Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/amisl/020102

Solution to Some Open Problems on Absorbant of Generalized de Bruijn Digraphs

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Received: 12 Jun. 2013, Revised: 9 Oct. 2013, Accepted: 13 Oct. 2013
Published online: 1 Jan. 2014

Abstract: The generalized de Bruijn digraph $G_B(n,d)$ has good properties as an interconnection network topology. The resource location problem in an interconnection network is one of the facility location problems. Finding absorbants of a digraph corresponds to solving a kind of resource location problem. In this paper, we solve some open problems given in the article ”Erfang Shan, T.C.E.Cheng, Liying Kang, Absorbant of generalized de Bruijn digraphs, Inform. Process. Lett. 105 (2007) 6-11”.

Keywords: Dominating set, Absorbant, Generalized de Bruijn digraphs, Interconnection networks

1 Introduction

In this paper, we deal with simple digraphs which admit self-loops but no multiple arcs. Let $D = (V,A)$ be a digraph with the vertex set V and the arc set A. There is an arc from x to y if $(x,y) \in A$. The vertex x is called a predecessor of y and y is called a successor of x. For a vertex $v \in V$, the out-neighborhood of v is $N^+(v) = w|(v,w) \in A$ and its in-neighborhood is the set $N^+(v) = u|(u,v) \in A$. The closed out-neighborhood of v is the set $N^+[v] = N^+(v) \cup v$ and its closed in-neighborhood is the set $N^-[v] = N^-(v) \cup v$. For $S \subseteq V$, its out-neighborhood is the set $N^+(S) = \bigcup_{s\in S} N^+(s)$ and its in-neighborhood is the set $N^-(S) = \bigcup_{s\in S} N^-(s)$. $N^+[S]$ and $N^-[S]$ are defined similarly. An absorbant of a digraph D is a set S of vertices of D such that for all $v \in V - S$, $N^+(v) \cap S \neq \emptyset$, i.e., $N^-[S] = V$. The absorbant number of D, denoted by $\gamma_a(D)$, is defined as the minimum cardinality of an absorbant of D. An absorbant of D of cardinality $\gamma_a(D)$ is called a γ_a-set. A set $S \subseteq V$ is a dominating set of D if for all $v \in V - S$, $N^-(v) \cap S \neq \emptyset$, i.e., $N^+[S] = V$. Similarly, the domination number of D, denoted by $\gamma(D)$, is defined as the minimum cardinality of a dominating set of D. For standard graph theory terminology not given here we refer to [4]. The resource location problem in an interconnection network is one of the facility location problems. Constructing the absorbants and dominating sets corresponds to solving two kinds of resource location problems [3,8]. For example, each vertex in an absorbant or a dominating set provides a service (file-server, and so on) for a network. In this case, every vertex has a direct access to file-servers. Since each file-server may cost a lot, the number of an absorbant or a dominating set has to be minimized. The generalized de Bruijn digraph $G_B(n,d)$ is defined in [2,5] by congruence equations as follows:

$$V(G_B(n,d)) = \{0,1,2,...,n-1\} \quad and$$

$$A(G_B(n,d)) = \{(x,y) : y \equiv dx + i(mod n), 0 \le x \le d-1\}$$

The following open problems are given in [7] 1. Is it true that if $G_B(n,d)$ is a generalized de Bruijn digraph with $d \ge 2$ and $n \ge d$, $\gamma(G_B(n,d)) \le \gamma_a(G_B(n,d))$? If it is not so, does there exist a generalized de Bruijn digraph $G_B(n,d)$ satisfying $\gamma_a(G_B(n,d)) < \gamma(G_B(n,d))$? 2. Find sufficient conditions for the absorbant number of $G_B(n,d)$ to be the lower bound $\left\lceil \frac{n}{d+1} \right\rceil$. 3. Find a sufficient and necessary condition for the absorbant number of $G_B(n,d)$ to be its domination number. 4. Is it true that $\gamma_a(G_B(8k-4,4k-3))$, for $k \ge 2$? 5. Is it true that $\gamma_a(G_B(6k,2k-1))$, for $k \ge 2$? ?

