Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040315

Solution of a Quadratic Non-Linear Oscillator by Elliptic Homotopy Averaging Method

A. M. El-Naggar[1] *and G. M. Ismail* [2,*]

[1] Department of Mathematics, Faculty of Science, Benha University, Egypt
[2] Department of Mathematics, Faculty of Science, Sohag University, 82524, Sohag, Egypt

Received: 7 May 2015, Revised: 21 Jul. 2015, Accepted: 23 Jul. 2015
Published online: 1 Sep. 2015

Abstract: In this paper, the periodic solutions of a strongly quadratic nonlinear oscillator whose motion is described with the generalized Van der Pol equation are studied. A new method based on homotopy and averaging is employed to determine the limit cycle motion. Three types of quadratic nonlinearity are considered: the coefficients of the linear and quadratic terms are positive, the coefficient of the linear term is positive and that of the quadratic term is negative and the opposite case. Comparison with the numerical solutions is also presented, revealing that the present method leads to accurate solutions.

Keywords: Periodic solutions, homotopy averaging method, elliptic functions, generalized Van der Pol equation

1 Introduction

Over the last century, perturbation methods based on circular functions have been successfully developed to accurately determine approximate solutions for weakly non linear oscillators in the form

$$\ddot{x} + c_1 x = \varepsilon f(x, \dot{x}). \tag{1}$$

Here c_1 is a constant, ε a small positive parameter. Classical methods, such as harmonic balance, Lindstedt–Poincaré, Krylov–Bogoliubov–Mitropolski, averaging and multiple scales [1,2,3,4], have been conducted to approximate periodic solutions of Eq. (1).

Recently, many authors have been developing various elliptic function methods such as elliptic harmonic balance method, elliptic Krylov-Bogoliubov method, elliptic averaging method, elliptic Galerkin method, elliptic Rayleigh method, elliptic perturbation method, elliptic Lindstedt-Poincaré method and elliptic homotopy averaging method [5,6,7,8,9,10,11,12,13,14]. However, most of these methods are related to cubic nonlinear oscillators, and very few of them have analyzed the equation with quadratic nonlinearity.

In this paper the elliptic homotopy averaging method was presented by authors [14] for certain oscillators having cubic nonlinearity will be used to analyze the periodic solutions of quadratic nonlinear oscillators of the form

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon f(x, \dot{x}) \tag{2}$$

which are associated with many physical systems such as betatron oscillators and vibration of shells. It is therefore also an important area of nonlinear vibration investigation. The analytical solution was enough to explain some of the phenomena which occur in the real systems. For example, in a Van der Pol electrical circuit the existence of a limit cycle was explained by the energy store in the capacitor during the slowly varying part of the motion, while during the abrupt changes the energy was being suddenly released. Unfortunately, the quantitative values obtained analytically were not enough accurate. This was the reason why the Van der Pol equation was extended with nonlinear terms. The generalized Van der Pol oscillator is

$$\ddot{x} + c_1 x + c_2 x^2 = \varepsilon \left(c_0 - c_3 x^2\right) \dot{x}, \tag{3}$$

where ε is a constant which is often assumed to be small $(\varepsilon \ll 1)$, c_i where $i = 0, \ldots, 3$ are constant coefficients and dots denote derivatives with respect to time t.

* Corresponding author e-mail: gamalm2010@yahoo.com

2 The solution of the generating equation

We first solve the so-called generating equation of Eq. (2).

$$\ddot{x}+c_1x+c_2x^2=0. \tag{4}$$

with initial conditions:

$$x(0)=q, \qquad \dot{x}(0)=0. \tag{5}$$

Eq. (4) has an exact analytical solution which can be expressed by Jacobian elliptic function. Let the solution be denoted by

