1+|\langle Nz,z\rangle|}\| + \\ & \|\frac{\langle Ny,z\rangle Nz}{1+|\langle Nz,z\rangle|} - \frac{\langle Ny,z\rangle Ny}{1+|\langle Ny,y\rangle|}\| \\ & \le C_2\|y-z\| + \\ & \|\frac{\langle Ny,z\rangle Ny}{1+|\langle Ny,y\rangle|} - \frac{\langle Ny,z\rangle Nz}{1+|\langle Nz,z\rangle|}\|,\ \ C_2 = 2C_1 \\ & \le C_2\|y-z\| + \|\frac{\langle Ny,z\rangle\left(Ny-Nz\right)}{1+|\langle Nz,z\rangle|}\| + \\ & \|\frac{\langle Ny,z\rangle Ny}{1+|\langle Ny,y\rangle|} - \frac{\langle Ny,z\rangle Ny}{1+|\langle Nz,z\rangle|}\| \\ & \le C_3\|y-z\| + \frac{\|\langle Ny,z\rangle Ny\|}{(1+|\langle Ny,y\rangle|)(1+|\langle Nz,z\rangle|)}\times \\ & |\langle Ny,y\rangle - \langle Nz,z\rangle|,\ \ C_3 = 3C_1 \\ & \le C_3\|y-z\| + \\ & C_4\left(|\langle Ny,y-z\rangle| + |\langle Ny-Nz,z\rangle|\right),\ \ C_4 = RC_1 \\ & \le C_5\|y-z\|,\ \ C_5 = C_3 + 2RC_4L_R.\end{aligned}$$

Hence g is locally Lipschitz. Then (see Theorem 1.2, p 184 in [15]), the system (14) admits a unique mild solution defined on a maximal interval $[0,t_{\max}[$, by the variation of constant formula :

$$y(t) = S(t)y_0 + \int_0^t S(t-s)g(y(s))ds. \tag{16}$$

Furthermore, using approximation techniques (see [14]) we get

$$\|y(t)\|^2 - \|y(s)\|^2 + 2\int_s^t \frac{|\langle Ny(\tau),y(\tau)\rangle|^2}{1+|\langle Ny(\tau),y(\tau)\rangle|}d\tau \le 0, \forall t,s \ge 0. \tag{17}$$

It follows from (17) that

$$\|y(t)\| \le \|y_0\|,\ \forall t \in [0,t_{\max}[, \tag{18}$$

which holds by density, for all $y_0 \in H$, and hence $y(t)$ is a global solution i.e $t_{\max} = +\infty$. Now, let us establish the estimate (13). From (16) and Schwartz's inequality, we get

$$\|y(t)-S(t)y_0\| \le L_{\|y_0\|}\|y_0\|\left(T\int_0^t \frac{|\langle y(s),Ny(s)\rangle|^2}{1+|\langle y(s),Ny(s)\rangle|}ds\right)^{\frac{1}{2}}\ \ \forall t \in [0,T]. \tag{19}$$

Using (18) and the fact that $S(t)$ is a semigroup of contractions, we deduce that

$$\begin{aligned}|\langle NS(s)y_0,S(s)y_0\rangle| \le\ & 2L_{\|y_0\|}\|y(s)-S(s)y_0\|\|y_0\| + \\ & |\langle Ny(s),y(s)\rangle|.\end{aligned} \tag{20}$$

Replacing y_0 by $y(t)$ in (19) and (20) and using the semigroup property of the solution $y(t)$, we obtain

$$\begin{aligned}|\langle NS(s)y(t),S(s)y(t)\rangle| \le\ & 2L^2_{\|y_0\|}\|\|y_0\|^2\times \\ & \left(T\int_t^{t+T}\frac{|\langle y(s),Ny(s)\rangle|^2}{1+|\langle y(s),Ny(s)\rangle|}ds\right)^{\frac{1}{2}} \\ & +|\langle Ny(t+s),y(t+s)\rangle|,\ \forall t,s \ge 0.\end{aligned}$$

Integrating this last inequality over the interval $[0,T]$ and using Schwartz's inequality, it follows that

$$\begin{aligned}&\int_0^T |\langle NS(s)y(t),S(s)y(t)\rangle|ds \le \\ &\left(2L^2_{\|y_0\|}\|y_0\|^2T^{\frac{3}{2}} + T\left(1+L_{\|y_0\|}\|y_0\|^2\right)\right)\times \\ &\left(\int_t^{t+T}\frac{|\langle y(s),Ny(s)\rangle|^2}{1+|\langle y(s),Ny(s)\rangle|}ds\right)^{\frac{1}{2}}.\end{aligned}$$

Which gives the estimate (13) ■

The following result concerns the strong stabilization of (1) by the control (7).

