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Modified Simple Equation Method and its Applications to Nonlinear Partial Differential Equations

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Abstract: In this paper, the modified simple equation method is used to construct exact periodic and soliton solutions of some nonlinear partial differential equations. Exact solutions of the nonlinear Schrödinger equation, the Hamiltonian amplitude equation, Klein-Gordon equation in 1+2 dimension, the coupled Klein-Gordon equation, the (2 + 1)-dimensional long-wave-short-wave resonance interaction equation, the modified KdV-KP equation and the modified Benjamin-Bona-Mahony equation are successfully obtained. These solutions may be important of significance for the explanation of some practical physical problems.

Keywords: Modified simple equation method, soliton, nonlinear evolution equation, exact solution

1 Introduction

Various physical phenomena in physics, engineering, mechanics, biology and chemistry are modeled by nonlinear partial differential equations (NPDEs). Searching for exact soliton solutions of NPDEs has been played significant role in the study of dynamics of observed phenomena [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19]. Many new methods for obtaining exact solutions to NPDEs have been proposed. Among these are inverse scattering method [20], Hirota's direct method [21], tanh method [22], extended tanh-function method [23], multiple exp-function method [24], transformed rational function method [25], first integral method [26], modified simple equation method [27,28,29,30,31,32,33], ansatz method [34,35,36,37,38] and so on. The modified simplest equation method is a very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations (ODEs). Recently, this useful method is developed successfully by Vitanov and the reference therein. The modified simple equation method [27,28,29] is based on the assumptions that the exact solutions can be expressed by a polynomial in $\frac{\Psi'}{\Psi}$, such that $\Psi = \Psi(z)$ satisfy in an unknown function to be determined later. The modified simple equation method [30,31,32,33] is based on the assumptions that the exact solutions can be expressed by a polynomial in Ψ, such that $\Psi = \Psi(z)$ satisfy in the equations of Bernoulli and Riccati which are well known nonlinear ordinary differential equations and their solutions can be expressed by elementary functions. Using this method in works [27,28,29] exact solutions of the nonlinear Fitzhugh-Nagumo equation, the Sharma-Tasso-Olver equation and the generalization of the Korteweg-de Vries equation were obtained. Also, in works [30,31,32,33] exact solutions of a class of equations that generalize the reaction-diffusion, the reaction- telegraph equation and the Fisher equation are discussed. In this paper, we apply the modified simple equation method [27,28,29] to seek the exact solutions of the second-order ODE $A\Phi''(z)+B\Phi(z)+C\Phi^3(z)=0$ and then by means of exact solutions this second-order ODE, we establish the exact solutions of the nonlinear Schrödinger equation, the Hamiltonian amplitude equation, the Klein-Gordon equation in 1+2 dimension, the coupled Klein-Gordon equation, the (2 + 1)-dimensional long-wave-short-wave resonance interaction equation, the modified KdV-KP equation and the modified Benjamin-Bona-Mahony equation

2 Modified simplest equation method

Step 1. Consider a general nonlinear PDE in the form

$$P(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0. \tag{1}$$

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Using a wave variable $z = x - ct$, we can write Eq. (1) in the following nonlinear ODE:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{2}$$

where the prime denotes the derivation with respect to z. If all terms contain derivatives, then Eq. (2) is integrated where integration constants are considered zeros.

Step 2. We suppose that Eq. (2) has the following formal solution:

$$u(z) = \sum_{l=0}^{N} A_l \left(\frac{\Psi'(z)}{\Psi(z)}\right)^l, \tag{3}$$

where A_i are arbitrary constants to be determined such that $A_N \neq 0$, while $\Psi(z)$ is an unknown function to be determined later.

Step 3. We determine the positive integer N in Eq. (3) by balancing the highest order derivatives and the nonlinear terms in Eq. (2).

Step 4. We substitute Eq. (3) into Eq. (2), we calculate all the necessary derivatives $u', u'', \ldots$ and then we account the function $\Psi(z)$. As a result of this substitution, we get a polynomial of $\frac{\Psi'(z)}{\Psi(z)}$ and its derivatives. In this polynomial, we equate with zero all the coefficients of it. This operation yields a system of equations which can be solved to find A_l and $\Psi(z)$. Consequently, we can get the exact solution of Eq. (1).

3 Exact solutions of second-order ODE $A\Phi''(z) + B\Phi(z) + C\Phi^3(z) = 0$:

Let us consider the following ODE

$$A\Phi''(z) + B\Phi(z) + C\Phi^3(z) = 0. \tag{4}$$

Balancing $\Phi''(z)$ with $\Phi^3(z)$ gives $N = 1$. This means that

$$\Phi(z) = A_0 + A_1 \frac{\Psi'(z)}{\Psi(z)}, \tag{5}$$

where A_0 and A_1 are constants to be determined such that $A_1 \neq 0$, while $\Psi(z)$ is an unknown function to be determined.