$$x=a\, ep^2(\omega t,k^2)+b, \tag{6}$$

here $ep(\omega t,k^2)$ denotes a convenient Jacobian elliptic function: $sn(\omega t,k^2)$, $cn(\omega t,k^2)$ or $dn(\omega t,k^2)$ according to the type of Eq. (4) which depends on the sign of c_1 and c_2. The constants a, ω and k^2 are called the amplitude, the angular frequency and the modulus of the elliptic function, respectively, and b is called the bias. (A survey of elliptic functions is given in the Appendix). The constants ω, b and k^2 are the known values which depend on a. Three types of Eq. (4) will be discussed in detail: (a) $c_1>0$ and $c_2>0$, (b) $c_1>0$ and $c_2<0$ and (c) $c_1<0$ and $c_2>0$. All the three types of equations have a physical meaning: case (a) corresponds to the oscillator with a hardening spring [1], case (b) to the oscillator with a softening spring [1] and case (c) is the first modal equation of transversal vibrations of a cantilever beam, for example, Refs. [15,16,17].

Type I : $\mathbf{c_1>0},\ \mathbf{c_2>0}$.

For this type of oscillator the generating function is as follows [11]:

$$x=a\, cn^2(\omega t,k^2)+b. \tag{7}$$

Substituting Eq. (7) into Eq. (4) and equating coefficients by the same order of function $cn\ \tau$, the values of k^2, a, b, and ω are obtained as:

$$a=\frac{6\omega^2k^2}{c_2}, \tag{8}$$

$$b=\frac{-\left[4\omega^2(2k^2-1)+c_1\right]}{2c_2}, \tag{9}$$

$$\omega^4=\frac{c_1^2}{16(k^4-k^2+1)}. \tag{10}$$

Type II : $\mathbf{c_1>0},\ \mathbf{c_2<0}$.

It is worth pointing out that when $\mathbf{c_1>0},\ \mathbf{c_2<0}$, the solution of Eq. (4) can be expressed by

$$x=a_1\, sn^2\tau+b_1, \tag{11}$$

where

$$a_1=-a,\ b_1=a+b. \tag{12}$$

It can be shown that Eq. (11) is indeed identical to Eq. (7), because

$$a\, cn^2\tau+b=a(1-sn^2\tau)+b=a_1\, sn^2\tau+b_1. \tag{13}$$

Type III : $\mathbf{c_1<0},\ \mathbf{c_2>0}$.

Similarly, when $\mathbf{c_1<0},\ \mathbf{c_2>0}$, the solution of Eq. (4) can be expressed by

$$x=a_2\, dn^2\tau+b_2, \tag{14}$$

here

$$a_2=\left(a/k^2\right),\ b_2=a+b-a/k^2. \tag{15}$$

It can also be proved that Eq. (14) is equivalent to Eq. (7). Therefore, one can use Eqs. (7) and $(8-10)$ as a unified solution of Eq. (4) later.

3 Basic idea of the elliptic homotopy averaging method

To explain this method, let us consider the following function:

$$A(x)-f(r)=0, \quad r\in\Omega, \tag{16}$$

with the boundary conditions of:

$$B(x,\partial x/\partial n)=0, \quad r\in\Gamma, \tag{17}$$

where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator A can be divided into two parts F and N where F is a linear, and N is nonlinear. Therefore, Eq. (16) can be written as follows:

$$F(x)+N(x)-f(r)=0. \tag{18}$$

By the homotopy technique see [18,19,20,21], we construct a homotopy of Eq. (16) $x(r,p):\Omega\times[0,1]\rightarrow R$ which satisfies:

$$\begin{aligned}H(x,p)&=(1-p)\left[F(x)-F(x_0)\right]\\&+p\left[A(x)-f(r)\right]=0,\ \ p\,\varepsilon[0,1],\ \ r\in\Omega,\end{aligned} \tag{19}$$

which is equivalent to

$$H(x,p)=F(x)-F(x_0)+pF(x_0)+p\left[N(x)-f(r)\right]=0, \tag{20}$$

where $p\ \varepsilon[0,1]$ is an embedding parameter, and x_0 is an initial approximation which satisfies the initial conditions. By introducing an embedding parameter p with values in the interval $[0,1]$, a transformation of the variable $x(t)$ to $X(t,p)$ is done. The homotopy transformed Eq. (3) is