In this article, we solve the problem 4 and we provide a partial solution to the problem 3 using problem 4.

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2 Solution of the problems

We use the following results subsequently in the proof of the problem. **Theorem A ([7]).** $\lceil\frac{n}{d+1}\rceil \leq \gamma_a(G_B(n,d)) \leq \lceil\frac{n}{d}\rceil$// **Theorem B ([7]).** If $d = 2,4$ and $(d+1)|n$ or $d = 3$ and $8|n$, then $\gamma_a(G_B(n,d)) = \frac{n}{d+1}$.// **Theorem C ([8]).** For domination number of generalized de Bruijn digraphs, we obtain

1.$\gamma(G_B(2s,2s-1)) = 2$.
2.$\gamma(G_B(8s-4,4s-3)) = 3$ and
3.$\gamma(G_B(6s,2s-1)) = 4$, where s is a natural number.

Theorem D ([2]). Let $gcd(n,d-1) = g$. Then the number of loops in $G_B(n,d)$ is $g.\left[\frac{d}{g}\right]$, where $[x]$denotes the smallest integer not less than x.// **Theorem E ([1]).** Assume $gcd(a,m) = d$. Then the linear congruence $ax \equiv b(modm)$ has solutions if, and only if, $d|b$.// **Theorem F ([1]).** Assume $gcd(a,m) = d$ and suppose that $d|b$. Then the linear congruence $ax \equiv b(modm)$ has exactly d solutions modulo m. These are given by $t, t+\frac{m}{d}$, $t+2\frac{m}{d}$,...,$t+(d-1)\frac{m}{d}$, where t is the solution, unique modulo $\frac{m}{d}$ of the linear congruence $\frac{a}{d}x \equiv \frac{b}{d}(mod\frac{m}{d})$.// **Theorem G ([6]).** If $n = d+1$, then $\gamma^*(G_B(n,d)) = 1$, d is even, $\gamma^*(G_B(n,d)) = 2$, d is odd.//

Theorem 2.1. Is it true that $\gamma_a(G_B(8k-4,4k-3))$, for $k \geq 2$. **Proof.** By Theorem A, $2 \leq \gamma_a(G_B(8k-4,4k-3)) \leq 3$. So we only need to show that $\gamma_a(G_B(8k-4,4k-3)) \neq 2$. Every vertex in $G_B(8k-4,4k-3)$ has d in-neighbors. Self-loop vertices increase the cardinality of an absorbant, so we are going to collect all the vertices, which do not have a self-loop. For that, we first develop a necessary and sufficient condition for a vertex to have a self-loop. In $G_B(8k-4,4k-3)$, the arcs are $y \equiv (4k-3)x+i(mod\ 8k-4), 0 \leq i \leq 4k-4$ and $0 \leq y \leq n-1$. Suppose x is a self-loop vertex in $G_B(8k-4,4k-3)$. We want to determine for how many $x = 0,1,2,\ldots,n-1$ the congruence $x \equiv (4k-3)x+i(mod\ 8k-4)$ or equivalently, the congruence $\frac{(4k-4)x}{g} \equiv \frac{-i}{g}\left(mod\ \frac{8k-4}{g}\right)$, where $g = gcd(n,d-1)$ is satisfied. Clearly

$$\begin{aligned} g &= gcd(8k-4,4k-4) \\ &= 4gcd(2k-1,k-1) \\ &= 4gcd(2k-1-k+1,k-1) \\ &= 4gcd(k,k-1) \\ &= 4, \end{aligned}$$