By substituting Eq. (5) into Eq. (4), we obtain

$$\begin{aligned} BA_0 + CA_0^3 &+ \frac{BA_1\Psi'(z) + AA_1\Psi'''(z) + 3CA_0^2A_1\Psi'(z)}{\Psi(z)} \\ &+ \frac{3CA_0A_1^2(\Psi'(z))^2 - 3AA_1\Psi'(z)\Psi''(z)}{\Psi^2(z)} \\ &+ \frac{2AA_1(\Psi'(z))^3 + CA_1^3(\Psi'(z))^3}{\Psi^3(z)} = 0. \end{aligned} \tag{6}$$

Equating expressions at $\Psi^0(z)$, $\Psi^{-1}(z)$, $\Psi^{-2}(z)$ and $\Psi^{-3}(z)$ to zero, we have the following equations:

$$BA_0 + CA_0^3 = 0, \tag{7}$$

$$BA_1\Psi'(z) + AA_1\Psi'''(z) + 3CA_0^2A_1\Psi'(z) = 0, \tag{8}$$

$$3CA_0A_1^2(\Psi'(z))^2 - 3AA_1\Psi'(z)\Psi''(z) = 0, \tag{9}$$

$$A_1 = \pm\sqrt{-\frac{2A}{C}}. \tag{10}$$

Eq. (7) directly implies following solutions

$$A_0 = 0, \quad A_0 = \pm\sqrt{-\frac{B}{C}}. \tag{11}$$

Case 1. If $A_0 = 0$, then we can obtain the trivial solution, which is rejected.

Case 2. If $A_0 = \pm\sqrt{-\frac{B}{C}}$, then we can deduce that

$$A\Psi'''(z) - 2B\Psi'(z) = 0, \tag{12}$$

$$-A\Psi''(z) + \sqrt{2AB}\Psi'(z) = 0. \tag{13}$$

By substituting Eq. (12) into Eq. (13), we get

$$\pm\sqrt{\frac{A}{2B}}\Psi'''(z) - \Psi''(z) = 0. \tag{14}$$

The general solution of Eq. (14) is

$$\Psi(z) = a_0 + a_1 z + a_2 e^{\pm\sqrt{\frac{2B}{A}}z}, \tag{15}$$

where $a_l (l = 0, 1, 2)$ are arbitrary constants.

Thus, we obtain the new exact solution of the ODE (4) in the following form

$$\Phi(z) = \pm\sqrt{-\frac{B}{C}}\left(1 + \sqrt{\frac{2A}{B}}\,\frac{a_1 \pm a_2\sqrt{\frac{2B}{A}}e^{\pm\sqrt{\frac{2B}{A}}z}}{a_0 + a_1 z + a_2 e^{\pm\sqrt{\frac{2B}{A}}z}}\right). \tag{16}$$

Theorem 3.1. The second-order ODE (4) has solutions described as follows.

1)- The new exact solution

$$\Phi(z) = \pm\sqrt{-\frac{B}{C}}\left(1 + \sqrt{\frac{2A}{B}}\,\frac{a_1 \pm a_2\sqrt{\frac{2B}{A}}e^{\pm\sqrt{\frac{2B}{A}}z}}{a_0 + a_1 z + a_2 e^{\pm\sqrt{\frac{2B}{A}}z}}\right), \tag{17}$$

where $a_0 \neq 0$, $a_1 \neq 0$ and $a_2 \neq 0$.

2)- The rational solution

$$\Phi(z) = \pm\sqrt{-\frac{B}{C}}\left(1 + \sqrt{\frac{2A}{B}}\,\frac{a_1}{a_0 + a_1 z}\right), \tag{18}$$

where $a_0 \neq 0,\ a_1 \neq 0$.

3)- The periodic solutions

$$\Phi(z) = \pm\sqrt{\frac{B}{C}}\tan\left(\sqrt{-\frac{B}{2A}}z\right), \tag{19}$$

and

$$\Phi(z) = \mp\sqrt{\frac{B}{C}}\cot\left(\sqrt{-\frac{B}{2A}}z\right), \tag{20}$$

where $\frac{B}{A} < 0$.

4)- The exact soliton solutions

$$\Phi(z) = \pm\sqrt{-\frac{B}{C}}\tanh\left(\sqrt{\frac{B}{2A}}z\right), \tag{21}$$

and

$$\Phi(z) = \pm\sqrt{-\frac{B}{C}}\coth\left(\sqrt{\frac{B}{2A}}z\right), \tag{22}$$

where $\frac{B}{A} > 0$.

