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Similar Transposed Bases of Polynomials in Clifford Analysis

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In this paper, we define a similar transposed base of special monogenic polynomials. The convergence properties in several domains in the higher dimensional space of that similar transposed base of monogenic polynomials are investigated. Certain inevitable normalizing conditions have been formulated to be undergone by the given base to ensure the existence of its convergence properties.

Keywords: Similar transposed base, effectiveness, Clifford analysis.

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1 Introduction

One of the fundamental problems in the theory of holomorphic function spaces is the existence of bases. Let $f(z)$ be holomorphic in some disc (centered at 0) and let $\{P_n(z) : z \in \mathbb{C}\}_{n\in\mathbb{N}}$ be a polynomial basis. Then there arises the question: Under what condition $f(z)$ may be expressed as a series with respect to the given basis?

Related to this problem is the theory of bases in nuclear Frèchet spaces and the theory of rings of infinite matrices. Thus, the two important problems that arise in the study of function spaces are:

1. Does the space under consideration posses a basis?
2. If this is the case, how can other bases of this space be characterized?

The subject of "bases of polynomials" or in the classical terminology of Whittaker [17], basic sets of polynomials, essentially deals with these two fundamental problems mentioned above, in the case where the function spaces considered admits a sequence of polynomials as a base. So, the polynomial bases problem is mainly devoted to study the representability of regular functions by infinite series in a given sequence of functions (or polynomials). Of course, as the theory of holomorphic functions in the plane allows generalizations to higher dimensions, analogous problems may be considered in the corresponding function spaces called monogenic function spaces. Given a base of special monogenic polynomials [1], it is of interest to derive from it, by some means, a new base; to study the properties of that new base; and to discover how far these properties are related to those of the original base. The present work is a new addition to the derived bases of special monogenic polynomials[1]. The similar[2] transposed bases of a given base are defined and their convergence properties are investigated. The significance of the problem treated in the present work lies in the fact that it considers some kind of product bases, since in general the product of two effective bases need not be effective [5, 12, 13] and also the inverse base of a given effective base is not necessarily effective [3].

More precisely, given the region of effectiveness of a base or of two bases it is required to determine the region of effectiveness of the similar transposed bases constructed by these two bases.

1.1 Notations and preliminaries

One useful approach to generalize complex analysis to higher dimensional spaces is the Cauchy-Riemann approach based on the consideration of functions that are the kernel of the generalized Cauchy-Riemann operator

$$D = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i}, \text{ in } \mathbb{R}^{m+1}.$$

In [8, 9] a theory of monogenic functions has been developed which generalizes in a natural way the theory of holomorphic functions of one complex variable to $(m+1)$-dimensional Euclidean space. The regular functions considered in the present work have values in a real Clifford algebra and are null-solutions of a linear differential operator which linearizes the Laplacian. The real Clifford algebra over $\mathbb{R}$ is defined as $\mathcal{A}_m = \{\sum_{A\subset\{1,\ldots,m\}} a_A e_A : a_A \in \mathbb{R}\}$, where $e_i = e_{\{i\}};\ i = 1,\ldots,m;\ e_0 = e_\phi = 1$ and $e_A = e_{\alpha_1}\ldots e_{\alpha_h},\ A = \{\alpha_1,\ldots,\alpha_h\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_h$.

The product in $\mathcal{A}_m$ is determined by the relations $e_k^2 = -1,\quad k = 1,\ldots,m$ and $e_i e_j + e_j e_i = 0,\ k \neq j\ , k, j = 1,\ldots,m$. The norm of a Clifford number is given by $|a|^2 = \sum_{A\subseteq M} |a_A|^2$, where M stands for $\{1,\ldots,m\}$. Since $\mathcal{A}_m$ is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$–norm $|a|$ and one sees easily that for any $a, b \in \mathcal{A}_m,\quad |ab| \leq 2^{m/2}\,|a|\,|b|$.

The elements $(x_0, \overrightarrow{x}) = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{m+1}$ will be identified with the Clifford numbers $x_0 + \overrightarrow{x} = x_0 + \sum_{j=0}^{m} e_i x_j$. Note that if $x = x_0 + \overrightarrow{x} \in \mathbb{R}^{m+1}$, its conjugate

[1] To avoid lengthy script we shall use in short sm-polynomials (or functions) where (sm)- stands for special monogenic.

