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Exact solutions for the nonlinear Schrödinger equation with power law nonlinearity

N. Taghizadeh** and **M. Mirzazadeh [*] *and **A.Samiei Paghaleh***

Department of Mathematics, Faculty of Mathematical Science, University of Guilan, P.O.Box 1914, Rasht, Iran

Abstract: In this paper, the nonlinear Schrödinger equation with power law nonlinearity is studied. The first integral method, the Riccati sub-ODE method are efficient methods to construct the exact solutions of nonlinear partial differential equations.By means of these methods, the periodic and solitary wave solutions of the nonlinear Schrödinger equation with power law nonlinearity are obtained.

Keywords: First integral method, Riccati sub-ODE method, Schrödinger equation with power law nonlinearity

1. Introduction

Recently, searching for exact traveling wave solutions of nonlinear partial differential equations has gained more and more popularity. Traveling wave solutions may be very helpful in understanding various phenomena in a number of fields, such as plasma physics, nonlinear optics, biology, among others. In recent decades, several powerful methods have been proposed to construct exact solutions for nonlinear partial differential equations, such as tanh method [13-14], multiple exp-function method [15], transformed rational function method [16], extended tanh-function method [17-18], first integral method [7-12] and so on.

In this paper, we will consider the nonlinear Schrödinger equation with power law nonlinearity [1] with following form

$$i\omega_t + \omega_{xx} + A|\omega|^{2n}\ \omega = 0, \tag{1}$$

where A is a real parameter and $\omega = \omega(x,t)$ is a complex -valued function of two real variables x, t. Eq. (1) has important application in various fields, such as nonlinear optics, plasma physics, superconductivity and quantum mechanics. More details are presented [3-6] .

In this paper, we would like to obtain the exact solutions of Eq. (1) by using the first integral method and the Riccati sub-ODE method. The first integral method is a powerful solution method for the computation of exact traveling wave solutions. This method is one of most direct and effective algebraic method for finding exact solutions of nonlinear partial differential equations. The first integral method, which is based on the ring theory of commutative algebra, was first proposed by Feng [7].

This method was further developed by the same author in [8-11]. The aim of this paper is to construct exact solutions for the nonlinear Schrödinger equation with power law nonlinearity.

2. The first integral method

The main steps of the first integral method are summarized as follows.

Step 1. Consider a general nonlinear PDE in the form

$$E(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \tag{2}$$

where $u = u(x,t)$ is a complex-valued function of two real variables x, t.

To find the travelling wave solutions to Eq. (2), we introduce the wave variable

$$\xi = k(x - ct), \quad \eta = \alpha x + \beta t, \tag{3}$$

so that

[*] *Corresponding author: e-mail:mirzazadehs2@guilan.ac.ir*

$$u(x,t)=e^{i\eta}u(\xi), \tag{4}$$

where k,c,α and β are constants, all of them are to be determined.

Based on this we use the following changes

$$\begin{aligned}
&\frac{\partial}{\partial x}(.)=e^{i\eta}(i\alpha+k\frac{\partial}{\partial\xi})(.),\\
&\frac{\partial}{\partial t}(.)=e^{i\eta}(i\beta-kc\frac{\partial}{\partial\xi})(.),\\
&\frac{\partial^2}{\partial x^2}(.)=e^{i\eta}(-\alpha^2+2i\alpha k\frac{\partial}{\partial\xi}+k^2\frac{\partial^2}{\partial\xi^2})(.),\\
&\frac{\partial^2}{\partial t\partial x}(.)=e^{i\eta}(-\beta\alpha+ik(\beta-c\alpha)\frac{\partial}{\partial\xi}-k^2c\frac{\partial^2}{\partial\xi^2})(.),
\end{aligned} \tag{5}$$

and so on for the other derivatives.