4 Applications

Example 4.1. The nonlinear Schrödinger (NLS) equation
Let us first consider the nonlinear Schrödinger equation [39,40]

$$iu_t + pu_{xx} + q|u|^2u = 0. \tag{23}$$

We introduce the transformation

$$u(x,t) = e^{i\theta}U(z),\ \ \theta = \alpha x + \beta t,\ \ z = x - 2p\alpha t, \tag{24}$$

where α and β are constants and $U(z)$ is real function. Substituting Eq. (24) into Eq. (23), we obtain ordinary differential equation:

$$-\left(\beta + p\alpha^2\right)U(z) + p\frac{d^2U(z)}{dz^2} + qU^3(z) = 0. \tag{25}$$

By using theorem 3.1, we obtain the exact solutions of ODE. (25) in the following forms:

$$U_1(z) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\Big(1+\sqrt{-\frac{2p}{\beta+p\alpha^2}} \times \frac{a_1 \pm a_2\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}z}}{a_0 + a_1 z + a_2 e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}z}}\Big), \tag{26}$$

$$U_2(z) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\left(1+\sqrt{-\frac{2p}{\beta+p\alpha^2}}\frac{a_1}{a_0+a_1 z}\right), \tag{27}$$

$$U_3(z) = \pm\sqrt{-\frac{\beta+p\alpha^2}{q}}\tan\left(\sqrt{\frac{\beta+p\alpha^2}{2p}}z\right), \tag{28}$$

$$U_4(z) = \mp\sqrt{-\frac{\beta+p\alpha^2}{q}}\cot\left(\sqrt{\frac{\beta+p\alpha^2}{2p}}z\right), \tag{29}$$

$$U_5(z) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\tanh\left(\sqrt{-\frac{\beta+p\alpha^2}{2p}}z\right), \tag{30}$$

$$U_6(z) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\coth\left(\sqrt{-\frac{\beta+p\alpha^2}{2p}}z\right). \tag{31}$$

In (x,t)-variables we have the following exact solutions of the nonlinear Schrödinger equation:
The new exact solution of Eq. (23):

$$u_1(x,t) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\Big(1+\sqrt{-\frac{2p}{\beta+p\alpha^2}} \times \frac{a_1 \pm a_2\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}(x-2p\alpha t)}}{a_0 + a_1(x-2p\alpha t) + a_2 e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{p}}(x-2p\alpha t)}}\Big) \times e^{i(\alpha x+\beta t)}. \tag{32}$$

The rational solution of Eq. (23):

$$u_2(x,t) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\Big(1+\sqrt{-\frac{2p}{\beta+p\alpha^2}} \times \frac{a_1}{a_0+a_1(x-2p\alpha t)}\Big)e^{i(\alpha x+\beta t)}. \tag{33}$$

The periodic solutions of Eq. (23) for $p(\beta + p\alpha^2) > 0$:

$$u_3(x,t) = \pm\sqrt{-\frac{\beta+p\alpha^2}{q}}\tan\left(\sqrt{\frac{\beta+p\alpha^2}{2p}}(x-2p\alpha t)\right) \times e^{i(\alpha x+\beta t)}, \tag{34}$$

and

$$u_4(x,t) = \mp\sqrt{-\frac{\beta+p\alpha^2}{q}}\cot\left(\sqrt{\frac{\beta+p\alpha^2}{2p}}(x-2p\alpha t)\right) \times e^{i(\alpha x+\beta t)}. \tag{35}$$

The exact soliton solutions of Eq. (23) for $-p(\beta + p\alpha^2) > 0$:

$$u_5(x,t) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\tanh\left(\sqrt{-\frac{\beta+p\alpha^2}{2p}}(x-2p\alpha t)\right) \times e^{i(\alpha x+\beta t)}, \tag{36}$$

and

$$u_6(x,t) = \pm\sqrt{\frac{\beta+p\alpha^2}{q}}\coth\left(\sqrt{-\frac{\beta+p\alpha^2}{2p}}(x-2p\alpha t)\right) \times e^{i(\alpha x+\beta t)}. \quad (37)$$

Example 4.2. New Hamiltonian amplitude equation
A new Hamiltonian amplitude equation [41]:

$$iu_x + u_{tt} + 2\sigma|u|^2 u - \varepsilon u_{xt} = 0, \quad (38)$$

where $\sigma = \pm 1$, $\varepsilon << 1$, was recently introduced by Wadati et al., [42]. This is an equation which governs certain instabilities of modulated wave trains, with the additional term $-\varepsilon u_{xt}$ overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinski equation which arises in dissipative systems and is apparently not integrable.
By making the transformation

$$u(x,t) = e^{i\theta}U(z), \ \ \theta = \alpha x - \beta t, \ \ z = k(x-\lambda t), \quad (39)$$

the Eq. (38) becomes

$$k^2(\lambda^2+\varepsilon\lambda)U''(z) + ik(1+2\beta\lambda+\varepsilon\alpha\lambda+\varepsilon\beta)U'(z) - (\alpha+\beta^2+\varepsilon\alpha\beta)U(z) + 2\sigma U^3(z) = 0. \quad (40)$$

If we take

$$\lambda = -\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha} \quad (41)$$