we have $(k-1)x \equiv \frac{-i}{4}(mod\ 2k-1)$. By Theorem E, this congruence has solutions only when $4|i$. That is $i = 2x, x$ is even. Also we have $2x \leq 4k-4 \Rightarrow x \leq 2k-2$. The congruence $\frac{(4k-4)x}{4} \equiv \frac{-i}{4}\left(mod\ \frac{8k-4}{4}\right)$ becomes $(2k-1)x \equiv 0(mod\ 2k-1)$. Clearly $x = 0,2,4,\ldots,2k-2$ are the solution of the congruence.
By Theorem F

$$\begin{aligned} x &= 0,2k-1,4k-2,6k-3 \\ x &= 2,2k+1,4k,6k-1 \\ x &= 4,2k+3,4k+2,6k+1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x &= 2k-2,4k-3,6k-4,8k-5 \end{aligned}$$

are also the solutions of the congruence. Therefore the number of self-loop vertices is $g\left(\frac{2k-2}{2}+1\right) = 4k$, which is also given in Theorem D. Conversely, any vertex x of the set $\{0,1,2,\ldots,8k-5\}$ satisfying the congruence relation $\frac{(4k-4)x}{4} \equiv \frac{-i}{4}\left(mod\ \frac{8k-4}{4}\right)$ is a self-loop vertex. By retracing the steps, we get the converse part. We construct a subset V_1 of $V(G_B(8k-4,4k-3))$ as follows:

$$V_1 = \bigcup_{i=0}^{3}\{(2k-1)i+1,(2k-1)i+3,\ldots,(2k-1)i+2k-3\}.$$

We claim that for every $v \in V_1, \{v\} \cap N^-(v) = \emptyset$. Any vertex in V_1 is of the form $j, 2k-1+j, 4k-2+j$ and $6k-3+j, j = 1,3,\ldots,2k-3$. Since a self-loop vertex is a solution of the congruence equation $\frac{(4k-4)x}{4} \equiv \frac{-i}{4}\left(mod\ \frac{8k-4}{4}\right)$, any vertex in V_1 is not a solution of this congruence equation. Therefore, any vertex in V_1 does not have a self-loop. For any two vertices $u,v \in V_1$, without loss of generality assume that $u > v$. Define $S = \{u,v\}$ and $m = u-v$. Construct the sets D_1 and D_2 as follows:
$D_1 = \bigcup_{i=0}^{k-2}\{4k-2+2i, 6k-3+2i\}$ and $D_2 = \bigcup_{i=0}^{k-3}\{4k+1+2i\}$.

Suppose that $m < 4k-3$. Then by definition of $G_B(n,d)$, there is a vertex w in $V(G_B(8k-4,4k-3))$ such that $w \in N^-(u) \cap N^-(v)$ and $|N^-(S) \cup S| \leq 2(4k-3)+2-1 = 8k-5 \neq 8k-4$. This shows that S is not an absorbant of $G_B(8k-4,4k-3)$.

Now we consider the case $m \geq 4k-3$. From the definition of V_1, let $u = (2k-1)x+y$ and $v = (2k-1)s+t$, for $x,s \in \{0,1,2,3\}$ and $y,t \in \{1,3,\ldots,2k-3\}$. If $m = (2k-1)(x-s)+(y-t)$ is odd, then $x-s$ is odd. It follows that neither both of x and s are odd nor even. If x and s are consecutive integers, then $m = 2k-1+y-t \leq 2k-1+2k-3-1 = 4k-5 < 4k-3$. The only possiblilty of x and s is 3 and 0 respectively and we get $m = 6k-3 \pm 2i, i = 0,1,2,\ldots,k-2$. Therefore $m = 6k-3+2i \in D_1, i = 0,1,2,\ldots,k-2$ and $m = 6k-3-2i \in D_2, i = 1,2,3,\ldots,k-2$.

