Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040112

On Some Soft Functions

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Received: 9 Jun. 2014, Revised: 22 Sep. 2014, Accepted: 28 Sep. 2014
Published online: 1 Jan. 2015

Abstract: Soft topological space is the mathematical formulation of approximate reasoning about information systems. Shabir et. al [35] and Cagman et. al [8] independently introduced the concept of soft topology in 2011 and studied several basic properties of soft topology. Some basic properties of soft continuous functions (also called soft pu-continuous functions) have been studied in [38]. In this paper, motivated by the findings of [38], we further establish fundamental and important characterizations of soft pu-continuous functions, soft pu-open functions and soft pu-closed functions via soft interior, soft closure, soft boundary and soft derived set. Finally, we give the relationships amongst soft pu-continuous, soft pu-open and soft pu-closed functions.

Keywords: Soft topology, soft interior(closure), soft boundary, soft derived set, soft pu-continuous, soft pu-open(pu-closed) functions.

1 Introduction

Soft topological spaces based on soft set theory which is a collection of information granules is the mathematical formulation of approximate reasoning about information systems. In 1999, Molodtsov [29] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems with incomplete information in engineering, physics, computer science, economics, social sciences and medical sciences. Soft set theory does not require the specification of parameters. Instead, it accommodates approximate description of an object as its starting point which makes it a natural mathematical formalism for approximate reasoning. So the application of soft set theory in other disciplines and real life problems are now catching momentum. In [30], Molodtsov applied successfully in directions such as, smoothness of functions, game theory, operations research, riemann-integration, perron integration, probability and theory of measurement. Maji et. al [27] applied soft sets in a multicriteria decision making problems. It is based on the notion of knowledge reduction of rough sets. They applied the technique of knowledge reduction to the information table induced by the soft set. In [28], they defined and studied several basic notions of soft set theory. A. Kharal and B. Ahmad [25], defined and discussed the several properties of soft images and soft inverse images of soft sets. They also applied these notions to the problem of medical diagnosis in medical systems. In 2005, Pei and Miao [32] discussed the relationship between soft sets and information systems. Chen et. al [9] focused discussions on parametrization reductions of soft sets and its applications. Many researchers have contributed towards the algebraic structure of soft set theory ([1,3,15,16,17, 18,19,20,21,22,23,33,36].
In 2011, Shabir and Naz [35] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as soft open and closed sets, soft subspace, soft closure, soft nbd of a point, soft T_i-spaces, for $i = 1,2,3,4$, soft regular and soft normal spaces and established their several properties. Cagman et. al [8] introduced and studied the basic properties of soft topological space defined on a soft set in 2011. Also in 2011, S. Hussain and B. Ahmad [13] continued investigating the properties of soft open(closed), soft nbd and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary. I. Zarlutana et. al [38] have defined soft continuity on soft topological spaces and have found several interesting and fundamental properties. Recently, S. Hussain [14], defined and explored the properties and characterizations of soft connectedness in soft topological spaces. They also discussed the behavior of soft connectedness under soft pu-continuous functions.

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2 Preliminaries

First, we recall some definitions and results.

Definition 1 [29]. Let X be an initial universe and E a set of parameters. Let $P(X)$ denotes the power set of X and A a non-empty subset of E. A pair (F,A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F,A). Clearly, a soft set is not a set.

For soft subsets, soft union , soft intersection, soft complement; we refer to [28,32,35].

Definition 2 [38]. A soft set (F,A) over X is said to be a null soft set, denoted by Φ_A , if for all $e \in A$, $F(e) = \phi$.

Definition 3 [38]. A soft set (F,A) over X is said to be an absolute soft set, denoted by X_A , if for all $e \in A$, $F(e) = X$. Clearly, $X_A^c = \Phi_A$ and $\Phi_A^c = X_A$.

Here we consider only soft sets (F,A) over a universe X in which all the parameters of set A are same. We denote the family of these soft sets by $SS(X)_A$.

Proposition 1 [38]. Let (F,A), $(G,A),(H,A)$, (S,A) $\in SS(X)_A$. Then the following are true.
(1) If $(F,A)\tilde{\cap}(G,A) = \Phi_A$, then $(F,A)\tilde{\subseteq}(G,A)^c$.
(2) $(F,A)\tilde{\cup}(F,A)^c = X_A$.
(3)If $(F,A)\tilde{\subseteq}(G,A)$ and $(G,A)\tilde{\subseteq}(H,A)$, then $(F,A)\tilde{\subseteq}(H,A)$.
(4) If $(F,A)\tilde{\subseteq}(G,A)$ and $(H,A)\tilde{\subseteq}(S,A)$, then $(F,A)\tilde{\cap}(H,A)\tilde{\subseteq}(G,A)\tilde{\cap}(S,A)$.
(5) If $(F,A)\tilde{\subseteq}(G,A)$ if and only if $(G,A)^c\tilde{\subseteq}(F,A)^c$.