Eq. (40) is transformed into

$$k^2(\lambda^2+\varepsilon\lambda)U''(z) - (\alpha+\beta^2+\varepsilon\alpha\beta)U(z) + 2\sigma U^3(z) = 0. \quad (42)$$

By using theorem 3.1, we get the exact solutions of ODE. (42) in the following forms:

$$U_1(z) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}}\Bigg(1+\sqrt{-\frac{2k^2(\lambda^2+\varepsilon\lambda)}{\alpha+\beta^2+\varepsilon\alpha\beta}} \times \frac{a_1 \pm a_2\sqrt{-\frac{2(\alpha+\beta^2+\varepsilon\alpha\beta)}{k^2(\lambda^2+\varepsilon\lambda)}}e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{k^2(\lambda^2+\varepsilon\lambda)}}z}}{a_0+a_1z+a_2e^{\pm\sqrt{-\frac{2(\alpha+\beta^2+\varepsilon\alpha\beta)}{k^2(\lambda^2+\varepsilon\lambda)}}z}}\Bigg), \quad (43)$$

$$U_2(z) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}}\Bigg(1+\sqrt{-\frac{2k^2(\lambda^2+\varepsilon\lambda)}{\alpha+\beta^2+\varepsilon\alpha\beta}} \times \frac{a_1}{a_0+a_1z}\Bigg), \quad (44)$$

$$U_3(z) = \pm\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \tan\left(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}z\right), \quad (45)$$

$$U_4(z) = \mp\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \cot\left(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}z\right), \quad (46)$$

$$U_5(z) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \tanh\left(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}z\right), \quad (47)$$

$$U_6(z) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \coth\left(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}z\right). \quad (48)$$

Combining (43)-(48) with (39), we obtain the exact solutions to Eq. (38) can be written as
The new exact solution of Eq. (38):

$$u_1(x,t) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}}\Bigg(1+\sqrt{-\frac{2k^2(\lambda^2+\varepsilon\lambda)}{\alpha+\beta^2+\varepsilon\alpha\beta}} \times \frac{a_1 \pm a_2\sqrt{-\frac{2(\alpha+\beta^2+\varepsilon\alpha\beta)}{k^2(\lambda^2+\varepsilon\lambda)}}e^{\pm\sqrt{-\frac{2(\beta+p\alpha^2)}{k^2(\lambda^2+\varepsilon\lambda)}}(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t))}}{a_0+a_1z+a_2e^{\pm\sqrt{-\frac{2(\alpha+\beta^2+\varepsilon\alpha\beta)}{k^2(\lambda^2+\varepsilon\lambda)}}(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t))}}\Bigg) \times e^{i(\alpha x-\beta t)}. \quad (49)$$

The rational solution of Eq. (38):

$$u_2(x,t) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}}\Bigg(1+\sqrt{-\frac{2k^2(\lambda^2+\varepsilon\lambda)}{\alpha+\beta^2+\varepsilon\alpha\beta}} \times \frac{a_1}{a_0+a_1\left(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t)\right)}\Bigg) \times e^{i(\alpha x-\beta t)}. \quad (50)$$

The periodic solutions of Eq. (38) for $\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)} > 0$:

$$u_3(x,t) = \pm\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \tan\left(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}\left(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t)\right)\right) \times e^{i(\alpha x-\beta t)}, \quad (51)$$

$$u_4(x,t) = \mp\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \cot\left(\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}\left(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t)\right)\right) \times e^{i(\alpha x-\beta t)}. \quad (52)$$

The exact soliton solutions of Eq. (38) for $-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)} > 0$:

$$u_5(x,t) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \tanh\left(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}\left(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t)\right)\right) \times e^{i(\alpha x-\beta t)}, \quad (53)$$

$$u_6(x,t) = \pm\sqrt{\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2\sigma}} \times \coth\left(\sqrt{-\frac{\alpha+\beta^2+\varepsilon\alpha\beta}{2k^2(\lambda^2+\varepsilon\lambda)}}\left(k(x+\frac{1+\varepsilon\beta}{2\beta+\varepsilon\alpha}t)\right)\right) \times e^{i(\alpha x-\beta t)}. \quad (54)$$

Example 4.3. Klein-Gordon equation in 1+2 dimension
Let us now consider the Klein-Gordon equation in 1+2 dimension [43]

$$q_{tt} - k^2(q_{xx}+q_{yy}) + aq - bq^3 = 0, \quad (55)$$

where k, a and b are real constants.
Using the transformation

$$q(x,y,t) = U(z), \quad z = B_1x+B_2y-vt, \quad (56)$$

and substituting Eq. (56) into Eq. (57) yields

$$(v^2-k^2(B_1^2+B_2^2))U''(z) + aU(z) - bU^3(z) = 0 \quad (57)$$

where B_1, B_2 and v are constants and the prime denotes the derivation with respect to z.
By using theorem 3.1 and similar to the previous examples, we can find the following exact solutions for Eq. (55):
The new exact solution of Eq. (55):