Using (5) changes the PDE (2) to an ODE

$$H(u,\frac{\partial u}{\partial\xi},\frac{\partial^2 u}{\partial\xi^2},...)=0, \tag{6}$$

where $u=u(\xi)$ is an unknown function, H is a polynomial in the variable u and its derivatives.

Step 2. Suppose the solution of ODE (6) can be written as follows:

$$u(x,t)=u(\xi)=X(\xi) \tag{7}$$

and furthermore, we introduce a new independent variable $Y=Y(\xi)$ such that

$$Y=X'(\xi) \tag{8}$$

Step 3.Under the conditions of Step 2, Eq. (6) can be converted to a system of nonlinear ODEs as follows

$$\begin{aligned}
&\frac{\partial X(\xi)}{\partial\xi}=Y(\xi),\\
&\frac{\partial Y(\xi)}{\partial\xi}=F(X(\xi),Y(\xi)).
\end{aligned} \tag{9}$$

If we can find the integrals to Eq. (9), then the general solutions to Eq. (9) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Eq. (9) which reduces Eq. (6) to a first order integrable ODE. An exact solution to Eq. (2) is then obtained by solving this equation.

Division Theorem.Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$, and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z)=P(w,z)G(w,z).$$

3. The Riccati sub-ODE method

Step 1.Suppose that the solutions $u(\xi)$ for Eq. (6) satisfy

$$u(\xi)=\sum_{i=0}^{N}a_i\phi^i+\sum_{i=1}^{N}b_i\phi^{-i}, \tag{10}$$

where a_i, b_i are constants to be determined later, N is a positive integer that can be determined by balancing the highest order linear term and the nonlinear terms in Eq. (6), $\phi=\phi(\xi)$ satisfies the known Riccati equation:

$$\phi_\xi=a\phi^2+b \quad (a\neq 0). \tag{11}$$

Eq. (11) admits the following exact solutions [2]:
When $b=0$,

$$\phi_1(\xi)=-\frac{1}{a\xi+\xi_0},$$

where ξ_0 is an arbitrary constant.
When $ab<0$,

$$\phi_2(\xi)=-\frac{\sqrt{-ab}}{a}\tanh(\sqrt{-ab}\,\xi-\frac{\varepsilon\ln\xi_0}{2}), \quad \xi_0>0, \quad (\varepsilon=\pm1).$$

$$\phi_3(\xi)=-\frac{\sqrt{-ab}}{a}\coth(\sqrt{-ab}\,\xi-\frac{\varepsilon\ln(-\xi_0)}{2}), \quad \xi_0<0.$$

When $ab>0$,

$$\phi_4(\xi)=\frac{\sqrt{ab}}{b}\tan(\sqrt{ab}\,\xi+\xi_0),$$

where ξ_0 is an arbitrary constant.

Step 2. Substituting Eq. (10) into Eq. (6), by use of Eq. (11), we can convert the left-hand side of Eq. (6) to a polynomial in ϕ. Equating each coefficient of ϕ^i to zero, we obtain a set of algebraic equations.

Step 3.Solving the equations in the last step, we can obtain the values of

$$a_i(i=0,1,\ldots,N), b_i(i=1,\ldots,N), a, b, c.$$

Step 4.Combining the results in Step 3 with the various solutions of Eq. (11), we can obtain a variety of exact solutions for Eq. (6).

4. Nonlinear Schrödinger equation with power law nonlinearity

We use the transformation

$$\omega(x,t)=e^{i(\alpha x+\beta t)}u(\xi), \quad \xi=k(x-2\alpha t), \tag{12}$$

where k, α and β are constants, all of them are to be determined.