Definition 4 [25]. Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. $u : X \to Y$ and $p : A \to B$ be mappings. Then a function $f_{pu} : SS(X)_A \to SS(Y)_B$ defined as :
(1) Let (F,A) be a soft set in $SS(X)_A$. The image of (F,A) under f_{pu} , written as $f_{pu}(F,A) = (f_{pu}(F), p(A))$, is a soft set in $SS(Y)_B$ such that

$$f_{pu}(F)(y) = \begin{cases} \bigcup_{x\in p^{-1}(y)\cap A} u(F(x)), & p^{-1}(y)\cap A \neq \phi \\ \phi, & \text{otherwise} \end{cases},$$

for all $y \in B$.
(2) Let (G,B) be a soft set in $SS(Y)_B$. Then the inverse image of (G,B) under f_{pu} , written as $f_{pu}^{-1}(G,B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SS(X)_A$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B \\ \phi, & \text{otherwise} \end{cases},$$

for all $x \in A$.

The soft function f_{pu} is called soft surjective, if p and u are surjective. The soft function f_{pu} is called soft injective, if p and u are injective.

Theorem 1 [25]. Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. For a function $f_{pu} : SS(X)_A \to SS(Y)_B$, the following statements are true.
(1) $f_{pu}(\Phi_A) = \Phi_B$.
(2) $f_{pu}(X_A)\tilde{\subseteq}Y_B$.
(3) $f_{pu}((F,A) \cup (G,A)) = f_{pu}(F,A) \cup f_{pu}(G,A)$ where $(F,A),(G,A) \in SS(X)_A$. In general $f_{pu}(\bigcup_i(F_i,A)) = \bigcup_i f_{pu}(F_i,A)$ where $(F_i,A) \in SS(X)_A$.
(4) If $(F,A)\tilde{\subseteq}(G,A)$, then $f_{pu}((F,A))\tilde{\subseteq} f_{pu}((G,A))$, where $(F,A),(G,A)\tilde{\in}SS(X)_A$.
(5) If $(G,B)\tilde{\subseteq}(H,B)$, then $f_{pu}^{-1}((G,B))\tilde{\subseteq} f_{pu}^{-1}((H,B))$, where $(G,B),(H,B)\tilde{\in}SS(Y)_B$.

Theorem 2 [38]. Let $SS(X)_A$ and $SS(Y)_B$ be families of soft sets. For a function $f_{pu} : SS(X)_A \to SS(Y)_B$, the following statements are true.
(1) $f_{pu}^{-1}(G,B)^c = (f_{pu}^{-1}(G,B))^c$, for any soft set (G,B) in $SS(Y)_B$.
(2) $f_{pu}(f_{pu}^{-1}(G,B))\tilde{\subseteq}(G,B)$, for any soft set (G,B) in $SS(Y)_B$. If f_{pu} is soft surjective, the equality holds.
(3) $(F,A)\tilde{\subseteq} f_{pu}^{-1}(f_{pu}(F,A))$, for any soft set (F,A) in $SS(X)_A$. If f_{pu} is soft injective, the equality holds.

Definition 5 [35]. Let τ be the collection of soft sets over X with the fixed set of parameters A. Then τ is said to be a soft topology on X, if
(1) Φ_A , X_A belong to τ,
(2) the union of any number of soft sets in τ belongs to τ,
(3) the intersection of any two soft sets in τ belongs to τ .
The triplet (X,τ,A) is called a soft topological space over X. The members of τ are called soft open sets. The soft complement of a soft open set A is called the soft closed sets.

Proposition 2 [35]. Let (X,τ,A) be a soft topological space over X. Then the collection $\tau_\alpha = \{F(\alpha) : (F,A)\tilde{\in}\tau\}$, for each $\alpha \in A$, defines a topology on X.

It is known that the intersection of two soft topological spaces over the same universe X is a soft topological space, whereas the union may or may not be a soft topological space as given in [35].

Definition 6 [38]. The soft set $(F,A)\tilde{\in}SS(X)_A$ is called a soft point in X_A , denoted by e_F , if for the element $e \in A$, $F(e) \neq \phi$ and $F(e') = \phi$, for all $e' \in A - \{e\}$.