$$q_1 = \pm\sqrt{\frac{a}{b}}(1+\sqrt{\frac{2(v^2-k^2(B_1^2+B_2^2))}{a}} \times \frac{a_1 \pm a_2\sqrt{\frac{2a}{v^2-k^2(B_1^2+B_2^2)}}e^{\pm\sqrt{\frac{2a}{v^2-k^2(B_1^2+B_2^2)}}z}}{a_0+a_1z+a_2e^{\pm\sqrt{\frac{2a}{v^2-k^2(B_1^2+B_2^2)}}z}}), \quad (58)$$

where $z = B_1x+B_2y-vt$.
The rational solution of Eq. (55):

$$q_2 = \pm\sqrt{\frac{a}{b}}(1+\sqrt{\frac{2(v^2-k^2(B_1^2+B_2^2))}{a}} \times \frac{a_1}{a_0+a_1(B_1x+B_2y-vt)}). \quad (59)$$

The periodic solutions of Eq. (55) for $a(k^2(B_1^2+B_2^2)-v^2) > 0$:

$$q_3 = \pm\sqrt{-\frac{a}{b}} \times \tan\left(\sqrt{\frac{a}{2(k^2(B_1^2+B_2^2)-v^2)}}(B_1x+B_2y-vt)\right), \quad (60)$$

$$q_4 = \mp\sqrt{-\frac{a}{b}} \times \cot\left(\sqrt{\frac{a}{2(k^2(B_1^2+B_2^2)-v^2)}}(B_1x+B_2y-vt)\right). \quad (61)$$

The exact soliton solutions of Eq. (55) for $a(v^2-k^2(B_1^2+B_2^2)) > 0$:

$$q_5 = \pm\sqrt{\frac{a}{b}} \times \tanh\left(\sqrt{\frac{a}{2(v^2-k^2(B_1^2+B_2^2))}}(B_1x+B_2y-vt)\right) \quad (62)$$

$$q_6 = \pm\sqrt{\frac{a}{b}} \times \coth\left(\sqrt{\frac{a}{2(v^2-k^2(B_1^2+B_2^2))}}(B_1x+B_2y-vt)\right) \quad (63)$$

Example 4.4. Coupled Klein-Gordon equation
In this section, we study the coupled Klein-Gordon equation [44]

$$u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0,$$

$$v_x - v_t - 4uu_t = 0. \quad (64)$$

The nonlinear coupled Klein-Gordon equation is very important equation in the area of Theoretical Physics. The nonlinear coupled Klein-Gordon equation was first studied in [45], and then Shang [46] and Yusufoglu and Bekir [47] gave further result by using the ideas of the tanh method and the general integral method. They also obtained the solutions and periodic solutions. In paper Sassaman and Biswas [48], the quasilinear coupled Klein-Gordon, which have several forms of power law nonlinearity, are well studied by using soliton perturbation theory.
We use the wave transformations

$$u(x,t) = u(z), \quad v(x,t) = v(z), \quad z = x-ct. \quad (65)$$

Substituting Eq. (65) into Eqs. (64), we have the ordinary differential equations (ODEs) for $u(z)$ and $v(z)$

$$(1-c^2)u''(z) - u(z) + 2u^3(z) + 2u(z)v(z) = 0,$$

$$(1+c)v'(z) + 4cu(z)u'(z) = 0. \quad (66)$$

By integrating the second equation with respect to z, and neglecting the constant of integration we obtain

$$v(z) = -\frac{2c}{1+c}u^2(z). \quad (67)$$

Substituting Eq. (67) into the first equation of Eqs. (66), we find

$$(1-c^2)u''(z) - u(z) + \frac{2(1-c)}{1+c}u^3(z) = 0. \quad (68)$$

By using theorem 3.1, we obtain the exact solutions of ODE. (68) in the following forms:

$$u_1(z) = \pm\sqrt{\frac{1+c}{2(1-c)}}(1+\sqrt{2(c^2-1)} \times \frac{a_1 \pm a_2\sqrt{\frac{2}{c^2-1}}e^{\pm\sqrt{\frac{2}{c^2-1}}z}}{a_0+a_1z+a_2e^{\pm\sqrt{\frac{2}{c^2-1}}z}}), \tag{69}$$

$$u_2(z) = \pm\sqrt{\frac{1+c}{2(1-c)}}\left(1+\sqrt{2(c^2-1)}\frac{a_1}{a_0+a_1z}\right), \tag{70}$$

$$u_3(z) = \pm\sqrt{\frac{1+c}{2(c-1)}}\tan\left(\frac{z}{\sqrt{2(1-c^2)}}\right), \tag{71}$$

$$u_4(z) = \mp\sqrt{\frac{1+c}{2(c-1)}}\cot\left(\frac{z}{\sqrt{2(1-c^2)}}\right), \tag{72}$$

$$u_5(z) = \pm\sqrt{\frac{1+c}{2(1-c)}}\tanh\left(\frac{z}{\sqrt{2(c^2-1)}}\right), \tag{73}$$

$$u_6(z) = \pm\sqrt{\frac{1+c}{2(1-c)}}\coth\left(\frac{z}{\sqrt{2(c^2-1)}}\right). \tag{74}$$