$$\begin{aligned}&(1-p)\left[(\ddot{X}+c_1X+c_2X^2)-(\ddot{x}_0+c_1x_0+c_2x_0^2)\right]\\&+p\left[(\ddot{X}+c_1X+c_2X^2)-\varepsilon(c_0\dot{X}-c_3X^2\,\dot{X})\right]=0,\end{aligned} \tag{21}$$

with transformed initial condition (5)

$$X(0,p)=q,\quad \dot{X}(0,p)=0, \tag{22}$$

where $x_0 \equiv x_0(t)$ is the initial approximate solution which has the form of (6)

$$x_0 = a\, ep^2(\omega t,k^2)+b \equiv a\, ep^2+b. \tag{23}$$

Using the Maclaurin series expansion

$$X(t,p)=x_0(t)+\sum_{n=1}^{\infty}\left(\frac{x_n}{n!}\right)p^n,\quad n=1,2,3,..., \tag{24}$$

where

$$x_n \equiv x_n(t)=\left(\frac{\partial X(t,p)}{\partial p^n}\right)_{p=0}, \tag{25}$$

the nonlinear differential Eq. (21) is transformed into the system of n linear differential equations

$$p^0:\ddot{x}_0+c_1x_0+c_2x_0^2=0, \tag{26}$$

$$\begin{aligned} p^1:\ &\ddot{x}_1+c_1x_1+2c_2x_0x_1+(\ddot{x}_0+c_1x_0+c_2x_0^2)\\ &=\varepsilon(c_0\dot{x}_0-c_3x_0^2\dot{x}_0). \end{aligned} \tag{27}$$

Applying Eq. (23) the differential Eq. (27) is transformed into the first order deformation equation

$$\begin{aligned} &\ddot{x}_1+c_1x_1+2c_2\left(a\, ep^2+b\right)x_1\\ &=\varepsilon\left(c_0a(\dot{ep^2})-c_3a\left(a\, ep^2+b\right)^2(\dot{ep^2})\right) \end{aligned} \tag{28}$$

The relation (28) is a nonlinear nonhomogeneous differential equation with time variable coefficient. To find the exact analytical solution for the Eq. (28) is not an easy task. Our aim is not to solve the equation but to determine the amplitude of steady-state motion.

Due to the property of the series expansion (25) and the form of the left side of the Eq. (27) the solution of (28) is assumed in the form of the first time derivative of the elliptic function in (23)

$$x_1=c(\dot{ep^2}), \tag{29}$$

where c is a constant. Substituting the assumed solution (29) into (28) we obtain

$$\begin{aligned} &c\left[(\dddot{ep^2})+c_1(\dot{ep^2})+2c_2\left(a\, ep^2+b\right)(\dot{ep^2})\right]\\ &=\varepsilon\left[c_0a(\dot{ep^2})-c_3a\left(a\, ep^2+b\right)^2(\dot{ep^2})\right]. \end{aligned} \tag{30}$$

This is the moment when the averaging procedure is introduced. The averaging is done for the period of elliptic function $4K(k^2)$, where $K(k^2)\equiv K$ is the complete elliptic integral of the first kind [22]. The averaged relation (30) is

$$\begin{aligned} &c\left[\left\langle(\dot{ep^2})(\dddot{ep^2})\right\rangle+c_1\left\langle\left[(\dot{ep^2})\right]^2\right\rangle+\left[(\dot{ep^2})\right]^2\right.\\ &\left.\times 2c_2(a\, ep^2+b)\right]=\varepsilon\left[c_0a\left\langle\left[(\dot{ep^2})\right]^2\right\rangle\right.\\ &\left.-c_3a\left\langle(a\, ep^2+b)^2\left[(\dot{ep^2})\right]^2\right\rangle\right], \end{aligned} \tag{31}$$

where $\langle...\rangle=\frac{1}{4K}\int_0^{4K}(...)d\tau,\ \tau=\omega t$. The left side of the Eq. (31) is always zero and the right side represents the condition for limit cycle motion

$$\begin{aligned} &(c_0-c_3b^2)\left\langle\left[(\dot{ep^2})\right]^2\right\rangle-2c_3ab\left\langle(ep)^2\left[(\dot{ep^2})\right]^2\right\rangle\\ &-c_3a^2\left\langle(ep)^4\left[(\dot{ep^2})\right]^2\right\rangle=0. \end{aligned} \tag{32}$$

Solving the system of algebraic Eqs. $(8-10)$ and (32), the constants a, ω, b and k^2 are obtained.