Definition 7 [38]. The soft point e_F is said to be in the soft set (G,A), denoted by $e_F\tilde{\in}(G,A)$, if for the element $e \in A$, $F(e) \subseteq G(e)$.

Proposition 3 [38]. Let $e_F\tilde{\in}X_A$ and $(G,A) \in SS(X)_A$. If

$e_F \tilde{\in} (G,A)$, then $e_F \tilde{\notin} (G,A)^c$.

Definition 8 [38]. Let (X,τ,A) be a soft topological space. Then a soft set (G,A) in $SS(X)_A$ is called a soft neighborhood (briefly: soft nbd) of the soft point $e_F \tilde{\in} X_A$, if there exists a soft open set (H,A) such that $e_F \tilde{\in} (H,A) \tilde{\subseteq} (G,A)$.
The soft neighborhood system of a soft point e_F , denoted by $N_\tau(e_F)$, is the family of all its soft neighborhoods.

Definition 9 [38]. Let (X,τ,A) be a soft topological space over X. Then a soft set (G,A) in $SS(X)_A$ is called a soft neighborhood (briefly: soft nbd) of the soft set (F,A), if there exists a soft open set (H,A) such that $(F,A) \tilde{\subseteq} (H,A) \tilde{\subseteq} (G,A)$.

Theorem 3 [38]. The soft neighborhood system $N_\tau(e_F)$ at a soft point e_F in a soft topological space (X,τ,A) has the following properties:
(1) If $(G,A) \tilde{\in} N_\tau(e_F)$, then $e_F \tilde{\in} (G,A)$,
(2) If $(G,A) \tilde{\in} N_\tau(e_F)$ and $(G,A) \tilde{\subseteq} (H,A)$, then $(H,A) \tilde{\in} N_\tau(e_F)$,
(3) If $(G,A),(H,A) \in N_\tau(e_F)$, then $(G,A) \tilde{\cap} (H,A) \tilde{\in} N_\tau(e_F)$,
(4) If $(G,A) \tilde{\in} N_\tau(e_F)$, then there is a $(M,A) \tilde{\in} N_\tau(e_F)$ such that $(G,A) \tilde{\in} N_\tau(e'_H)$ for each $e'_H \tilde{\in} (M,A)$.

Definition 10 [13]. Let (X,τ,A) be a soft topological space over X and (F,A) a soft set in $SS(X)_A$. The soft interior of soft set (F,A) is denoted by $(F,A)^\circ$ and is defined as the union of all soft open sets contained in (F,A). Clearly $(F,A)^\circ$ is the largest soft open set contained in (F,A).

Definition 11 [38]. Let (X,τ,E) be a soft topological space over X and (G,A) a soft set in $SS(X)_A$. The soft point $e_F \tilde{\in} X_A$ is called a soft interior point of a soft set (G,A), if there exists a soft open set (H,A) such that $e_F \tilde{\in} (H,A) \tilde{\subseteq} (G,A)$.

Proposition 4 [38]. Let (X,τ,A) be a soft topological space over X and (G,A) a soft set in $SS(X)_A$. Then $(G,A)^\circ = \bigcup_{e\in A}\{e_F : e_F$ is any soft interior point of (G,A) for $e \in A\}$.

Definition 12 [35]. Let (X,τ,A) be a soft topological space over X with the fixed set of parameters A and (F,A) a soft set over X. Then the soft closure of (F,A), denoted by $\overline{(F,A)}$ is the intersection of all soft closed supersets of (F,A). Clearly $\overline{(F,A)}$ is the smallest soft closed set over X which contains (F,A).

Definition 13 [13]. Let (X,τ,A) be a soft topological space over X. Then soft boundary of a soft set (F,A) in $SS(X)_A$ is denoted by $(F,A)^b$ and is defined as: $(F,A)^b = \overline{(F,A)} \tilde{\cap} \overline{(F,A)^c}$.

3 Soft pu-Continuous Functions

Definition 14 [38]. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively. Let $u : X \to Y$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a function and $e_F \tilde{\in} X_A$. The function f_{pu} is soft pu-continuous at $e_F \tilde{\in} X_A$, if for each $(G,B) \tilde{\in} N_{\tau^*}(f_{pu}(e_F))$, there exists a $(F,A) \tilde{\in} N_\tau(e_F)$ such that $f_{pu}(F,A) \tilde{\subseteq} (G,B)$.
f_{pu} is soft pu-continuous on X_A, if f_{pu} is soft pu-continuous at each soft point in X_A.

