A New Extension of Some Well Known Fixed Point Theorems in Metric Spaces

Mehmet KIR[1,] and Hukmi KIZILTUNC[2,*]*

Department of Mathematics, Faculty of Science, Ataturk University, 25240, Erzurum, Turkey

Abstract: In this paper, our purpose is to give a new generalization of Kannan's type and Chatterjea's type fixed point theorems in metric spaces. We have two main ideas. Our first idea is applying the logic of Choudhury [5] to the Kannan type contraction mappings, the second is applying the logic of Dutta and Choudhury [6] to the Kannan type contraction mappings and Chatterjea type contraction mappings.

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1 Introduction and Preliminaries

Fixed point theory is one of the most important topic in development of nonlinear analysis. Also, fixed point theory has been used effectively in many other branch of science, such as chemistry, biology, economics, computer science, engineering et,. It is a long time many mathematicians have studied on fixed point theory. The authors developed very useful results in this area. Now, we briefly recall some of those results.

A mapping $T : X \rightarrow X$ where (X,d) is a metric space, is said to be a contraction if there exists $k \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq kd(x,y). \tag{1}$$

If the metric space (X,d) is complete then the mapping satisfying (1) has a unique fixed point. Inequality (1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

Kannan [2] established the following result in which the above question has been answered in the affirmative.

If a mapping $T : X \rightarrow X$ where (X,d) is a complete metric space, satisfies the inequality

$$d(Tx,Ty) \leq a[d(x,Tx)+d(y,Ty)] \tag{2}$$

where $a \in \left[0,\frac{1}{2}\right)$ and $x,y \in X$. Then, T has a unique fixed point. The mappings satisfying (2) are called Kannan type mappings.

A similar contractive condition has been introduced by Chatterjea [3] as follows:

If $T : X \rightarrow X$ where (X,d) is a complete metric space, satisfies the inequality

$$d(Tx,Ty) \leq b[d(x,Ty)+d(y,Tx)] \tag{3}$$

where $b \in \left[0,\frac{1}{2}\right)$ and $x,y \in X$. Then, T has a unique fixed point. The mappings satisfying (3) are called Chatterjea type mapping.

Another generalization of the contraction principle was suggested by Rhoades [4] as following:

Definition 1. *[4] (weakly contractive mapping). A mapping $T : X \rightarrow X$, where (X,d) is a metric space, is said to be weakly contractive if*

$$d(Tx,Ty) \leq d(x,y) - \psi(d(x,y)) \tag{4}$$

where $x,y \in X$ and $\psi : [0,\infty) \rightarrow [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Rhoades [4] showed that a weakly contractive mapping have a unique fixed point in complete metric space (X,d). Also, it is clear that taken if $\psi(t) = kt$ where $k \in [0,1)$

* Corresponding author e-mail: mehmetkir04@gmail.com, hukmu@atauni.edu.tr

then, (4) reduces to (1). For this reason, the concept of Rhoades [4] is more general than the concept of Banach [1].

Choudhury [5] introduced a generalization of Chatterjea type contraction as follows:

Definition 2. *[5] A self-mapping $T : X \to X$, on a metric space (X,d), is said to be a weakly $C-$contractive (or weak Chatterjea type contraction) if for all $x,y \in X$,*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)) \tag{5}$$

where $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(x,y) = 0$ if and only if $x = y = 0$.

Choudhury [5] showed that a mapping satisfying the inequality (5) has unique fixed point in complete metric space and proved that there is no requirement of continuity of the $C-$contraction. Also, taken if $\psi(x,y) = k(x+y)$ where $k \in (0,\frac{1}{2}]$, then (5) reduces to (3). For this reason, result of Choudhury is more general than the result of Chatterjea [3].

Dutta and Choudhury [6] introduced a new generalization of the Banach contraction mapping principle. This new generalization is more general than the concept of Rhoades in [4]. The result of Dutta and Choudhury is following:

Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequality

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)), \tag{6}$$

where

1) $\psi,\phi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions,
2) $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.
Then, T has a unique fixed point.

Other than the above, very recently, various generalizations of contraction mappings have been studied extensively by Karapınar, [7],[9],[10],[11].

In present study, we have two main ideas. Our first idea is applying the logic of Choudhury [5] to the Kannan type contraction mappings, the second is applying the logic of Dutta and Choudhury [6] to the Kannan type contraction mappings and Chatterjea type contraction mappings. For this reason our results more general than the some results of [1],[2],[3],[4],[5],[6].