4 A study of the type I of generalized Van der Pol oscillator

As an application of the elliptic homotopy averaging method, type I of the generalized Van der Pol oscillator is studied in detail.

Oscillator type I : $\mathbf{c_1>0}$, $\mathbf{c_2>0}$ For this type of oscillator the generating solution is

$$x=a\, cn^2(\omega t,k^2)+b=a\, cn^2+b. \tag{33}$$

According to the aforementioned procedure the solution of (27) is assumed

$$x_1=c(\dot{cn^2})=-2c\omega cn\, sn\, dn, \tag{34}$$

where c is a constant. Substituting (34) into relation (32) we obtain

$$(c_3q_3)a^2+(2c_3b\,q_2)a-\left(c_0-c_3b^2\right)q_1=0, \tag{35}$$

where

$$\begin{aligned} q_1&=M_2-(k^2+1)M_4+k^2M_6\\ &=\tfrac{4}{15k^2}\left[(-2+3k^2-k^4)K+2(1-k^2+k^4)E\right],\\ q_2&=M_2-(k^2+2)M_4+(2k^2+1)M_6-k^2M_8\\ &=\tfrac{4}{105k^6}\left[\left(8-23k^2+18k^4-3k^6\right)K\right.\\ &\quad\left.+\left(-8+19k^2-9k^4+6k^6\right)E\right],\\ q_3&=M_2-(k^2+3)M_4+(3k^2+3)M_6\\ &\quad-(3k^2+1)M_8+k^2M_{10}\\ &=\tfrac{4}{315k^8}\left[\left(-16+64k^2-93k^4+50k^6-5k^8\right)K\right.\\ &\quad\left.+\left(-16-56k^2+66k^4-20k^6+10k^8\right)E\right], \end{aligned}$$

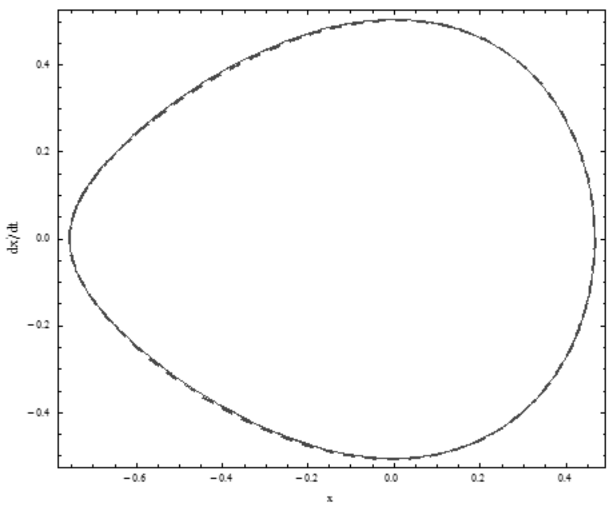

Fig. 1: Limit cycle solutions of Eq. (37) obtained analytically (—) and numerically (- - -).

here M_{2n}, $n = 1,...,5$ are the averaged elliptic functions which are given in the Appendix, and $E \equiv E(k^2)$ is the complete elliptic integral of the second kind [23].

From the amplitude modulation Eq. (35), the stationary amplitude is obtained by solving the algebraic Eq. (35). Thus, the stationary amplitude a, which must be positive, is given by

$$a = \frac{-(2c_3bq_2) \pm \sqrt{(2c_3bq_2)^2 + 4(c_3q_3)(c_0 - c_3b^2)q_1}}{2c_3q_3}. \tag{36}$$

Solving Eq. $(8-10)$, and (36) the parameters of the orbital motion k^2, a, b, and ω are obtained.