Theorem 4 [38]. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function and $e_F \tilde{\in} X_A$. Then the following statements are equivalent:
(1) f_{pu} is soft pu-continuous at e_F .
(2) For each $(G,B) \in N_{\tau^*}(f_{pu}(e_F))$, there exists a $(H,A) \in N_\tau(e_F)$ such that $(H,A) \tilde{\subseteq} f_{pu}^{-1}(G,B)$.
(3) For each $(G,B) \in N_{\tau^*}(f_{pu}(e_F))$, $f_{pu}^{-1}(G,B) \in N_\tau(e_F)$.

Theorem 5 [38]. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively. For a soft function $f_{pu} : SS(X)_A \to SS(Y)_B$, consider the following statements:
(1) f_{pu} is soft pu-continuous.
(2) for each soft set (F,A) in $SS(X)_A$, the inverse image of every soft nbd of $f_{pu}(F,A)$ is a soft nbd of (F,A).
(3) for each soft set (F,A) in $SS(X)_A$ and each soft nbd (H,B) of $f_{pu}(F,A)$, there is a soft nbd (G,A) of (F,A) such that $f_{pu}(G,A) \tilde{\subseteq} (H,B)$.
Then we have $(1) \Leftrightarrow (2) \Leftrightarrow (3)$.

Theorem 6 [38]. Let (X,τ,A) and (Y,τ^*,B) be soft topological spaces over X and Y respectively and $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft function. Then the following statements are equivalent:
(1) f_{pu} is soft pu-continuous.
(2) For each $(G,B) \tilde{\in} \tau^*$, $f_{pu}^{-1}(G,B) \tilde{\in} \tau$.
(3) For (G,B) soft closed in (Y,τ^*,B). $f_{pu}^{-1}(G,B)$ is soft closed in (X,τ,A).

Now we prove the following:
Theorem 7. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be mappings. Then the following are equivalent:
(1) A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-continuous.
(2) $f_{pu}^{-1}(G,B)^o \tilde{\subseteq} (f_{pu}^{-1}(G,B))^o$.
(3) $\overline{f_{pu}^{-1}(G_1,B)} \tilde{\subseteq} f_{pu}^{-1}\overline{(G_1,B)}$.
Proof. $(1) \Rightarrow (2)$. Let f_{pu} be soft pu-continuous. Let $e_F \tilde{\in} f_{pu}^{-1}(G,B)^o$. Then $f_{pu}(e_F) \in (G,B)^o$. Therefore there exists a soft open set (H,B) such that $f_{pu}(e_F) \tilde{\in} (H,B) \tilde{\subseteq} (G,B)$. Since f_{pu} is soft pu-continuous , there exists a soft open set (F,A) such that $e_F \tilde{\in} (F,A)$ and

$f_{pu}(F,A)\tilde{\subseteq}(H,B)$. By Theorem 2(3), this gives $(F,A)\tilde{\subseteq} f_{pu}^{-1}(H,B)$, which implies $e_F\tilde{\in}(f_{pu}^{-1}(G,B))^o$. This proves (2).
(2) $\Rightarrow$ (3). Take $(G,B) = (\overline{(G_1,B)})^c\tilde{\subseteq}(G_1,B)^c$. Then $(G,B)^o = (\overline{(G_1,B)})^c$. Then by supposition , Theorem 1(5) and Proposition 1, we have $f_{pu}^{-1}((\overline{(G_1,B)})^c\tilde{\subseteq}(f_{pu}^{-1}(\overline{(G_1,B)})^c)^o\tilde{\subseteq}(f_{pu}^{-1}(G_1,B)^c)^o = ((\overline{f_{pu}^{-1}(G_1,B)})^c$
or $f_{pu}^{-1}(\overline{(G_1,B)})^c\tilde{\subseteq}(\overline{f_{pu}^{-1}(G_1,B)})^c$. , Thus we have $\overline{f_{pu}^{-1}(G_1,B)}\tilde{\subseteq} f_{pu}^{-1}(\overline{(G_1,B)})$. This proves (3).
(3) $\Rightarrow$ (1). Let $e_F\tilde{\in}X_A$ and (G,B) a soft open nbd of $f_{pu}(e_F)$ in (Y,τ^*,B). Put $(G_1,B) = (G,B)^c$. Then (G_1,B) is soft closed which implies $(G_1,B) = \overline{(G_1,B)}$ and $f_{pu}(e_F)\tilde{\notin}(G_1,B)$. Then $\overline{f_{pu}^{-1}(G_1,B)}\tilde{\subseteq} f_{pu}^{-1}\overline{(G_1,B)}$ gives $\overline{f_{pu}^{-1}(G_1,B)}\tilde{\subseteq} f_{pu}^{-1}(G_1,B)$ implies $f_{pu}^{-1}(G_1,B)$ is soft closed. Put $(F,A) = (f_{pu}^{-1}(G_1,B))^c$ and therefore $e_F\tilde{\in}(F,A)$. Then (F,A) is a soft open nbd of e_F and $(F,A) = (f_{pu}^{-1}(G_1,B))^c$ gives $(F,A) = f_{pu}^{-1}(G_1,B)^c = f_{pu}^{-1}(G,B)$. By Theorems 1(4) and 2(2), we have $f_{pu}(F,A) = f_{pu}f_{pu}^{-1}(G,B)\tilde{\subseteq}(G,B)$ or $f_{pu}(F,A)\tilde{\subseteq}(G,B)$. This proves (1). □