Theorem 2.2. Let A generate a semigroup $S(t)$ of contractions on H and let N be locally Lipschitz such that (4) holds. Then the feedback (7) strongly stabilizes (1) with the following decay estimate

$$\|y(t)\| = O\left(\frac{1}{\sqrt{t}}\right),\ \text{as } t \to +\infty. \tag{21}$$

Proof. Using (17), we obtain

$$\|y(kT)\|^2 - \|y((k+1)T)\|^2 \ge 2\int_{kT}^{(k+1)T}\frac{|\langle Ny(t),y(t)\rangle|^2}{1+|\langle Ny(t),y(t)\rangle|}dt\ \ \forall k \in \mathbb{N},$$

and using (4), we deduce that

$$\|y(kT)\|^2 - \|y((k+1)T)\|^2 \ge M_1\|y(kT)\|^4, \tag{22}$$

where $M_1 = \dfrac{\delta^2}{2\left(2L^2_{\|y_0\|}\|y_0\|^2T^{\frac{3}{2}} + T\left(1+L_{\|y_0\|}\|y_0\|^2\right)\right)^2}.$

Since $\|y(kT)\|$ is a decreasing function, it follows that

$$\|y(kT)\|^2 - \|y((k+1)T)\|^2 \ge M_1\|y((k+1)T)\|^4. \tag{23}$$

Setting $s_k = \|y(kT)\|^2$ and $u(s) = M_1 s^2$, we get $u(s_{k+1}) + s_{k+1} \leq s_k$. Applying Lemma 2 with $\gamma = 1$, we deduce that $s_k \leq X(k)$, where $X(t)$ is the solution of $X'(t) + v(X(t)) = 0$, $X(0) = s_0$. Furthermore, it is easy to see that $v(s) = M_1 s^2 + o(s^2)$. It follows that

$$X'(t) \sim -M_1 X^2(t) \cdot \tag{24}$$

Integrating (24)and using the fact that $s_k \leq X(k)$ we get $s_k = O(k^{-1})$. Since $\|y(t)\|$ is decreasing in times, then the last discrete estimate implies the continuous one (21) ■

Remark . Note that for all initial states $y_0 \in H$, we have $|p(t)| \leq 1$, for all $t \geq 0$, and if the system (1) is subject to the control constraint $|p(t)| \leq M$, then one may consider the pondered control $Mp(t)$.

2.2 Robustness

In this part, we exhibit a class of allowed perturbations under which, the stability of the closed loop (14) is preserved. We consider the perturbed system

$$\frac{dy(t)}{dt} = Ay(t) - \frac{\langle y(t), Ny(t)\rangle}{1+|\langle y(t), Ny(t)\rangle|} Ny(t) + \xi(y(t)), \tag{25}$$

where ξ maps H to it self. A common question in application is: how large can the perturbation ξ be that leaves the strong stability of the dynamics (25)? This analysis is called the robustness analysis in control systems literature [16,17,18]. In this context, we establish the following result.

Theorem 2.3. Let assumptions of Theorem 2.1 hold. Then the estimate (21) is preserved under the perturbation ξ provided that ξ is locally Lipschitz and

$$\|\xi(y)\| \leq \frac{|\langle y, Ny\rangle|^2}{\|y\|\left(1+|\langle y, Ny\rangle|\right)} \cdot \tag{26}$$

Proof. First let us note that 0 remains an equilibrium of the perturbed system (25), which can be written in the form

$$\frac{dy(t)}{dt} = Ay(t) + g(y(t)),\ y(0) = y_0, \tag{27}$$

where $g = f + \xi$ and

$$f(y) = \begin{cases} -\dfrac{\langle y, Ny\rangle}{1+|\langle y, Ny\rangle|} Ny, & y \neq 0 \\ 0, & y = 0. \end{cases}$$