If m is even, then $x-s$ is even. Also since $x-s$ is even, both of x and s are either odd or even. The number $x-s$ to be even, the only possibilities are either $x = 2$ and $y = 0$ or $x = 3$ and $y = 1$. Thus we get $m = 4k-2 \pm 2i, i = 0,1,2,\ldots,k-2$. Therefore

$m = 4k-2+2i \in D_1, i = 0,1,2,\ldots,k-2$ and $m = 4k-2-2i < 4k-3$, if $i = 1,2,\ldots,k-2$. Therefore if $m \geq 4k-3$, then either $m \in D_1$ or $m \in D_2$.

Case 1. Suppose that $m \in D_1$ is even.

Then choose $m = 4k-2$ and $u = (2k-1)x+y, x = 0,1,2,3$ and $y = 1,3,5,\ldots,2k-3$. Since $m = 4k-2$, v is of the form $v = (2k-1)x+y+4k-2$. We claim that, u is an in-neighbor of v. For that, it is enough to prove that v is a out-neighbor of u. From the definition of $G_B(n,d)$, we have,

$$\begin{aligned} N^+(u) = \{&(4k-3)((2k-1)x+y)(mod\ 8k-4),\\ &((4k-3)((2k-1)x+y)+1)(mod\ 8k-4),\ldots,\\ &((4k-3)((2k-1)x+y)+4k-4)(mod\ 8k-4)\}\\ = \{&(4k-3)(2k-1)x+y(4k-3)(mod\ 8k-4),\\ &((4k-3)(2k-1)x+y(4k-3)+1)(mod\ 8k-4),\ldots,\\ &((4k-3)(2k-1)x+y(4k-3)+4k-4)(mod\ 8k-4)\}. \end{aligned}$$

Since $(2k-1)(4k-3)(mod\ 8k-4) = (k-1)(8k-4) + (2k-1)(mod\ 8k-4) \equiv (2k-1)(mod\ 8k-4)$, we have

$$\begin{aligned} N^+(u) = \{&((2k-1)x+y(4k-3))(mod\ 8k-4),\\ &((2k-1)x+y(4k-3)+1)(mod\ 8k-4),\ldots,\\ &((2k-1)x+y(4k-3)+4k-4)(mod\ 8k-4)\}. \end{aligned}$$

When

$y = 1, ((2k-1)x+(4k-3)y+2)(mod\ 8k-4) \equiv ((2k-1)x+y+4k-2)(mod\ 8k-4)$

$y = 3, ((2k-1)x+(4k-3)y+6)(mod\ 8k-4) \equiv ((2k-1)x+y+4k-2)(mod\ 8k-4)$

....

$y = 2k-3, ((2k-1)x+(4k-3)y+4k-7)(mod\ 8k-4) \equiv ((2k-1)x+y+4k-2)(mod\ 8k-4)$.

From the above argument, v is an out-neighbor of u and $u \in N^-(v)$. Similarly we can prove that $v \in N^-(u)$. For $m > 4k-3$, by the above argument, we can prove that $u \in N^-(v)$ and $v \in N^-(u)$.

This shows that $S \cap N^-(S) = S$ and $|N^-(S) \cup S| = 2(4k-3)+2-2 = 8k-6 \neq 8k-4$. Therefore S is not an absorbant of $G_B(8k-4,4k-3)$.

Case 2. Suppose that $m \in D_1$ is odd.

Then choose $m = 6k-3, u = (2k-1)x+y$ and $v = (2k-1)s+t$, where $s,x = 0,1,2,3$ and $y,t = 1,3,5,\ldots,2k-3$. We claim that $2k-2 \in N^-(u) \cap N^-(v)$. For that, it is enough to prove that $u,v \in N^+(2k-2)$. From the definitions of $G_B(n,d)$, we have

$$\begin{aligned} N^+(2k-2) = \{&(4k-3)(2k-2)(mod\ 8k-4),\\ &((4k-3)(2k-2)+1)(mod\ 8k-4),\ldots,\\ &((4k-3)(2k-2)+4k-4)(mod\ 8k-4)\} \end{aligned}$$