Theorem 8. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be mappings. Then the following are equivalent:
(1) A soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-continuous.
(2) $f_{pu}(\overline{(F,A)})\tilde{\subseteq}\overline{(f_{pu}(F,A))}$, for any soft subset (F,A) in $SS(X)_A$.
(3) $(f_{pu}^{-1}(G,B))^b\tilde{\subseteq} f_{pu}^{-1}(G,B)^b$, for any soft subset (G,B) in $SS(Y)_B$.
Proof. (1) $\Rightarrow$ (2). Let (F,A) be any soft subset in $SS(X)_A$. Since $\overline{f_{pu}(F,A)}$ is soft closed in (Y,τ^*,B), then f_{pu} is soft pu-continuous implies $f_{pu}^{-1}(\overline{f_{pu}(F,A)})$ is soft closed in (X,τ,A), which contains (F,A). Thus $(F,A)\tilde{\subseteq} f_{pu}^{-1}(\overline{f_{pu}(F,A)})$ gives $\overline{(F,A)}\tilde{\subseteq}\overline{f_{pu}^{-1}(\overline{f_{pu}(F,A)})} = f_{pu}^{-1}\overline{f_{pu}(F,A)}$. Therefore by Theorem 1(4) $f_{pu}\overline{(F,A)}\tilde{\subseteq} f_{pu}(f_{pu}^{-1}\overline{f_{pu}(F,A)})$. Consequently by Theorem 2(2) we have, $f_{pu}\overline{(F,A)}\tilde{\subseteq}\overline{f_{pu}(F,A)}$. This gives (2).
(2) $\Rightarrow$ (1). Suppose $f_{pu}\overline{(F,A)}\tilde{\subseteq}\overline{f_{pu}(F,A)}$, for any soft subset (F,A) in $SS(X)_A$. To prove (1), we use Theorem 6. Let (G,B) be a soft closed subset in (Y,τ^*,B). We show that $f_{pu}^{-1}(G,B)$ is soft closed. By our hypothesis and Theorem 2(2),
$f_{pu}\overline{f_{pu}^{-1}(G,B)}\tilde{\subseteq}\overline{(f_{pu}f_{pu}^{-1}(G,B))}\tilde{\subseteq}\overline{(G,B)} = (G,B)$ … $(*)$
By Theorem 2(3) and (*), $\overline{f_{pu}^{-1}(G,B)}\tilde{\subseteq} f_{pu}^{-1}(f_{pu}\overline{f_{pu}^{-1}(G,B)})\tilde{\subseteq} f_{pu}^{-1}(G,B)$ or $\overline{f_{pu}^{-1}(G,B)}\tilde{\subseteq} f_{pu}^{-1}(G,B)$ implies $f_{pu}^{-1}(G,B)$ is soft closed in (X,τ,A). Thus f_{pu} is soft continuous. This proves (1).
(1) $\Rightarrow$ (3). Suppose that f_{pu} is soft pu-continuous. Let $G,B)$ be any soft subset in $SS(Y)_B$. Since f_{pu} is soft pu-continuous, therefore by Theorems 7(3) and 2(1), we have $(f_{pu}^{-1}(G,B))^b = \overline{f_{pu}^{-1}(G,B)}\tilde{\cap}\overline{(f_{pu}^{-1}(G,B))^c}$ $\tilde{\subseteq} f_{pu}^{-1}(\overline{(G,B)})\tilde{\cap} f_{pu}^{-1}(\overline{(G,B)^c} = f_{pu}^{-1}(\overline{(G,B)})\tilde{\cap}(\overline{(G,B)^c}) = f_{pu}^{-1}(G,B)^b$. Therefore, $(f_{pu}^{-1}(G,B))^b\tilde{\subseteq} f_{pu}^{-1}((G,B)^b$. This proves (3).
(3) $\Rightarrow$ (1) . Let (G,B) be soft closed in (Y,τ^*,B). We show that $f_{pu}^{-1}(G,B)$ is soft closed in (X,τ,A). By hypothesis and Theorem 6(2) [13], $(f_{pu}^{-1}(G,B))^b\tilde{\subseteq} f_{pu}^{-1}(G,B)^b\tilde{\subseteq} f_{pu}^{-1}(G,B))$ implies $(f_{pu}^{-1}(G,B))^b\tilde{\subseteq} f_{pu}^{-1}(G,B)$. By Theorem 6(3)[13], $f_{pu}^{-1}(G,B)$ is soft closed in (X,τ,A). Hence f_{pu} is soft pu-continuous. This proves (1). □