Since f and ξ are locally Lipschitz, then so is g. Also g is dissipative: $\langle g(y), y\rangle \leq 0$, $\forall y \in H \cdot$ Then from the proof of Theorem 2.1 the system (25) admits a unique mild solution such that

$$y(t) = S(t)y_0 + \int_0^t S(t-s)g(y(s))ds,\ \forall t \geq 0 \cdot \tag{28}$$

Furthermore, we have

$$\|y(t)\|^2 - \|y(s)\|^2 + 2\int_s^t \frac{|\langle y(\tau), Ny(\tau)\rangle|^2}{1+|\langle y(\tau), Ny(\tau)\rangle|} d\tau - 2\times \int_s^t Re\left(\langle \xi(y(\tau)), y(\tau)\rangle\right) d\tau \leq 0,\ \forall t, s \geq 0. \tag{29}$$

Remarking that (26) implies that :

$$|\langle \xi(y), y\rangle| \leq \frac{|\langle y, Ny\rangle|^2}{1+|\langle y, Ny\rangle|},$$

it follows from similar techniques as in the proof of the Theorem 2.1 that

$$\int_0^T |\langle NS(s)y(t), S(s)y(t)\rangle| ds \leq C_1 \left(\int_t^{t+T} \frac{|\langle y(s), Ny(s)\rangle|^2}{1+|\langle y(t), Ny(t)\rangle|} ds\right)^{\frac{1}{2}},\ C_1 = C_1(\|y_0\|) > 0,$$

and using (29), we deduce that

$$s_k - s_{k+1} \geq 2\int_{kT}^{(k+1)T} \frac{|\langle y(\tau), Ny(\tau)\rangle|^2}{1+|\langle y(\tau), y(\tau)\rangle|} d\tau - 2\times \int_{kT}^{(k+1)T} Re(\langle \xi(y(\tau)), y(\tau)\rangle) d\tau,$$

where $s_k = \|y(kT\|^2$, $\forall k \geq 0$. Using (4), (13) and the fact that ξ is locally Lipschitz we obtain

$$s_k - s_{k+1} \geq C_2\left(s_k^2 - s_k\right),\ C_2 = C(\|y_0\|).$$

Witch may be written as:

$$\gamma s_k \geq u(s_{k+1}) + s_{k+1},\ k \geq 0,$$

where $\gamma = 1 + C_2 > 0$, $u(t) = C_2 t^2$. Applying Lemma 2 and proceeding as in the proof of Theorem 2.1, we obtain the estimate of the perturbed system.

Remark .

1. The robustness of the control (7) to the perturbation ξ can be regarded as a robustness to the perturbation of A by ξ.
2. The problem of robustness of the control (2) and (5) has been studied in [19] and [11], respectively.

2.3 Weak stabilization

Our result concerning the weak stabilization is stated as follow:

Theorem 2.1. Let A generate a semigroup $S(t)$ of contractions on H and let N be locally Lipschitz and sequentially continuous from H_w to H, such that (3) holds.
Then

1. The feedback (7) weakly stabilizes (1).
2. The system (25) remains weakly stable under any perturbation ξ, which is sequentially continuous and satisfies : $-\frac{|\langle y,Ny\rangle|^2}{1+|\langle y,Ny\rangle|} \leq \langle \xi(y),y\rangle \leq 0\cdot$

Proof. Let f and g be defined as in the proof of Theorem 2.1

1. From Theorem 2.1, there is a unique mild solution for the system (1). Since the function $N : H_w \to H$ is sequentially continuous, then so is f. Moreover we have $\langle f(y),y\rangle \leq 0$, $\forall y \in H$. Then the weak stability of (1) follows from Theorem 2.4 of Ball [4].
2. Remarking that the assumption $-\frac{|\langle y,Ny\rangle|^2}{1+|\langle y,Ny\rangle|} \leq \langle \xi(y),y\rangle \leq 0$ together with (3) guarantees the following implication
$$\langle g(S(t)y),S(t)y\rangle = 0 \Longrightarrow y = 0,$$
the conclusion follows from the same arguments as in the above point.