Now, we give our main results.

2 Main Results

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping satisfying the inequality*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx)+d(y,Ty)] - \psi(d(x,Tx),d(y,Ty)) \tag{7}$$

where $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $\psi(x,y) = 0$ if and only if $x = y = 0$. Then, T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and for all $n \geq 1$, $x_{n+1} = Tx_n$.

$$\begin{aligned}d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &\quad - \psi(d(x_{n-1},x_n),d(x_n,x_{n+1})) \\ &\leq \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]. \end{aligned} \tag{8}$$

From (8), we have

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n).$$

Thus, $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists $r \in \mathbb{R}$ such that $d(x_n,x_{n+1}) \to r$, as $n \to \infty$.

$$\begin{aligned}d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\ &\quad - \psi(d(x_{n-1},x_n),d(x_n,x_{n+1})). \end{aligned} \tag{9}$$

Letting $n \to \infty$ in (9), we obtain that

$$r \leq r - \psi(r,r). \tag{10}$$

The equation (10) implies that $r = 0$. Thus, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{11}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{12}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (11). Then,

$$d(x_{m(k)},x_{n(k)-1}) < \varepsilon. \tag{13}$$

Then, by putting (11) in (12), we have

$$\begin{aligned}\varepsilon &\leq d(x_{m(k)},x_{n(k)}) \\ &\leq d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{n(k)}) \\ &< \varepsilon + d(x_{n(k)-1},x_{n(k)}). \end{aligned} \tag{14}$$

Letting $k \to \infty$ and using (14),

$$d(x_{m(k)},x_{n(k)}) = \varepsilon. \tag{15}$$

Also, we have

$$d(x_{m(k)},x_{n(k)}) \leq d(x_{n(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{m(k)})+d(x_{m(k)-1},x_{m(k)}) \tag{16}$$

and

$$d\left(x_{n(k)-1},x_{m(k)-1}\right) \le d\left(x_{n(k)-1},x_{n(k)}\right) + d\left(x_{n(k)},x_{m(k)}\right) + d\left(x_{m(k)},x_{m(k)-1}\right). \quad (17)$$

Letting $k \to \infty$ in (16) and (17),

$$\lim_{k\to\infty} d\left(x_{n(k)-1},x_{m(k)-1}\right) = \varepsilon.$$

Also, by using (14), we obtain that

$$\begin{aligned}
\varepsilon &\le d\left(x_{m(k)},x_{n(k)}\right) \\
&= d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right) \\
&\le \frac{1}{2}\left[d\left(x_{m(k)-1},x_{m(k)}\right) + d\left(x_{n(k)-1},x_{n(k)}\right)\right] \\
&\quad - \psi\left(d\left(x_{m(k)-1},x_{m(k)}\right),d\left(x_{n(k)-1},x_{n(k)}\right)\right). \quad (18)
\end{aligned}$$

Letting $k \to \infty$ in (18), we obtain $\varepsilon \le -\psi(0,0)$ and this implies that $\varepsilon \le 0$ but this case is a contradiction for $\varepsilon > 0$. Consequentially, we obtain that $\{x_n\}$ is a Cauchy sequence and hence there exists $x^* \in X$ such that $\{x_n\} \to x^*$.

Now, we will show that $x^* \in X$ is the fixed point of T. Indeed, we have

$$\begin{aligned}
d\left(x^*,Tx^*\right) &\le d\left(x^*,x_{n+1}\right) + d\left(x_{n+1},Tx^*\right) \\
&\le d\left(x^*,x_{n+1}\right) + \frac{1}{2}\left[d\left(x_n,x_{n+1}\right) + d\left(x^*,Tx^*\right)\right] \\
&\quad - \psi\left(d\left(x_n,x_{n+1}\right),d\left(x^*,Tx^*\right)\right). \quad (19)
\end{aligned}$$

If we take $n \to \infty$ in (19), we obtain

$$\frac{1}{2}d\left(x^*,Tx^*\right) \le -\psi\left(0,d\left(x^*,Tx^*\right)\right). \quad (20)$$

The inequality (20) is conradiction for definition of ψ, unless $d\left(x^*,Tx^*\right) = 0$. Hence x^* is the fixed point of T. Also, it is easy to see that fixed point of T is unique. Assume that x' is an other fixed point of T.

$$\begin{aligned}
d\left(x^*,x'\right) &= d\left(Tx^*,Tx'\right) \\
&\le \frac{1}{2}\left[d\left(x^*,Tx^*\right) + d\left(x',Tx'\right)\right] \\
&\quad - \psi\left(d\left(x^*,Tx^*\right) + d\left(x',Tx'\right)\right). \quad (21)
\end{aligned}$$

Finally, we obtain from (21), $x^* = x'$. This completes the proof.