4 Soft pu-Open and Soft pu-Closed Functions

In this section, we define and discuss the characterizations of soft pu-open, soft pu-closed functions. We also explore the relationships amongst soft pu-open, soft pu-closed and soft pu-continuous functions.

Definition 15. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be mappings. Then a soft function $f_{pu} : SS(X)_A \to SS(Y)_B$ is called soft pu-open (respt. soft pu-closed), if for each soft open set (F,A) in $SS(X)_A$, $f_{pu}(F,A)$ is soft open (respt. soft closed) in (Y,τ^*,B).

Theorem 9. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be functions. Let $f_{pu} : SS(X)_A \to SS(Y)_B$ be a soft pu-closed (respt. soft pu-open) function. Then for any soft set (G,B) in $SS(Y)_B$ and any for any soft open (respt. soft closed) set (F,A) in (X,τ,A) containing $f_{pu}^{-1}(G,B)$, there exists a soft open (respt. soft closed) set (G_1,B) containing (G,B) such that $f_{pu}^{-1}(G_1,B)\tilde{\subseteq}(F,A)$.
Proof. Let $(G_1,B) = (f_{pu}((F,A)^c))^c$. Then calculations show that
$f_{pu}^{-1}(G,B)\tilde{\subseteq}(F,A)$ implies $(G,B)\tilde{\subseteq}(G_1,B)$. Since f_{pu} is soft pu-closed, therefore (G_1,B) is soft open in (Y,τ^*,B) and by Theorem 1(1)and (3), $f_{pu}^{-1}(G_1,B) = f_{pu}^{-1}(f_{pu}(F,A)^c)^c = (f_{pu}^{-1}f_{pu}(F,A)^c)^c$ $\tilde{\subseteq}((F,A)^c)^c = (F,A)$ or $f_{pu}^{-1}(G_1,B)\tilde{\subseteq}(F,A)$. □

The following theorem gives the characterizations of soft pu-open functions:
Theorem 10. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u : X \to Y$ and $p : A \to B$ be mappings. Then the following are equivalent:
(1) $f_{pu} : SS(X)_A \to SS(Y)_B$ is soft pu-open.

(2) $f_{pu}(F,A)^o\tilde{\subseteq}(f_{pu}(F,A))^o$, for any soft subset (F,A) in $SS(X)_A$.
(3) For each $e_F \in X_A$ and a soft nbd (F_1,A) of e_F, there exists a soft nbd (G,B) of $f_{pu}(e_F)$ such that $(G,B)\tilde{\subseteq} f_{pu}(F_1,A)$.
Proof. (1) $\Rightarrow$ (2). Since $(F,A)^o\tilde{\subseteq}(F,A)$, therefore by Theorem 1(4), we have $f_{pu}(F,A)^o\tilde{\subseteq} f_{pu}(F,A)$. Since f_{pu} is soft pu-open, therefore $f_{pu}F,A)^o$ is soft open in (Y,τ^*,B) and is contained in $f_{pu}(F,A)$. But $(f_{pu}(F,A))^o$ is the largest soft open set contained in $f_{pu}(F,A)$. Therefore $f_{pu}(F,E)^o\tilde{\subseteq}((f_{pu}(F,E))^o$. This proves (2).
(2) $\Rightarrow$ (3). Let $e_F\tilde{\in}X_A$ and (F_1,A) a soft nbd of e_F. Then $e_F\tilde{\in}(F_1,A)^o$. Since $f_{pu}(F_1,A)^o\tilde{\subseteq}(f_{pu}(F_1,A))^o$ and so $f_{pu}(e_F)\tilde{\in} f_{pu}(F_1,A)^o\tilde{\subseteq}(f_{pu}(F_1,A))^o\tilde{\subseteq} f_{pu}(F_1,A)$. Put $(G,B)=(f_{pu}(F_1,A))^o$. Then (G,B) is a soft open nbd of $f_{pu}(e_F)$ such that $(G,B)\tilde{\subseteq} f_{pu}(F_1,A)$. This proves (3).
(3) $\Rightarrow$ (1). Let (F_1,A) be a soft open nbd of e_F in X_A. By (3), there exists a soft open nbd (G,B) of $f_{pu}(e_F)$ such that $(G,B)\tilde{\subseteq} f_{pu}(F_1,A)$. Thus
$f_{pu}(F_1,A)=\bigcup\{(G,B) : f_{pu}(e_F)\tilde{\in} f_{pu}(F_1,A)\}$ shows that $f_{pu}(F_1,A)$ is soft open in (Y,τ^*,B). □