Substituting (12) into (1), we obtain ordinary differential equation:

$$-(\beta+\alpha^2)u+k^2u''+Au^{2n+1}=0. \tag{13}$$

4.1. The first integral method:

Due to the difficulty in obtaining the first integral of Eq. (13), we propose a transformation denoted by $u = v^{\frac{1}{n}}.$

Then Eq. (13) is converted to

$$-(\beta+\alpha^2)v + Av^3 + \frac{k^2(1-n)}{n^2}\frac{(v')^2}{v} + \frac{k^2}{n}v'' = 0 \tag{14}$$

Rewrite this equation as follows

$$-\frac{n(\beta+\alpha^2)}{k^2}v + \frac{nA}{k^2}v^3 + (\frac{1}{n}-1)\frac{(v')^2}{v} + v'' = 0. \tag{15}$$

If we let $X = v, \quad Y = \frac{dv}{d\xi},$ the Eq. (15) is

equivalent to the two dimensional autonomous system

$$\begin{cases} X' = Y, \\ Y' = \frac{n(\beta+\alpha^2)}{k^2}X - \frac{nA}{k^2}X^3 + (1-\frac{1}{n})\frac{Y^2}{X}. \end{cases} \tag{16}$$

Making the following transformation

$$d\eta = \frac{d\xi}{X}, \tag{17}$$

then system (16) becomes

$$\begin{cases} \frac{dX}{d\eta} = XY, \\ \frac{dY}{d\eta} = \frac{n(\beta+\alpha^2)}{k^2}X^2 - \frac{nA}{k^2}X^4 + (1-\frac{1}{n})Y^2. \end{cases} \tag{18}$$

Now, we are applying the Division Theorem to seek the first integral to system (18). Suppose that $X = X(\eta), \quad Y = Y(\eta)$ are the nontrivial solutions to (18), and

$$q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in $C[X,Y],$ such that

$$q(X(\eta),Y(\eta)) = \sum_{i=0}^{m} a_i(X(\eta))Y^i(\eta) = 0, \tag{19}$$

where $a_i(X)(i = 0,1,...,m),$ are polynomials of X and $a_m(X) \neq 0.$ Eq. (19) is called the first integral to system (18). We start our study by assuming $m=1$ in (19). Note that $\frac{dq}{d\eta}$ is a polynomial in X and $Y,$ and $q[X(\eta),Y(\eta)] = 0$ implies $\left.\frac{dq}{d\eta}\right|_{(18)} = 0.$ According to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$ in $C[X,Y]$ such that

$$\left.\frac{dq}{d\eta}\right|_{(18)}=\left.\left(\frac{dq}{dX}\cdot\frac{dX}{d\eta}+\frac{dq}{dY}\cdot\frac{dY}{d\eta}\right)\right|_{(18)}=\left(\sum_{i=0}^{l}a'_i(X)Y\right)(XY)+$$
$$\left(\sum_{i=0}^{l}ia_i(X)Y^{i-1}\right).\left(\frac{n(\beta+\alpha^2)}{k^2}-\frac{nA}{k^2}X^4+\left(1-\frac{1}{n}\right)Y^2\right)$$
$$=\left(\mathrm{g}(\mathrm{X})+\mathrm{h}(\mathrm{X})\mathrm{Y}\right)\sum_{i=0}^{l}a_i(X)Y^i \tag{20}$$

where prime denotes differentiation with respect to the variable X. By comparing with the coefficients of $Y^i\ (i=2,1,0)$ of both sides of (20), we have

$$Xa_1'(X)=h(X)a_1(X)-(1-\frac{1}{n})a_1(X), \tag{21}$$
$$Xa_0'(X)=g(X)a_1(X)+h(X)a_0(X), \tag{22}$$
$$a_1(X)[\frac{n(\beta+\alpha^2)}{k^2}X^2-\frac{nA}{k^2}X^4]=g(X)a_0(X). \tag{23}$$