Remark. In Theorem 1, if we take $\psi(t) = kt$ where $k \in \left(0,\frac{1}{2}\right]$ then we obtain well-known Kannan fixed point theorem.

Theorem 2. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying the inequalty*

$$\psi(d\left(Tx,Ty\right)) \le \psi\left[\tfrac{1}{2}(d\left(x,Tx\right) + d\left(y,Ty\right))\right] - \phi\left[d\left(x,Tx\right),d\left(y,Ty\right)\right] \quad (22)$$

where,

1) $\psi : [0,\infty) \to [0,\infty)$ is continuous and monotone nondecreasing function such that

$\psi(t) = 0$ if and only if $t = 0$.
2) $\phi : [0,\infty)^2 \to [0,\infty)$ is continuous and monotone nondecreasing function such that
$\phi(x,y) = 0$ if and only if $x = y = 0$.
Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ and for all $n \ge 1$, $x_{n+1} = Tx_n$.

$$\begin{aligned}
\psi(d\left(x_n,x_{n+1}\right)) &= \psi(d\left(Tx_{n-1},Tx_n\right)) \\
&\le \psi\left[\frac{d\left(x_{n-1},x_n\right) + d\left(x_n,x_{n+1}\right)}{2}\right] \\
&\quad - \phi\left[d\left(x_{n-1},x_n\right),d\left(x_n,x_{n+1}\right)\right] \quad (23) \\
&\le \psi\left[\frac{d\left(x_{n-1},x_n\right) + d\left(x_n,x_{n+1}\right)}{2}\right] \\
&\le \psi\left[\frac{d\left(x_{n-1},x_n\right) + d\left(x_{n-1},x_n\right) + d\left(x_n,x_{n+1}\right)}{2}\right] \\
&\le \psi(d\left(x_{n-1},x_n\right)). \quad (24)
\end{aligned}$$

Since ψ is nondecreasing, we obtain that

$$d\left(x_n,x_{n+1}\right) < d\left(x_{n-1},x_n\right). \quad (25)$$

Thus, $\{d\left(x_n,x_{n+1}\right)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists $r \in \mathbb{R}$ such that $d\left(x_n,x_{n+1}\right) \to r$, as $n \to \infty$.
Letting $n \to \infty$ in (23),

$$\psi(r) \le \psi(r) - \phi(r,r). \quad (26)$$

The inequality (26) implies that $\phi(r,r) = 0$ and this implies $r = 0$. Thus, we obtain that

$$\lim_{n\to\infty} d\left(x_n,x_{n+1}\right) = 0. \quad (27)$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d\left(x_{m(k)},x_{n(k)}\right) \ge \varepsilon.$$

Also, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (27). Then,

$$d\left(x_{m(k)},x_{n(k)-1}\right) < \varepsilon.$$

Again, by taking (22) into account with (27), we have

$$\begin{aligned}
\varepsilon &\le d\left(x_{m(k)},x_{n(k)}\right) \\
&= d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right) \\
&\le \psi\left[\frac{\left[d\left(x_{m(k)-1},x_{m(k)}\right) + d\left(x_{n(k)-1},x_{n(k)}\right)\right]}{2}\right] \\
&\quad - \phi\left[d\left(x_{m(k)-1},x_{m(k)}\right),d\left(x_{n(k)-1},x_{n(k)}\right)\right]. \quad (28)
\end{aligned}$$

Letting $k \to \infty$, in (28)and using (27), we obtain that

$$\psi(\varepsilon) \le \psi(0) - \phi(0,0). \quad (29)$$

The inequality (29) implies that $\varepsilon \leq 0$. But, this case is a contradiction for $\varepsilon > 0$. Thus, we obtain $\{x_n\}$ is a Cauchy sequence and hence convergent in complete metric space (X,d). Let $\{x_n\} \to x^*$ as $n \to \infty$, thus, we have

$$d(x^*,Tx^*) \leq d(x^*,x_{n+1}) + d(x_{n+1},Tx^*)$$
$$\leq d(x^*,x_{n+1}) + \psi\left(\frac{[d(x_n,x_{n+1}) + d(x^*,Tx^*)]}{2}\right)$$
$$- \phi(d(x_n,x_{n+1}),d(x^*,Tx^*)). \qquad (30)$$

If we take $n \to \infty$, we obtain

$$d(x^*,Tx^*) \leq -\psi(0) - \phi(0,d(x^*,Tx^*)). \qquad (31)$$

The inequality (31) implies that $d(x^*,Tx^*) = 0$. Thus, we have $Tx^* = x^*$. It is easy to see that the fixed point is unique. This completes the proof.