3 Applications

Example 3.1. In this example, we give an application of Theorem 2.1 to a finite dimensional bilinear system, and concerns the stabilization of a single oscillatory motion by means of suitable damping. Such motion is described by two dimensional system of ordinary differential equations of Lienard's type like:

$$y'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y(t) + p(t)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} y(t) \qquad (30)$$

In [20], the quadratic feedback $p(t) = -y_2^2$ where $y(t) = (y_1(t),y_2(t))$ has been used to obtain the estimate (21). However this feedback law is not bounded with respect to initial states. Applying Theorem 2.1, we deduce that the bounded control $p(t) = -\frac{y_2^2}{1+y_2^2}$ strongly stabilizes the system (30). Here $p(t)Ny(t)$ models a damping device of structure described by the matrix $N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ with gain $p(t) = p(y_1(t),y_2(t))$. The matrix A admits the two eigenvalues $\lambda_1 = -i$ and $\lambda_2 = \overline{\lambda_1}$ (the conjugate of λ_1), associated with the eigenvectors $\varphi_1 = (i)$ and $\overline{\varphi_1}$ respectively. Setting $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H := \mathbb{R}^2$ we obtain

$$S(t)y = e^{-it}\langle y,\varphi_1\rangle\varphi_1 + e^{it}\langle y,\overline{\varphi_1}\rangle\overline{\varphi_1}\cdot$$

Then

$$\langle S(t)y,NS(t)y\rangle = e^{-2it}(y_1 - iy_2)^2 - 2(y_1^2+y_2^2) + e^{2it}(y_1+iy_2)^2,$$

and hence $\int_0^{\pi} |\langle S(t)y,NS(t)y\rangle|dt = 2\|y\|^2$, $\forall t \geq 0$, so (4) holds.

Example 3.2. In this section, $\Omega \subset \mathbb{R}^n$ denotes a bounded open domain with C^∞ boundary and $Q = \Omega\times]0,+\infty[$. Let us consider the following system

$$\begin{cases} \dfrac{d^2z(x,t)}{\partial t^2} = \Delta z(x,t) + p(t)z(x,t), \ in\ Q \\ z(\xi,t) = 0, \qquad on\ \partial\Omega\times]0,+\infty[. \end{cases} \qquad (31)$$

Here, A and H are defined as in the above example, while N is defined by $N = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. The operator N is compact (see [4]). Then the feedback given by

$$p(t) = -\frac{\int_\Omega z(x,t)\frac{\partial z(x,t)}{\partial t}dx}{1+|\int_\Omega z(x,t)\frac{\partial z(x,t)}{\partial t}dx|},$$

ensures the weak stabilization of (31).

Remark .

1. In [4], a weak stabilization result of (31) has been given using the quadratic control (2).
2. Note that in the above example, the operator N is not self-adjoint, so the results of [19,6] are not applicable to obtain the feedback stabilization of (31).

Example 3.3. In this section, $\Omega \subset \mathbb{R}^n$ denotes a bounded open domain with C^∞ boundary and $Q = \Omega\times]0,+\infty[$. Let us consider the system

$$\begin{cases} \dfrac{\partial^2z(x,t)}{\partial t^2} = \Delta z(x,t) + p(t)a(x)\dfrac{\partial z(x,t)}{\partial t}, \ in\ Q \\ z(\xi,t) = 0, \qquad on\ \partial\Omega\times]0,+\infty[. \end{cases} \qquad (32)$$

where $a \in L^\infty(\Omega)$ is such that $a(x) \geq 0$, a.e on Ω and $a(x) \geq c > 0$ on a non-empty open subset ω of Ω. This system has the form (1) if we set

$$y = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \in H = H_0^1(\Omega)\times L^2(\Omega),\ A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with

$$D(A) = [H^2(\Omega)\bigcap H_0^1(\Omega)]\times H_0^1(\Omega),\ N = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}$$

where I is the identity operator Δ is the Laplacian operator and G is defined for all $u \in L^2(\Omega)$ by

$$Gu(x) = a(x)u(x) \ \text{ a.e on } \Omega.$$

With the inner product

$$\langle (y_1,z_1),(y_2,z_2)\rangle = \langle (y_1,y_2)\rangle_{H_0^1(\Omega)} + \langle (z_1,z_2)\rangle_{L^2(\Omega)}$$

the operator A generates a semigroup of contractions (see [4]) and (4) holds (see [19]).
Applying Theorem 2.1, we obtain the strong stabilizability of the system (32) by the control

$$p(t) = -\frac{\int_{\Omega} a(x)(\frac{\partial z(x,t)}{\partial t})^2 dx}{1+|\int_{\Omega} a(x)(\frac{\partial z(x,t)}{\partial t})^2 dx|},$$

and we have the estimate

$$\int_{\Omega}(\nabla z(x,t))^2 dx + \int_{\Omega}(\frac{\partial z(x,t)}{\partial t})^2 dx = O(\frac{1}{t}),\ as\ t \to +\infty.$$

Let us now see the simulations of the above example for $\Omega = (0,1)$, $a(x) = x+1$ and $y_0(x) = x-1$, $\frac{\partial y(x,0)}{\partial t} = 3$ in Ω. Then we obtain the results shown in Figures 1-6.