Combining (69)- (74) with Eq. (67), we obtain the exact solutions to Eqs. (64) can be written as
The new exact solution of Eqs. (64):

$$u_1(x,t) = \pm\sqrt{\frac{1+c}{2(1-c)}}(1+\sqrt{2(c^2-1)} \times \frac{a_1 \pm a_2\sqrt{\frac{2}{c^2-1}}e^{\pm\sqrt{\frac{2}{c^2-1}}(x-ct)}}{a_0+a_1z+a_2e^{\pm\sqrt{\frac{2}{c^2-1}}(x-ct)}}), \tag{75}$$

$$v_1(x,t) = \frac{c}{c-1}(1+\sqrt{2(c^2-1)} \times \frac{a_1 \pm a_2\sqrt{\frac{2}{c^2-1}}e^{\pm\sqrt{\frac{2}{c^2-1}}(x-ct)}}{a_0+a_1z+a_2e^{\pm\sqrt{\frac{2}{c^2-1}}(x-ct)}})^2.$$

The rational solution of Eqs. (64):

$$u_2(x,t) = \pm\sqrt{\frac{1+c}{2(1-c)}}(1+\sqrt{2(c^2-1)} \times \frac{a_1}{a_0+a_1(x-ct)}), \tag{76}$$

$$v_2(x,t) = \frac{c}{c-1}(1+\sqrt{2(c^2-1)}\frac{a_1}{a_0+a_1(x-ct)})^2.$$

The periodic solutions of Eqs. (64) for $1-c^2>0:$

$$u_3(x,t) = \pm\sqrt{\frac{1+c}{2(c-1)}}\tan\left(\frac{x-ct}{\sqrt{2(1-c^2)}}\right), \tag{77}$$

$$v_3(x,t) = \frac{c}{1-c}\tan^2\left(\frac{x-ct}{\sqrt{2(1-c^2)}}\right).$$

$$u_4(x,t) = \mp\sqrt{\frac{1+c}{2(c-1)}}\cot\left(\frac{x-ct}{\sqrt{2(1-c^2)}}\right), \tag{78}$$

$$v_4(x,t) = \frac{c}{1-c}\cot^2(\frac{x-ct}{\sqrt{2(1-c^2)}}).$$

The exact soliton solutions of Eqs. (64) for $c^2-1>0:$

$$u_5(x,t) = \pm\sqrt{\frac{1+c}{2(1-c)}}\tanh\left(\frac{x-ct}{\sqrt{2(c^2-1)}}\right), \tag{79}$$

$$v_5(x,t) = \frac{c}{c-1}\tanh^2\left(\frac{x-ct}{\sqrt{2(c^2-1)}}\right).$$

$$u_6(x,t) = \pm\sqrt{\frac{1+c}{2(1-c)}}\coth\left(\frac{x-ct}{\sqrt{2(c^2-1)}}\right), \tag{80}$$

$$v_6(x,t) = \frac{c}{c-1}\coth^2\left(\frac{x-ct}{\sqrt{2(c^2-1)}}\right).$$

Example 4.5. (2 + 1)- dimensional long- wave-short- wave resonance interaction equation
Next, we consider the (2 + 1)-dimensional long-wave-short-wave resonance interaction equation [49, 50]

$$i(u_t+u_y)-u_{xx}+uv=0, \qquad (a) \tag{81}$$

$$v_t-2(|u|^2)_x=0. \qquad (b)$$

where u and v denote the the short surface wave packets and long interfacial wave respectively. Eqs. (81) describe the long and short waves propagating at an angle of each other in a two-layer fluid. This system has been demonstrated to have both bright and dark two-soliton solutions.
Using the wave transformations

$$u(x,y,t) = e^{i(px+qy+kt)}U(z), \quad v(x,y,t) = V(z), \quad z = x+(a-2p)y+at, \tag{82}$$

where p,q,k and a are real constant.
Substituting (82) into (81), we have

$$U''(z)+(q+k-p^2)U(z)-U(z)V(z)=0, \qquad (a) \tag{83}$$

$$aV'(z)-2(U^2(z))'=0, \qquad (b)$$

Integrating Eq. (83b) with respect to z and taking the integration constant as zero yields

$$V=\frac{2}{a}U^2. \tag{84}$$

Substituting Eq. (84) into Eq. (83a) yields

$$U''+(q+k-p^2)U-\frac{2}{a}U^3=0. \tag{85}$$

By using theorem 3.1, and similar to the previous examples, we can find the following exact solutions for Eqs. (81):
The new exact solution of Eqs. (81):