Now, we give some results of the Theorem 2.

Remark. In Theorem 2, if we choose $\psi(t) = t$ then, we obtain Theorem 1.

Remark. Also, if we choose $\psi(t) = t$ and take $\phi(x,y) = k(x+y)$ where $k \in \left(0,\frac{1}{2}\right]$, we obtain well known Kannan's fixed point theorem [2].

Theorem 3. *Let* (X,d) *be a complete metric space.* $T : X \to X$ *be a self-mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \psi\left[\frac{1}{2}(d(x,Ty) + d(y,Tx))\right] - \phi[d(x,Ty),d(y,Tx)] \qquad (32)$$

where

1) $\psi : [0,\infty) \to [0,\infty)$ *is continuous and monotone nondecreasing function such that*
$\psi(t) = 0$ *if and only if* $t = 0$.

2) $\phi : [0,\infty)^2 \to [0,\infty)$ *is continuous and monotone nondecreasing function such that*
$\phi(x,y) = 0$ *if and only if* $x = y = 0$.
Then, T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and for all $n \geq 1$, $x_{n+1} = Tx_n$.

$$\psi(d(x_n,x_{n+1})) = \psi(d(Tx_{n-1},Tx_n))$$
$$\leq \psi\left(\frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{2}\right)$$
$$- \phi(d(x_{n-1},x_{n+1}),d(x_n,x_n))$$
$$\leq \psi\left(\frac{d(x_{n-1},x_{n+1})}{2}\right)$$
$$\leq \psi\left(\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}\right)$$
$$\leq \psi(d(x_{n-1},x_n)). \qquad (33)$$

Since ψ is nondecreasing, we obtain that

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n). \qquad (34)$$

Thus, $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists $r \in \mathbb{R}$ such that $d(x_n,x_{n+1}) \to r$, as $n \to \infty$.

Also, by using (32), we have

$$\psi(d(x_n,x_{n+1})) = \psi(d(Tx_{n-1},Tx_n))$$
$$\leq \psi\left(\frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{2}\right)$$
$$- \phi(d(x_{n-1},x_{n+1}),d(x_n,x_n))$$
$$\leq \psi\left(\frac{d(x_{n-1},x_{n+1})}{2}\right) - \phi(d(x_{n-1},x_{n+1}),0) \qquad (35)$$
$$\leq \psi\left(\frac{d(x_{n-1},x_{n+1})}{2}\right) \qquad (36)$$
$$\leq \psi\left(\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}\right). \qquad (37)$$

Letting $n \to \infty$, we see by (36) and (37)

$$\psi(r) \leq \lim_{n\to\infty} \psi\left(\frac{d(x_{n-1},x_{n+1})}{2}\right) \leq \psi(r).$$

Since ψ is nondecreasing, we obtain that

$$r \leq \lim_{n\to\infty} \frac{d(x_{n-1},x_{n+1})}{2} \leq r$$

and we obtain that

$$\lim_{n\to\infty} d(x_{n-1},x_{n+1}) = 2r. \qquad (38)$$

Letting $n \to \infty$ in (35) and using (38) and continuity of ϕ, we have

$$\psi(r) \leq \psi(r) - \phi(2r,0). \qquad (39)$$

The inequality (39) implies that $r = 0$. Thus, we obtain that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (40)$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon. \qquad (41)$$

Also, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (40). Then,

$$d(x_{m(k)},x_{n(k)-1}) < \varepsilon. \qquad (42)$$

Again

$$\varepsilon \leq d(x_{m(k)},x_{n(k)})$$
$$\leq d(x_{m(k)},x_{n(k)-1}) + d(x_{n(k)-1},x_{n(k)})$$
$$\leq \varepsilon + d(x_{n(k)-1},x_{n(k)}). \qquad (43)$$

Letting $k \to \infty$ in (43), we obtain

$$\lim_{n\to\infty} d(x_{m(k)},x_{n(k)}) = \varepsilon \text{ and } \lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1}) = \varepsilon. \qquad (44)$$