Since $(2k-2)(4k-3) = (k-2)(8k-4)+(6k-2) \equiv (6k-2)$ $(mod\ 8k-4)$, we have

$$\begin{aligned} N^+(2k-2) &= \{6k-2, 6k-1, 6k, \ldots, 2k-2\}\\ &= \{3(2k-1)+1, 3(2k-1)+2, \ldots, 3(2k-1)+4k-1\}\\ &= \{3(2k-1)+1, 3(2k-1)+2, \ldots, 2k-4, 2k-3, 2k-2\}. \end{aligned}$$

By the above argument, if $m \in D_1$ is odd. Then $x = 3$ and $s = 0$. Therefore $2k-2 \in N^-(u)$ and $2k-2 \in N^-(v)$. For $m > 6k-3$, by the above argument we can prove that $2k-2 \in N^-(v)$ and $2k-2 \in N^-(u)$. This shows that $|N^-(S) \cup S| = 2(4k-3)+2-1 = 8k-5 \neq 8k-4$. Therefore S is not an absorbant of $G_B(8k-4,4k-3)$. **Case 3.** $m \in D_2$.

Then by a similar argument as in Case 2, we have $2k-2 \in N^-(u) \cap N^-(v)$. This implies that $|N^-(S) \cup S| = 2(4k-3)+2-2 = 8k-6 \neq 8k-4$. Therefore S is not an absorbant of $G_B(8k-4,4k-3)$.

The above argument forces us to conclude that $\gamma_a(G_B(8k-4,4k-3)) \neq 2$. By Theorem A, we have $\gamma_a(G_B(8k-4,4k-3)) \leq 3$. Therefore $\gamma_a(G_B(8k-4,4k-3)) = 3$.

Now we explain the steps given in the proof of the above Theorem 2.1 by giving an example. Consider the graph $G_B(8k-4,4k-3) = G_B(20,9)$, for $k = 3$.

$$V_1 = \bigcup_{i=0}^{3}\{5i+1, 5i+3\} = \{1,3,6,8,11,13,16,18\}$$

$$D_1 = \bigcup_{i=0}^{1}\{10+2i, 15+2i\} = \{10,15,12,17\}$$

$$D_2 = \bigcup_{i=0}^{0}\{13+2i\} = \{13\}.$$

In Table 2.2, the elements of D_1 are represented in the first row and the common in-neighbors of u and v in $G_B(20,9)$ are given in the third row. The elements of D_2 are mentioned in fourth row along with their common in-neighbors in $G_B(20,9)$. Some pairs in V_1 with $m = u-v$ are not mentioned in the table because in that case $m < 4k-3$.

Theorem 2.2 The absorbant number of $\gamma_a(G_B(n,d))$ and the domination number of $\gamma(G_B(n,d))$ are equal, if any one of the following conditions hold.

(a)$n = d+1$
(b)$n = d$

Table 1: The vertices in V_1 and their in-neighbors of $G_B(20,9)$. The vertices and their in-neighbors in bold face are the vertices of the set V_1.

Vertex	In neighbors								
0	0	2	4	6	8	11	13	15	17
1	**0**	**2**	**4**	**6**	**9**	**11**	**13**	**15**	**17**
2	0	2	4	6	9	11	13	15	18
3	**0**	**2**	**4**	**7**	**9**	**11**	**13**	**15**	**18**
4	0	2	4	7	9	11	13	16	18
5	0	2	5	7	9	11	13	16	18
6	**0**	**2**	**5**	**7**	**9**	**11**	**14**	**16**	**18**
7	0	3	5	7	9	11	14	16	18
8	**0**	**3**	**5**	**7**	**9**	**12**	**14**	**16**	**18**
9	1	3	5	7	9	12	14	16	18
10	1	3	5	7	10	12	14	16	18
11	**1**	**3**	**5**	**7**	**10**	**12**	**14**	**16**	**19**
12	1	3	5	8	10	12	14	16	19
13	**1**	**3**	**5**	**8**	**10**	**12**	**14**	**17**	**19**
14	1	3	6	8	10	12	14	17	19
15	1	3	6	8	10	12	15	17	19
16	**1**	**4**	**6**	**8**	**10**	**12**	**15**	**17**	**19**
17	1	4	6	8	10	13	15	17	19
18	**2**	**4**	**6**	**8**	**10**	**13**	**15**	**17**	**19**
19	2	4	6	8	11	13	15	17	19