The following theorem gives the characterization of soft pu-closed functions:
Theorem 11. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u: X\to Y$ and $p: A\to B$ are mappings. A soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is soft pu-closed if and only if $\overline{f_{pu}(F,A)}\tilde{\subseteq} f_{pu}\overline{(F,A)}$, for any soft subset (F,A) in $SS(X)_A$.
Proof. Since $(F,A)\tilde{\subseteq}\overline{(F,A)}$, therefore by Theorem 1(4), we have $f_{pu}(F,A)\tilde{\subseteq} f_{pu}\overline{(F,A)}$. By hypothesis, $f_{pu}\overline{(F,A)}$ is soft closed which contains $f_{pu}(F,A)$. Since $\overline{f_{pu}(F,A)}$ is the smallest soft closed set containing $f_{pu}(F,A)$, therefore $\overline{f_{pu}(F,A)}\tilde{\subseteq} f_{pu}\overline{(F,A)}$.
Conversely, suppose that (F,A) is a soft closed set in (X,τ,A). Then by supposition $f_{pu}(F,A)=f_{pu}\overline{(F,A)}\tilde{\supseteq}\overline{f_{pu}(F,A)}$ or $\overline{f_{pu}(F,A)}\tilde{\subseteq} f_{pu}(F,A)$. This implies that $f_{pu}(F,A)$ is soft closed in (Y,τ^*,B). This proves that f_{pu} is soft pu-closed. □

We recall that a soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is soft bijective, if $u: X\to Y$ and $p: A\to B$ are soft bijective mappings. Now we prove the following:
Theorem 12. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively. If a soft bijective soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is soft pu-continuous, then for any soft subset (F,A) in $SS(X)_A$, we have $(f_{pu}(F,A))^o\tilde{\subseteq} f_{pu}(F,A)^o$.
Proof. Since $(f_{pu}(F,A))^o$ is soft open in (Y,τ^*,B), therefore by supposition $f_{pu}^{-1}(f_{pu}(F,A))^o$ is soft open in (X,τ,A). Since f_{pu} is soft injective, therefore by Theorem 2(2), we have $f_{pu}^{-1}(f_{pu}(F,A))^o\tilde{\subseteq} f_{pu}^{-1}f_{pu}(F,A)=(F,A)$ or $f_{pu}^{-1}(f_{pu}(F,A))^o\tilde{\subseteq}(F,A)$ implies $f_{pu}^{-1}(f_{pu}(F,A))^o\tilde{\subseteq}(F,A)^o$ ... $(**)$.
Since f_{pu} is soft surjective, therefore by Theorem 2(2) and (**) , we have
$(f_{pu}(F,A))^o=f_{pu}(f_{pu}^{-1}(f_{pu}(F,A))^o)\tilde{\subseteq} f_{pu}(F,A)^o$. □

Proposition 5. If a soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is soft bijective, then for any soft subset (F,A) in $SS(X)_A$, we have $f_{pu}(F,A)^c=(f_{pu}(F,A))^c$.
Proof. Since by definition $(F,A)^c=(F^c,A)$, $F^c: A\to P(X)$ such that $F^c(x)=X-F(x)$, for all $x\in A$ and $u: X\to Y$ and $p: A\to B$ are bijective, therefore for all $y\in B$, we have
$f_{pu}(F^c,A)(y)=\bigcup_{x\in p^{-1}(y)\cap A}u(F^c(x)),\qquad p^{-1}(y)\cap A\neq\phi.$