Also,

$$d(x_{m(k)},x_{n(k)}) \leq d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)}) + d(x_{n(k)},x_{n(k)-1}). \qquad (45)$$

and

$$d\left(x_{m(k)-1},x_{n(k)}\right)\leq d\left(x_{m(k)-1},x_{m(k)}\right)+d\left(x_{m(k)},x_{n(k)}\right).\tag{46}$$

Making $k\to\infty$ *in* (45) *and* (46), *we get*

$$\lim_{k\to\infty}d\left(x_{m(k)-1},x_{n(k)}\right)=\varepsilon.\tag{47}$$

Also, by using (43), (44), (47), *we obtain that*

$$\begin{aligned}\psi(\varepsilon)&\leq\psi\left(d\left(x_{m(k)},x_{n(k)}\right)\right)\\&=\psi\left(d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right)\right)\\&\leq\psi\left(\frac{d\left(x_{m(k)-1},x_{n(k)}\right)+d\left(x_{n(k)-1},x_{m(k)}\right)}{2}\right)\\&\quad-\phi\left(d\left(x_{m(k)-1},x_{n(k)}\right),d\left(x_{n(k)-1},x_{m(k)}\right)\right).\end{aligned}\tag{48}$$

Letting $k\to\infty$, *in* (48) *and using* (47),(44), *we obtain that*

$$\psi(\varepsilon)\leq\psi(\varepsilon)-\phi(\varepsilon,\varepsilon).$$

This implies that $\phi(\varepsilon,\varepsilon)\leq 0$ *but this case is a contradiction for* $\varepsilon>0$. *Thus, we obtain* $\{x_n\}$ *is a Cauchy sequence and hence convergent in complete metric space* (X,d). *Let* $\{x_n\}\to x^*$ *as* $n\to\infty$. *Thus, we have*

$$d\left(x^*,Tx^*\right)\leq d\left(x^*,x_{n+1}\right)+d\left(x_{n+1},Tx^*\right).\tag{49}$$

Since ψ *is nondecreasing, we have*

$$\begin{aligned}\psi\left(d\left(x^*,Tx^*\right)\right)&\leq\psi\left(d\left(x_{n+1},Tx^*\right)\right)\\&=\psi\left(d\left(Tx_n,Tx^*\right)\right)\\&\leq\psi\left(\frac{d\left(x_n,Tx^*\right)+d\left(x^*,Tx_n\right)}{2}\right)\tag{50}\\&\quad-\phi\left(d\left(x_n,Tx^*\right),d\left(x^*,Tx_n\right)\right)\tag{51}\end{aligned}$$

If we take $n\to\infty$ *in* (51) *and by using continuity of* ϕ, *we obtain that*

$$\psi\left(d\left(x^*,Tx^*\right)\right)\leq\psi\left(d\left(x^*,Tx^*\right)\right)-\phi\left(d\left(x^*,Tx^*\right),d\left(x^*,Tx^*\right)\right).$$

This implies that $\phi\left(d\left(x^*,Tx^*\right),d\left(x^*,Tx^*\right)\right)=0$. *Thus,* $d\left(x^*,Tx^*\right)=0$ *and hence* $x^*=Tx^*$.

　　Now, we show that the fixed point is unique. Assume that x' *is an other fixed point of* T. *Thus,* $Tx'=x'$ *and we have*

$$\begin{aligned}\psi\left(d\left(x^*,x'\right)\right)&=\psi\left(d\left(Tx^*,Tx'\right)\right)\\&\leq\psi\left(\frac{d\left(x^*,Tx'\right)+d\left(x',Tx^*\right)}{2}\right)\\&\quad-\phi\left(d\left(x^*,Tx'\right),d\left(x',Tx^*\right)\right)\\&=\psi\left(d\left(x^*,x'\right)\right)-\phi\left(d\left(x^*,x'\right),d\left(x',x^*\right)\right).\end{aligned}\tag{52}$$

By the definition of ϕ, *we hold that* $d\left(x^*,x'\right)=0$ *and hence* $x^*=x'$. *This completes the proof.*

　　Now, we give some results of the Theorem 3.

Remark. In Theorem 3, if we choose $\psi(t)=t$, then we obtain the result of Choudhury [5].

Remark. Also, if we choose $\psi(t)=t$ and take $\phi(x,y)=k(x+y)$ where $k\in\left(0,\frac{1}{2}\right]$, we obtain Chatterjea's fixed point theorem [3].

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