Since $a_i(X)(i=0,1)$ are polynomials, then from (21) we deduce that $a_1(X)$ is constant and $h(X)=1-\frac{1}{n}.$ For simplicity, take $a_1(X)=1.$ Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X))=\deg(a_0(X))=2.$ Suppose that

$$\begin{aligned}&g(X)=A_0+A_1X+A_2X^2,\\&a_0(X)=B_0+B_1X+B_2X^2,\\&(A_2\neq 0,B_2\neq 0),\end{aligned} \tag{24}$$

where A_0,A_1,A_2,B_0,B_1,B_2 are all constants to be determined . Substituting (24) into Eq. (22), we obtain

$$g(X)=((\frac{1}{n}+1)B_2)X^2+(\frac{B_1}{n})X+(\frac{1}{n}-1)B_0.$$

Substituting $a_0(X),a_1(X)$ and $g(X)$ in (23), and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain the following solutions

$$\begin{aligned}&B_0=B_1=0,\quad B_2=\pm\frac{n}{k}\sqrt{-\frac{A}{n+1}},\\&\beta=-\alpha^2,\end{aligned} \tag{25}$$

where k and α are arbitrary constants. Using the conditions (25) in (19), we obtain

$$Y\pm\frac{n}{k}\sqrt{-\frac{A}{n+1}}X^2=0.$$

Combining this first integral with (18), the second order differential Eq. (14) can be reduced to

$$\frac{dv}{d\xi}=\mp\frac{n}{k}\sqrt{-\frac{A}{n+1}}v^2. \tag{26}$$

Solving Eq. (26) and changing to the original variables, we obtain the complex rational function solutions to the nonlinear Schrödinger equation with power law nonlinearity in the following form

$$\omega_1(x,t)=e^{i(\alpha x-\alpha^2 t)}\left[\pm\frac{k}{n}\sqrt{-\frac{n+1}{A}}\cdot\frac{1}{k(x-2\alpha t)+\xi_0}\right]^{\frac{1}{n}} \tag{27}$$

where ξ_0 is an arbitrary constant.

Now we assume that $m=2$ in (19). By the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in $C[X,Y]$ such that

$$\begin{aligned}\left.\frac{dq}{d\eta}\right|_{(18)}&=\left.\left(\frac{dq}{dX}\cdot\frac{dX}{d\eta}+\frac{dq}{dY}\cdot\frac{dY}{d\eta}\right)\right|_{(18)}=\left(\sum_{i=0}^{2}a'_i(X)Y\right)(XY)+\\&\left(\sum_{i=0}^{2}ia_i(X)Y^{i-1}\right)\cdot\left(\frac{n(\beta+\alpha^2)}{k^2}-\frac{nA}{k^2}X^4+\left(1-\frac{1}{n}\right)Y^2\right)\\&=\left(\mathrm{g(X)+h(X)Y}\right)\sum_{i=0}^{2}a_i(X)Y^i\end{aligned} \tag{28}$$

On equating the coefficients of $Y^i\,(i=3,2,1,0)$ on both sides of (28), we have

$$Xa_2'(X)=h(X)a_2(X)-(2-\frac{2}{n})a_2(X), \tag{29}$$

$$Xa'_1(X)=g(X)a_2(X)+h(X)a_1(X)-(1-\frac{1}{n})a_1(X), \tag{30}$$

$$Xa'_0(X)=-2a_2(X)\left[\frac{n(\beta+\alpha^2)}{k^2}X^2-\frac{nA}{k^2}X^4\right]+g(X)a_2(X)+h(X)a_0(X), \tag{31}$$

$$a_1(X)\left[\frac{n(\beta+\alpha^2)}{k^2}X^2-\frac{nA}{k^2}X^4\right]=g(X)a_0(X). \tag{32}$$

Since $a_i(X)(i=0,1,2)$ are polynomials, then from (29) we deduce that $a_2(X)$ is constant and $h(X)=2-\frac{2}{n}$. For simplicity, take $a_2(X)=1$. Balancing the degrees $g(X),a_1(X)$ and $a_0(X)$, we conclude that

$\deg(g(X))=2,\quad \deg(a_1(X))=2,\quad \deg(a_0(X))=4.$

Suppose that

$$\begin{aligned}&g(X)=A_0+A_1X+A_2X^2,\\&a_1(X)=B_0+B_1X+B_2X^2,\\&(A_2\neq0,B_2\neq0),\\&a_0(X)=c_0+c_1X+c_2X^2+c_3X^3+c_4X^4,\quad(c_4\neq0)\end{aligned} \tag{33}$$

where $A_0,A_1,A_2,B_0,B_1,B_2,c_0,c_1,c_3,c_4$ are all constants to be determined.