Table 2: The sets D_1, D_2, and their common in-neighbors in $G_B(20,9)$

<table>
<tr><td>$d=u-v$</td><td>10</td><td>12</td><td>15</td><td>17</td></tr>
<tr><td rowspan="4">$\{u,v\}$</td><td>11, 1</td><td rowspan="2">13,1</td><td rowspan="2">16, 1</td><td rowspan="4">18,1</td></tr>
<tr><td>13,3</td></tr>
<tr><td>16,6</td><td rowspan="2">18,6</td><td rowspan="2">18,3</td></tr>
<tr><td>18,8</td></tr>
<tr><td>Common in-neighbors of u and v</td><td colspan="2">u is an in-neighbor of v
v is an in-neighbor of u</td><td colspan="2">$2k-2=4$</td></tr>
<tr><td colspan="5">When $\{u,v\}=\{16,3\}$, then $d=13$.
The common in-neighbors of u and v are $2k-2=4$.</td></tr>
</table>

(c)$d=1$
(d)$d|n$ and $n \leq d^2$
(e)$d=2,4,(d+1)|n$ and 2 does not divide n
(f)$n=8k-4$ and $d=4k-3$ for $k \geq 2$
(g)$n=6k$ and $d=2k-1$ for $k \geq 2$

Proof. (a) Suppose that $n=d+1$. By using Theorem G, we have $\gamma_a(G_B(n,d))=\gamma(G_B(n,d))$. (b) Suppose that $n=d$.

For any $v \in V, N^+(v)=N^-(v)=\{0,1,2,\ldots,d-1\}$. Clearly $|N^+(v)|=|N^-(v)|=d=n-1$. Define $S=\{u\}$ for any $u \in V$. Then $N^+[u]=N^-[u]=V$.

(c) Suppose that $d=1$.

Let $V=\{0,1,2,\ldots,n-1\}$ be the vertex set of $G_B(n,d)$. When $d=1$, $N^+[v]=N^-[v]=\{v\}$ for any $v \in V$. Thus the set $S=\{0,1,2,\ldots,n-1\}$ is a dominating set and an absorbant of $G_B(n,d)$.

(d) Suppose that $d|n$ and $n \leq d^2$.

Let S be an absorbant or a dominating set of $G_B(n,d)$. Then by the definition of $G_B(n,d), |S|+d|S| \geq n$. This shows that $\gamma_a(G_B(n,d)) \geq \left\lceil \frac{n}{d+1} \right\rceil$ and $\gamma(G_B(n,d)) \geq \left\lceil \frac{n}{d+1} \right\rceil$. If $d|n$, then

$$V=\bigcup_{i=0}^{d-1}\left\{\tfrac{n}{d}i, \tfrac{n}{d}i+1, \tfrac{n}{d}i+2, \ldots, \tfrac{n}{d}i+\left(\tfrac{n}{d}-1\right)\right\}.$$