$$\begin{aligned} u_1 &= \pm\sqrt{\frac{a}{2}(q+k-p^2)}(1+\sqrt{\frac{2}{q+k-p^2}} \\ &\times \frac{a_1 \pm a_2\sqrt{2(q+k-p^2)}e^{\pm\sqrt{2(q+k-p^2)}z}}{a_0+a_1z+a_2e^{\pm\sqrt{2(q+k-p^2)}z}}) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{86}$$

$$\begin{aligned} v_1 &= (q+k-p^2)(1+\sqrt{\frac{2}{q+k-p^2}} \\ &\times \frac{a_1 \pm a_2\sqrt{2(q+k-p^2)}e^{\pm\sqrt{2(q+k-p^2)}z}}{a_0+a_1z+a_2e^{\pm\sqrt{2(q+k-p^2)}z}})^2, \end{aligned}$$

where $z = x+(a-2p)y+at$.
The rational solution of Eqs. (81):

$$\begin{aligned} u_2 &= \pm\sqrt{\frac{a}{2}(q+k-p^2)}(1+\sqrt{\frac{2}{q+k-p^2}} \\ &\times \frac{a_1}{a_0+a_1(x+(a-2p)y+at)}) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{87}$$

$$\begin{aligned} v_2 &= (q+k-p^2)(1+\sqrt{\frac{2}{q+k-p^2}} \\ &\times \frac{a_1}{a_0+a_1(x+(a-2p)y+at)})^2. \end{aligned}$$

The periodic solutions of Eqs. (81) for $p^2 > q+k$:

$$\begin{aligned} u_3 &= \pm\sqrt{\frac{a}{2}(p^2-q-k)} \\ &\times \tan\left(\sqrt{\frac{p^2-q-k}{2}}(x+(a-2p)y+at)\right) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{88}$$

$$v_3 = (p^2-q-k)\tan^2\left(\sqrt{\frac{p^2-q-k}{2}}(x+(a-2p)y+at)\right).$$

$$\begin{aligned} u_4 &= \mp\sqrt{\frac{a}{2}(p^2-q-k)} \\ &\times \cot\left(\sqrt{\frac{p^2-q-k}{2}}(x+(a-2p)y+at)\right) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{89}$$

$$v_4 = (p^2-q-k)\cot^2\left(\sqrt{\frac{p^2-q-k}{2}}(x+(a-2p)y+at)\right).$$

The exact soliton solutions of Eqs. (81) for $q+k > p^2$:

$$\begin{aligned} u_5 &= \pm\sqrt{\frac{a}{2}(q+k-p^2)} \\ &\times \tanh\left(\sqrt{\frac{q+k-p^2}{2}}(x+(a-2p)y+at)\right) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{90}$$

$$v_5 = (q+k-p^2)\tanh^2\left(\sqrt{\frac{q+k-p^2}{2}}(x+(a-2p)y+at)\right).$$

$$\begin{aligned} u_6 &= \pm\sqrt{\frac{a}{2}(q+k-p^2)} \\ &\times \coth\left(\sqrt{\frac{q+k-p^2}{2}}(x+(a-2p)y+at)\right) \\ &\times e^{i(px+qy+kt)}, \end{aligned} \tag{91}$$

$$v_6 = (q+k-p^2)\coth^2\left(\sqrt{\frac{q+k-p^2}{2}}(x+(a-2p)y+at)\right).$$

Example 4.6. Modified KdV-KP equation
Using the idea of Kadomtsev and Petviashvili, who relaxed the restriction that the waves be strictly one-dimensional in the KdV equation, leads to the (2+1)-dimensional modified KdV-KP equation [51]:

$$(u_t - \frac{3}{2}u_x + 6u^2u_x + u_{xxx})_x + u_{yy} = 0. \tag{92}$$

This equation was investigated in the literature because it is used tomodel a variety of nonlinear phenomena. Functional variable methods was used to construct traveling wave solutions of this equation in [51].
We use the wave transformation

$$u(x,y,t) = U(z), \quad z = k(x+ly-\lambda t), \tag{93}$$

where k, l and λ are constants.
Substituting (93) into (92), we obtain the following ODE:

$$(-(\lambda+\frac{3}{2})U'(z)+6U^2(z)U'(z)+k^2U'''(z))'+l^2U''(z) = 0. \tag{94}$$

Integrating Eq. (**??**) twice with respect to z and neglecting constants of integration, we find

$$\left(l^2 - \frac{(2\lambda+3)}{2}\right)U(z)+2U^3(z)+k^2U''(z) = 0. \tag{95}$$

Using theorem 3.1 and proceeding as before we find the following exact solutions for the modified KdV-KP equation:
The new exact solution of Eq. (92):

$$\begin{aligned} u_1 &= \pm\frac{\sqrt{2\lambda+3-2l^2}}{2}(1+\frac{2k}{\sqrt{2l^2-2\lambda-3}} \\ &\times \frac{a_1 \pm a_2\sqrt{\frac{2l^2-2\lambda-3}{k^2}}e^{\pm\sqrt{\frac{2l^2-2\lambda-3}{k^2}}z}}{a_0+a_1z+a_2e^{\pm\sqrt{\frac{2l^2-2\lambda-3}{k^2}}z}}), \end{aligned} \tag{96}$$

where $z = k(x+ly-\lambda t)$.