Substituting (33) into Eqs. (30)- (32), and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it we obtain

$$A_0 = A_1 = 0, \quad A_2 = \pm\frac{2}{k}\sqrt{-A(n+1)},$$
$$B_0 = B_1 = 0, B_2 = \pm\frac{2n}{k}\sqrt{-\frac{A}{n+1}}, \tag{34}$$
$$c_0 = c_1 = c_2 = c_3 = 0,$$
$$c_4 = -\frac{n^2 A}{k^2(n+1)}, \quad \beta = -\alpha^2,$$

where k and α are arbitrary constants.

Now, taking the solution set (34) into account, Eq. (19) becomes

$$-\frac{n^2 A}{k^2(n+1)}X^4 \pm \frac{2n}{k}\sqrt{-\frac{A}{n+1}}X^2 Y + Y^2 = 0, \tag{35}$$

which is a first integral of Eq. (18). Solving Eq. (35), we get

$$Y = \pm\frac{n}{k}\sqrt{-\frac{A}{n+1}}X^2. \tag{36}$$

Combining (36) with (18), we obtain the exact solution to Eq. (14) and then the complex rational function solutions to the nonlinear Schrödinger equation with power law nonlinearity can be written as

$$\omega_1(x,t) = e^{i(\alpha x - \varepsilon^2 t)}\left[\pm\frac{k}{n}\sqrt{-\frac{n+1}{A}}.\ \frac{1}{k(x-2\varepsilon t)+\xi_0}\right]^{\frac{1}{n}} \tag{37}$$

where ξ_0 is an arbitrary constant.

4.2. The Riccati sub-ODE method:

Rewrite Eq. (14) as follows

$$-(\beta+\alpha^2)n^2 v^2 + An^2 v^4 + k^2(1-n)(v')^2 + nk^2 v v'' = 0$$

Suppose that the solutions of Eq. (38) can be expressed by a polynomial in ϕ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{-i}, \tag{39}$$

where $\phi = \phi(\xi)$ satisfies Riccati equation (11) .

Balancing the order of vv'' and v^4 in Eq. (38), we have $N = 1$. Therefore; Eq. (39) can be rewritten as

$$v(\xi) = a_0 + a_1\phi + \frac{b_1}{\phi} \tag{40}$$

where a_0, a_1, b_1 are constants to be determined late

Substituting Eq. (40) into Eq. (38) we can obtain an equation with respect to ϕ^i, $i = 0, \pm 1, \pm 2, \ldots$ collecting all the terms with the same power of ϕ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$\phi^4 = k^2(1-n)a^2a_1^2 + n^2Aa_1^4 + 2nk^2a^2a_1^2 = 0$$
$$\phi^3 = 4nAa_0a_1^3 + 2nk^2a^2a_0a_1 = 0,$$

(41)

$$\phi = 12n^2Aa_0a_1^2b_1 - 2(\beta+\alpha^2)n^2a_0a_1 + 2nk^2aba_0a_1 + 4n^2Aa_0^3a_1^2 = 0,$$
$$\phi^0 = k^2(1-n)a^2b_1^2 + 6n^2Aa_1^2b_1^2 - 4k^2(1-n)aba_1b_1 - 2(\beta+\alpha^2)n^2Aa_1b_1$$
$$+12n^2Aa_0^2a_1b_1 + k^2(1-n)b^2a_1^2 + n^2Aa_0^4 - (\beta+\alpha^2)n^2a_0^2 = 0,$$