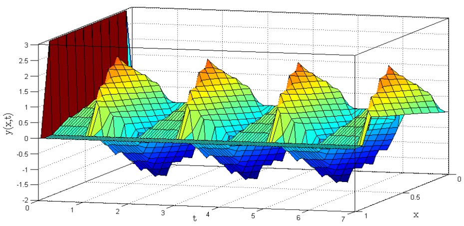

Fig. 1: First component of the free state

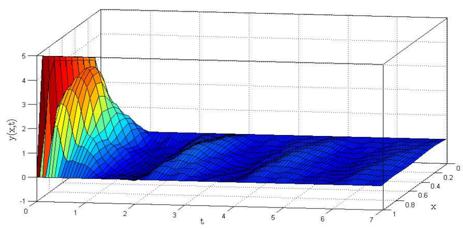

Fig. 2: First component of the stabilized state

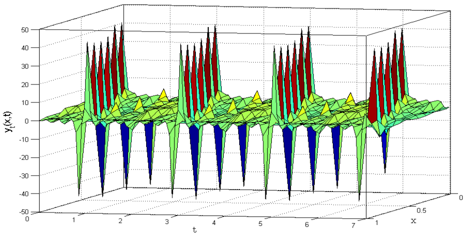

Fig. 3: Second component of the free state

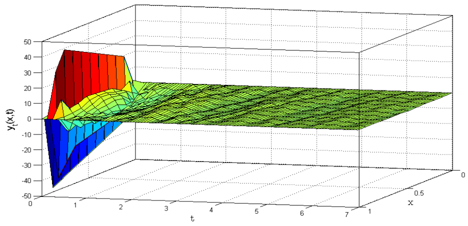

Fig. 4: Second component of the stabilized state

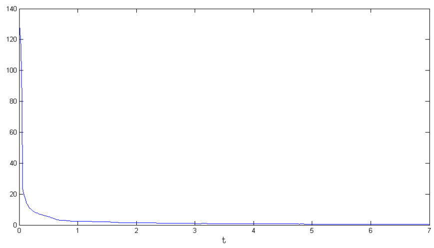

Fig. 5: The energy of the stabilized state

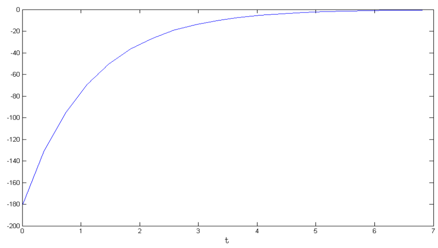

Fig. 6: The stabilized control

4 Conclusion

Under observation-like assumptions, weak and strong stabilization of constrained distributed semilinear systems have been studied. A decay estimate for the stabilized state is given. Also, the robustness of the constrained controller has been studied. The paper leaves the open question of whether the established estimate (21) can be improved.

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Azzeddine Tsouli is a researcher, preparing his Doctorat in Applied Mathematics at the University Moulay Ismail, of Meknes in Morocco. His studies are focused on analysis and control of distributed systems and numerical analysis. At present he is a Ph.D student at team STI (System Theory and Informatics), MACS Laboratory at University Moulay Ismail of Meknes Morocco.

Mohamed Ouzahra received his Doctorat in applied mathematics from Moulay Ismail University, Morocco, in 2004, the agrgation degree in mathematics from the ENS, Casablanca, Morocco, in 1999. He joined the AMES team Department of Mathematics, ENS, University of Sidi Mohamed Ben Abdellah, Fes, Morocco, where he is currently a Professor. His current research interests include controllability and feedback stabilization of continuous and discrete time bilinear systems. He is a member of the team AMES (Applied Mathematics and Engineering Sciences) in the Department of Mathematics, ENS, University of Sidi Mohamed Ben Abdellah.

Ali Boutoulout is a professor at the University Moulay Ismail of Meknes in Morocco. He obtained his Doctorat dEtat in System Regional Analysis (2000) at University Moulay Ismail. Professor Boutoulout has published many paper in the area of system analysis and control. Now he is the head of the research Laboratory MACS (Modelling, Analysis and Control Systems) and a director of Master System Theory and Informatics, in department of Mathematics and Informatics of Faculty of Sciences at the University Moulay Ismail, of Meknes in Morocco.