5 Application.

Consider the equation

$$\ddot{x} + 0.9x + 0.9x^2 = \varepsilon\left(0.1 - x^2\right)\dot{x}. \tag{37}$$

From Eqs. $(8-10)$ and (36), we have $\omega = 0.501186$, $a = 1.22047$, $b = -0.755452$ and $k^2 = 0.728825$. Using the analytical solution in the first approximation

$$x = 1.22047cn^2(0.501186t, 0.728825) - 0.755452. \tag{38}$$

The approximate solution (38) and the solution obtained by fourth-order Runge-Kutta method are compared in Figure 1 for $\varepsilon = 0.1$, the results of our computations show that the two solutions are in good agreement.

6 Conclusion

The elliptic homotopy averaging method applied is an efficient tool for calculating periodic solutions to strongly quadratic nonlinear oscillatory systems. Illustrative example show that the results of the present method are in excellent agreement with those obtained by a fourth order Runge-Kutta method.

Appendix: Elliptic functions

For the convenience of our readers, we collect some facts on Jacobian elliptic functions (see ref [22]) for details. Jacobian elliptic functions satisfy algebraic relations which are analogous to those for trigonometric functions. The fundamental three elliptic functions are $cn(\tau,k)$, $sn(\tau,k)$, and $dn(\tau,k)$. Each of the elliptic functions depends on the modulus k as well as the argument τ. Note that the elliptic functions sn and cn may be thought of as generalizations of sin and cos where their period depends on the modulus k.

The elliptic functions satisfy the following identities, which are analogous to $\sin^2+\cos^2=1$:

$$sn^2 + cn^2 = 1,\ k^2sn^2 + dn^2 = 1,\ k^2cn^2 + 1 - k^2 = dn^2.$$

Before the averaging it is very convenient to transform all the elliptic functions to sinus elliptic function

$$\begin{aligned}
&sn^2cn^2 = sn^2 - sn^4,\\
&sn^2dn^2 = sn^2 - k^2sn^4,\\
&cn^2dn^2 = 1 - (1+k^2)\ sn^2 + k^2sn^4,\\
&sn^2\ cn^2\ dn^2 = sn^2 - (k^2+1)\ sn^4 + k^2sn^6,\\
&sn^2\ cn^4\ dn^2 = sn^2 - (k^2+2)\ sn^4 + (2k^2+1)\ sn^6 - k^2sn^8,\\
&sn^2\ cn^6\ dn^2 = sn^2 - (k^2+3)\ sn^4 + (3k^2+3)\ sn^6\\
&\qquad -(3k^2+1)sn^8 + k^2sn^{10}.
\end{aligned}$$

Averaging the sinus elliptic functions according to [22] one gets

$$\begin{aligned}
M_2 &= \int_0^{4K} sn^2\, d\tau = \frac{4}{k^2}[K-E],\\
M_4 &= \int_0^{4K} sn^4\, d\tau = \frac{4}{3k^4}\left[(2+k^2)K - 2(1+k^2)E\right],\\
M_6 &= \int_0^{4K} sn^6\, d\tau = \frac{4}{15k^6}[(8+3k^2+4k^4)K\\
&\qquad -(8+7k^2+8k^4)E],\\
M_{2m+2} &= \int_0^{4K} sn^{2m+2}\, d\tau = \frac{2m(1+k^2)M_{2m} + (1-2m)M_{2m-2}}{(2m+1)k^2}.
\end{aligned}$$

References

[1] A. H. Nayfeh and D. T. Mook, Nonlinear Oscillators, Wiley, New York 1979.

[2] A. H. Nayfeh, Introduction to Perturbation Techniques, Wiley, New York 1981.

[3] N. N. Bogoliubov and Y. S. Mitropolski, Asymptotic Methods in the Theory of Non-linear Oscillations, Gordon and Breach, New York 1961.