$$\phi^{-1} = 12n^2Aa_0a_1^2b_1^2 - 2(\beta+\alpha^2)n^2a_0b_1 + 2nk^2aba_0b_1 + 4n^2Aa_0^3b_1 = 0,$$
$$\phi^{-2} = 4n^2Aa_1b_1^3 - (\beta+\alpha^2)n^2b_1^2 + 2nk^2abb_1^2 + 2k^2(1-n)abb_1^2 + 6n^2Aa_0^2b_1^2 + 2nk^2b^2a_1b_1$$
$$-2k^2(1-n)b_2a_1b_1 = 0,$$
$$\phi^{-3} = 2nk^2b^2a_0b_1 + 4n^2Aa_0^3b_1^3 = 0,$$
$$\phi^{-4} = k^2(1-n)b^2b_1 + 2nk^2b^2b_1^2 + n^2Ab_1^4 = 0,$$

With the aid of Maple, we shall find the special solution of the above system
where

$$a_0 = 0, \quad a_1 = \pm\frac{ak}{n}\sqrt{-\frac{n+1}{A}},$$
$$b_1 = \mp\frac{kb}{n}\sqrt{-\frac{n+1}{A}},$$
$$\beta = -\alpha^2 - \frac{4k^2ab}{n^2},$$

and α are arbitrary constants. a, b, k
From the conditions (42) we obtain

$$v(\xi) = \pm\frac{k}{n}\sqrt{\frac{n+1}{A}\left(a\phi(\xi) - \frac{b}{\phi(\xi)}\right)} \tag{42}$$

Then combining with the solutions of Riccati equation (11) we can obtain the exact solutions as follows. When $b = 0,$ we obtain the complex rational function solutions to the nonlinear Schrödinger equation with power law nonlinearity in the following form

$$\omega_1(x,t) = \left\{\mp\frac{k}{n}\sqrt{-\frac{n+1}{A}}\left(\frac{a}{ak(x-2\alpha t)+\xi_0}\right)\right\}^{\frac{1}{n}} e^{i(\alpha x-\alpha^2 t)}.$$

When $ab > 0,$ we obtain the complex trigonometric function solutions to the nonlinear Schrödinger equation with power law nonlinearity in the following form

$$\omega_2(x,t)=\left[\pm\frac{k}{n}\sqrt{-\frac{n+1}{abA}}.\left[a^2\tan(\sqrt{ab}k(x-2\alpha t)+\xi_0)-b^2\cot(\sqrt{ab}k(x-2\alpha t)+\xi_0\right]\right]^{\frac{1}{n}}e^{-i\left[\alpha x-(\varepsilon^2+\frac{4k^2ab}{n^2})t\right]}$$

When $ab<0,$ we obtain the complex hyperbolic function solutions to the nonlinear Schrödinger equation with power law nonlinearity in the following form

$$\omega_3(x,t)=\left[\pm\frac{k}{n}\sqrt{-\frac{n+1}{abA}}.\left[a^2\tanh(\sqrt{-ab}k(x-2\alpha t)-\frac{\varepsilon\ln\xi_0}{2})+b^2\cot(\sqrt{-ab}k(x-2\alpha t)-\frac{\varepsilon\ln\xi_0}{2})\right]\right]^{\frac{1}{n}}\times e^{-i\left[\alpha x-(\varepsilon^2+\frac{4k^2ab}{n^2})t\right]}$$

5. Conclusion

In this paper, some traveling wave solutions of the nonlinear Schrödinger equation with power law nonlinearity are successfully constructed by use of the first integral method and the Riccati sub-ODE method respectively. From the results we have obtained, it turns out that the Riccati sub-ODE method is more effective than the first integral method in handling the presented problem, and more general solutions are constructed by the Riccati sub-ODE method. The methods proposed in this paper can also be extended to solve some nonlinear evolution equations in mathematical physics.

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