The rational solution of Eq. (92):

$$u_2 = \pm\frac{\sqrt{2\lambda+3-2l^2}}{2}(1+\frac{2k}{\sqrt{2l^2-2\lambda-3}} \times \frac{a_1}{a_0+a_1k(x+ly-\lambda t)}). \tag{97}$$

The periodic solutions of Eq. (92) for $2\lambda+3 > 2l^2$:

$$u_3 = \pm\frac{\sqrt{2l^2-2\lambda-3}}{2} \times \tan\left[\frac{\sqrt{2\lambda+3-2l^2}}{2k}(k(x+ly-\lambda t))\right], \tag{98}$$

$$u_4 = \mp\frac{\sqrt{2l^2-2\lambda-3}}{2} \times \cot\left[\frac{\sqrt{2\lambda+3-2l^2}}{2k}(k(x+ly-\lambda t))\right]. \tag{99}$$

The exact soliton solutions of Eq. (92) for $2l^2 > 2\lambda+3$:

$$u_5 = \pm\frac{\sqrt{2\lambda+3-2l^2}}{2} \times \tanh\left[\frac{\sqrt{2l^2-2\lambda-3}}{2k}(k(x+ly-\lambda t))\right], \tag{100}$$

$$u_6 = \pm\frac{\sqrt{2\lambda+3-2l^2}}{2} \times \coth\left[\frac{\sqrt{2l^2-2\lambda-3}}{2k}(k(x+ly-\lambda t))\right]. \tag{101}$$

Example 4.7. Modified Benjamin-Bona-Mahony equation

Consider the modified Benjamin-Bona-Mahony equation (mBBM) [52]

$$u_t + u_x + u^2u_x + u_{xxt} = 0. \tag{102}$$

This equation models long waves in a nonlinear dispersive system. The existence of the solutions of initial value problems for the mBBM equation has been considered in [52,53]. Yusufoglu and Bekir [54] used the tanh and the sine-cosine methods to obtain exact solutions of the mBBM equation. By the exp-function method, Yusufoglu [55] obtained new solitary solutions for the mBBM equations. Layeni and Akinola [56] used the hyperbolic auxiliary function method and reported some new exact solutions of the mBBM equation. Abbasbandy and Shirzadi used the first integral method in this equation [57].

We use the wave transformation

$$u(x,t) = U(z), \quad z = x-ct, \tag{103}$$

where c is constant.

Substituting (103) into (102), we obtain ordinary differential equation:

$$(1-c)U'(z) + U^2(z)U(z) - cU'''(z) = 0. \tag{104}$$

Integrating Eq. (104) with respect to z and considering the zero constants for integration, we obtain

$$(1-c)U(z) + \frac{1}{3}U^3(z) - cU''(z) = 0. \tag{105}$$

Using theorem 3.1, and proceeding as before we find the following exact solutions for the modified KdV-KP equation:

The new exact solution of Eq. (102):

$$u_1(x,t) = \pm\sqrt{3(c-1)}(1+\sqrt{\frac{2c}{c-1}} \times \frac{a_1 \pm a_2\sqrt{\frac{2(c-1)}{c}}e^{\pm\sqrt{\frac{2(c-1)}{c}}(x-ct)}}{a_0+a_1(x-ct)+a_2e^{\pm\sqrt{\frac{2(c-1)}{c}}(x-ct)}}). \tag{106}$$

The rational solution of Eq. (102):

$$u_2 = \pm\sqrt{3(c-1)}(1+\sqrt{\frac{2c}{c-1}} \times \frac{a_1}{a_0+a_1(x-ct)}). \tag{107}$$

The periodic solutions of Eq. (102) for $c(1-c) > 0$:

$$u_3 = \pm\sqrt{3(1-c)}\tan\left(\sqrt{\frac{1-c}{2c}}(x-ct)\right), \tag{108}$$

$$u_4(x,t) = \mp\sqrt{3(1-c)}\cot\left(\sqrt{\frac{1-c}{2c}}(x-ct)\right). \tag{109}$$

The exact soliton solutions of Eq. (102) for $c(1-c) < 0$:

$$u_5 = \pm\sqrt{3(c-1)}\tanh\left(\sqrt{\frac{c-1}{2c}}(x-ct)\right), \tag{110}$$

$$u_6 = \pm\sqrt{3(c-1)}\coth\left(\sqrt{\frac{c-1}{2c}}(x-ct)\right). \tag{111}$$

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