[4] A. H. Nayfeh, Perturbation Method,Wiley, New York 1973.
[5] P. G. D. Barkhan and A. C. Soudack, An extension to the method of Kryloff and Bogoliuboff, Int J Control, **10,** 337-392 (1969)
[6] S. B. Yuste, Cubication of non-linear oscillators using the principle of harmonic balance, Int. J. Non-Linear Mech, **27**, 347-356 (1992).
[7] S. H. Chen and Y. K. Cheung. An elliptic perturbation method for certain strongly non-linear oscillators, J Sound Vib, **192**, 453-464 (1996).
[8] S. H. Chen and Y. K. Cheung, An elliptic Lindstedt-Poincaré method for analysis of certain strongly non-linear oscillators, Nonlinear Dyn, **12**, 199-213 (1997).
[9] S. H. Chen, X. M. Yang, Y. K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic Lindstedt-Poincaré method, J Sound Vib, **227**, 1109-1118 (1999).
[10] S. H. Chen, X. M. Yang, Y. K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic perturbation method, J Sound Vib, **212**, 771-780 (1998).
[11] F. Lakrad and M. Belhaq, Periodic solutions of strongly non-linear oscillators by the multiple scales method, J Sound Vib, **258**, 677-700 (2002).
[12] A. Elias-Zuniga, Exact solution of the cubic-quintic Duffing oscillator, Appl Math Modelling, **37**, 2574-2579 (2013).
[13] Y. M. Chen and J.K. Liu, Elliptic harmonic balance method for two degree-of-freedom self-excited oscillators, Commun Nonlinear Sci Numer Simul, **14**, 916-922 (2009).
[14] L. Cveticanin, G. M. Abd El-Latif, A. M. El-Nagger, G. M. Ismail, Periodic solution of the generalized Rayleigh equation, J Sound Vib, **318,** 580-591 (2008).
[15] P. J. Holmes and J. Marsden, A partial differential equation with infinitely many periodic orbits: Chaotic oscillations of a forced beam, Arch Rational Mech Analy, **76**, 135-166 (1981).
[16] F. C. Moon and P.J. Holmes, Strange attractors and chaos in nonlinear systems, ASME, J Appl Mech, **50**, 1021-1032 (1983).
[17] F. C. Moon and S.W. Shaw, Chaotic vibrations of a beam with nonlinear boundary conditions, Int. J. Non-Linear Mech, **18**, 465-477 (1983).
[18] J. H. He, Homotopy perturbation technique, Comp Meth Appl Mech Eng, **178**, 257–262 (1999).
[19] J. H. He, Coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-Linear Mech, **35**, 37-43 (2000).
[20] L. Cveticanin, The homotopy–perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity, J Sound Vib, **285**, 1171-1179 (2005).
[21] L. Cveticanin, Homotopy-perturbation method for pure nonlinear differential equation, Chaos Solitons & Fractals, **30**, 1221-1230 (2006).
[22] P. Byrd and M. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists, Berlin, Springer-Verlag 1954.
[23] M. Abramowitz and A. Stegun, Handbook of Mathematical Functions, New York, Dover 1972.

A. M. El-Naggar Professor of Mathematics at Benha University. Faculty of Science. His main research interests are: non-linear second order differential equations (weakly non-linear, strongly non-linear), topological study of periodic solutions of types (Harmonic, sub-harmonic of even and odd order, super-harmonic of even and odd order), analytical study of periodic solutions of different types (perturbation methods), dynamical systems (strongly non-linear or weakly non-linear), theory of elasticity (dynamical problems), theory of generalized thermo-elasticity or thermo-visco-elasticity, supervisor on about 35 thesis (M. Sc. and PhD) in the above subjects, has about 75 published papers in the previous fields.

G. M. Ismail Lecturer of Mathematics at Sohag University, Faculty of Science. He is referee and editor of several international journals in the frame of pure and applied mathematics. His main research interests are: nonlinear differential equations, nonlinear oscillators, applied mathematics, analytical methods, perturbation methods, analysis of nonlinear differential equations.