Define $S=\{0,d,2d,3d,\ldots,(\frac{n}{d}-1)d\}$. We claim that S is an absorbant. For every vertex $u \in V$, $N^-(u)=\{\frac{u}{d}, \frac{u}{d}+\frac{n}{d}, \frac{u}{d}+2\frac{n}{d}, \ldots, \frac{u}{d}+(d-1)\frac{n}{d}\}$. Let v be any vertex in $V-S$. It is enough to prove that v has an out-neighbor in S. Any vertex v in $V(G_B(n,d))$ is of the form $v=\frac{n}{d}i+r$ for $0 \leq i \leq d-1$ and $0 \leq r \leq \frac{n}{d}-1$. Then there is a vertex $rd \in V(G_B(n,d))$ such that $N^+(\frac{n}{d}i+r) \cap S=\{rd\}$. This shown that S is absorbant of $G_B(n,d)$. Therefore, $\gamma_a(G_B(n,d)) \leq \frac{n}{d}=\left\lceil \frac{n}{d+1} \right\rceil$.

Define $S=\left\{0,1,2,3,\ldots,\left(\frac{n}{d}-1\right)\right\}$. The out-neighbors of any vertex u in S is $N^+(u)=\{ud,ud+1,ud+2,\ldots,ud+d-1\} \bigcup\limits_{u \in S} N^+(u)$ $=\{0,1,2,\ldots,n-1\}=V$. This shows that S is a dominating set of $G_B(n,d)$. Therefore, $\gamma(G_B(n,d)) \leq \frac{n}{d}=\left\lceil \frac{n}{d+1} \right\rceil$.

(e) Suppose that $d=2,4,(d+1)|n$ and 2 does not divide n.

Since $V=\bigcup\limits_{i=0}^{\frac{n}{3}-1}\{3i,3i+1,3i+2\}$. Let $S=\{3i+1|i=0,1,2,\ldots, \frac{n}{3}-1\}$. First we claim that $S \cap N^+[S]=\emptyset$.

Since $N^+(x)=\{6i+2,6i+3\}(mod\ n)$ for any $x \in S$, $6i+2,6i+3 \neq 3j+1\,(j \in \{0,1,\ldots, \frac{n}{3}-1\})$, which implies that $S \cap N^+(x)=\emptyset$. Hence $S \cap N^+(x)=\emptyset$. Further we claim that $N^+(x) \cap N^+(y)=\emptyset$, for any two distinct vertices $x,y \in S$. Suppose not. Then there are two distinct vertices $x,y \in S$, such that $N^+(x) \cap N^+(y) \neq \emptyset$. Then $1 \leq |N^+(x) \cap N^+(y)| \leq d=2$.

Suppose that $1 \leq |N^+(x) \cap N^+(y)| \leq d-1=1$. Then $|N^+(x) \cup N^+(y)|=3$. Note that $N^+(x) \cup N^+(y)$ consists of three consecutive integers. Then there exists a vertex $z \in N^+(x) \cup N^+(y)$ such that $z \in S$. This contradicts the earlier fact.

Suppose that $|N^+(x) \cap N^+(y)|=d=2$. Then $N^+(x)=N^+(y)$. Hence we have $2(y-x) \equiv 0(mod\ n)$. Since 2 does not divide $n, y=x$, which is contrary to $x \neq y$. So our claim follows. This implies that $N^+(S) \cup S=V$, and so S is an dominating set of $G_B(n,d)$.

Suppose that $d=4$. Since $(d+1)|n, V=\bigcup\limits_{i=0}^{\frac{n}{5}-1}\{5i,5i+1,5i+2,5i+3,5i+4\}$. Let $S=\left\{5i+2|i=0,1,2,\ldots,\frac{n}{5}-1\right\}$. By a similar argument, we can prove that S is a dominating set of $G_B(n,d)$. By Theorem B, the desired result follows.

(f) From Theorem 2.1. and Theorem C, the result follows.

3 Conclusion

In this article, we have solved the open problem 4 and have given a partial solution to the problem 3 which are appeared in article Erfang Shan, T.C.E.Cheng, Liying Kang, Absorbant of generalized de Bruijn digraphs, Inform. Process. Lett. 105 (2007) 6-11. One can try to solve the problem 1, problem 2 and problem 5 given in the same article.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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