$=\bigcup_{x\in p^{-1}(y)\cap A}u(X-F(x)),\ \ p^{-1}(y)\cap A\neq\phi.$
$=Y-\bigcup_{x\in p^{-1}(y)\cap A}F(x),\qquad p^{-1}(y)\cap A\neq\phi.$
$=(f_{pu}(F,A))^c.$ □

We use Proposition 5 to prove the following:
Theorem 13. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively. A soft bijective soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is a soft pu-open function if and only if f_{pu} is soft pu-closed.
Proof. (F,A) is soft closed (respt. soft open) in (X,τ,A) if and only is $(F,A)^c$ is soft open (respt. soft closed). By supposition, $f_{pu}(F,A)^c$ is soft open (respt. soft closed) in (Y,τ^*,B). Since f_{pu} is soft bijective, therefore by Proposition 5, we have $f_{pu}(F,A)^c=(f_{pu}(F,A))^c$ is soft open(respt. soft closed) in (Y,τ^*,B) if and only if $f_{pu}(F,A)$ is soft closed(respt. soft open) in (Y,τ^*,B). □

Theorem 14. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively. A soft bijective soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is a soft pu-open function if and only if f_{pu}^{-1} is soft pu-continuous.
Proof. Necessity. Since f_{pu} is soft bijective, therefore f_{pu}^{-1} exists. Suppose f_{pu} is soft pu-open. Let (F,A) be soft open in (X,τ,A). Then by supposition $f_{pu}(F,A)$ is soft pu-open in (Y,τ^*,B). But $(f_{pu}^{-1})^{-1}(F,A)=f_{pu}(F,A)$ gives $(f_{pu}^{-1})^{-1}(F,A)$ is soft open in (Y,τ^*,B) under f_{pu}^{-1}. This implies f_{pu}^{-1} is a soft pu-continuous.
Sufficiency. Suppose f_{pu}^{-1} is a soft pu-continuous. Then by Theorem 6(2), (F,A) is soft open in (X,τ,A) implies $(f_{pu}^{-1})^{-1}(F,A)=f_{pu}(F,A)$ is soft open in (Y,τ^*,B). This proves that f_{pu} is soft pu-open. □.

Definition 16. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively and $u: X\to Y$ and $p: A\to B$ are mappings. A soft function $f_{pu}: SS(X)_A\to SS(Y)_B$ is called a soft pu-homeomorphism, if
(1) f_{pu} is soft bijective.
(2) f_{pu} is soft pu-continuous.
(3) f_{pu}^{-1} is soft pu-continuous.
Two soft topological spaces (X,τ,A) and (Y,τ^*,B) are soft homeomorphic, if there is a soft pu-homeomorphism between them and we write $(X,\tau,A)\tilde{\cong}(Y,\tau^*,B)$.

Combining Theorems 11, 13 and 14, we have:

Theorem 15. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively. Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be soft bijective soft function. Then the following are equivalent:
(1) f_{pu} is soft pu-open.
(2) f_{pu} is soft pu-closed.
(3) f_{pu}^{-1} is soft pu-continuous.
(4) $\overline{f_{pu}(F,A)} \tilde{\subseteq} f_{pu}\overline{(F,A)}$, for any soft subset (F,A) in $SS(X)_A$.

Finally, by Theorem 15, we have;

Theorem 18. Let (X,τ,A) and (Y,τ^*,B) be two soft topological spaces over X and Y respectively. Let $f_{pu}: SS(X)_A \to SS(Y)_B$ be soft bijective soft function. Then the following are equivalent:
(1) f_{pu} and f_{pu}^{-1} are soft pu-continuous.
(2) f_{pu} is soft pu-continuous and soft pu-open.
(3) f_{pu} is soft pu-continuous and soft pu-closed.
(4) $\overline{f_{pu}(F,A)} = f_{pu}\overline{(F,A)}$, for any soft subset (F,A) in $SS(X)_A$.

5 Conclusion

The study of soft sets and soft topology indicate possible applications in classical and non classical logic. Soft topological spaces based on soft set theory which is a collection of information granules is the mathematical formulation of approximate reasoning about information systems.We continued to investigate soft pu-continuity in soft topological spaces. We also defined and explored soft pu-open, soft pu-closed functions in soft topological spaces. The notions of soft mappings have been applied to the problem of medical diagnosis in medical expert systems in [25]. We hope that the findings in this paper can be applied to problems of many fields that contains uncertainties. It will also promote and enhance the study on soft topology and will provide general framework for the applications in practical life.

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