[2] It must be noted that similar does not mean linked by an isomorphism, as is sometimes met in the literature.

is $\overline{x} = x_0 - \overrightarrow{x}$. The $\mathcal{A}_m$-valued solutions of $Df = 0$ (resp. $fD = 0$), where D is the generalized Cauchy-Riemann, are called left (resp. right) monogenic functions on an open set Ω. The right $\mathcal{A}_m$-module $\mathcal{A}_m[x]$, defined by $\mathcal{A}_m = \text{span}_{\mathcal{A}_m}\{z_n(x) : n \in \mathbb{N}\}$ is called the space of special monogenic polynomials, if the polynomial $z_n(x)$ is given by (see [1])

$$z_n(x) = \sum_{i+j=m} \frac{\big((m-1)/2\big)_i \big((m+1)/2\big)_j}{i!j!} \, \overline{x}^i x^j,$$

where for $b \in \mathbb{R}^{m+1}$, $(b)_l = b(b+1)\cdots(b+l-1)$, $\overline{x}$ is the conjugate of x, and $\mathbb{R}^{m+1}$ is identified with a subset of $\mathcal{A}_m$. If $P_n(x)$ is a homogeneous special monogenic polynomial of degree n in x then (see [1]) $P_n(x) = z_n(x)\alpha$, where α is some constant in $\mathcal{A}_m$, and

$$\|z_n(x)\|_r = \sup_{|x|=r} |z_n(x)| = \binom{m+n-1}{n} r^n = \frac{(m)_n}{n!} r^n,$$

where $m_n/n! = (m+n-1)/[n!(m-1)!]$.

Definition 1.1 (Special monogenic function)**.** Let Ω be a connected open subset of $\mathbb{R}^{m+1}$ containing 0, then a monogenic function in Ω is said to be *special monogenic* in Ω iff its Taylor series near zero (which is known to exist) has the form $f(x) = \sum_{n=0}^{\infty} z_n(x)c_n,\ c_n \in \mathcal{A}_m$.

A function f is said to be a special monogenic function on $\overline{B}(r)$ if it is special monogenic on some connected open neighborhood Ω_f of $\overline{B}(r)$.

The fundamental references for special monogenic functions are [10] and [16].

1.2 Bases of Clifford polynomials and their convergence properties

The theory of basic set (bases) of polynomials in one complex variable as was introduced by Whittaker [17] has been already generalized to the setting of Clifford analysis (see [1–5]) in the following framework.

Let $SM_{[x]}$ be the vector space of all special monogenic polynomials in the Clifford variable x with Clifford coefficients. By the base (or basic set) we mean a linearly independent spanning set. Let $\{P_n(x)\}$ be a sequence of sm-polynomials in x that forms a base for $SM_{[x]}$, then we have:

1. The set $\{P_n(x)\}$ is linearly independent in the space $SM_{[x]}$.
2. $\text{span}\{P_n(x) : n \in \mathbb{N}\} = SM_{[x]}$.

Since $\{P_n(x)\}$ is a base for $SM_{[x]}$, there exists a row-finite infinite matrix $P = (P_{nk})$ such that

$$P_n(x) = \sum_k z_k(x) P_{nk}, \quad P_{nk} \in \mathcal{A}_m \tag{1.1}$$

and consequently we have

$$z_n(x) = \sum_k P_k(x)\overline{P}_{nk}, \quad \overline{P}_{nk} \in \mathcal{A}_m. \tag{1.2}$$

In fact one can see in view of [1] that $\{P_n(x)\}$ is a base of $SM_{[x]}$ iff $\overline{P}P = I$, where I is the unit matrix, P is called the matrix of coefficients in $P_n(x)$ and $\overline{P} = (\overline{P}_{nk})$ is the matrix of operators. A base $\{P_n(x)\}$ of sm-polynomials is said to be *simple* if for all n, $P_n(x)$ is of degree n, and a simple base is called *monic* if for all n the coefficients of $z_n(x)$ in $P_n(x)$ is unity. The base $\{\overline{P}_n(x)\}$ is called the *inverse base* of $\{P_n(x)\}$, if its matrix of coefficients is the matrix $\overline{P}$.

1.3 Effectiveness property of the base

The base is said to be effective in the monogenic function space $U_n(\Omega_1)$ iff each function $f \in U(\Omega_1)$ admits a series expansion in terms of the elements of the base $\{P_n(x)\}$. This means that, given a sm-function $f(x) = \sum_{n=0}^{\infty} z_n(x)c_n$ (near 0), $c_n \in \mathcal{A}_m$, there is formally an associated basic series $\sum_{k=0}^{\infty} P_k(x)\left(\sum_{n=0}^{\infty} \overline{P}_{nk}c_n\right)$. When this associated basic series converges normally to$f(x)$ in some domain it is said to represent $f(x)$ in that domain, in other words, the base $\{P_n(x)\}$ will be effective in that domain. The convergence properties of bases are classified according to the classes of functions represented by their associated basic series and also to the domain in which they are represented.

1.4 Characterization of effectiveness property

We write

$$\Omega_n(r) = \sum_k \sup_{|x|=r} \left|P_k(x)\overline{P}_{nk}\right|. \tag{1.3}$$

The convergence properties of a base $\{P_n(x)\}$ depend entirely on the value of the expression

$$\Omega(r) = \limsup_{n\to\infty} \left[\Omega_n(r)\right]^{1/n}, \tag{1.4}$$

where $\Omega_n(r)$ and $\Omega(r)$ are called the Cannon sum and the Cannon function respectively. The necessary and sufficient condition for a base $\{P_n(x)\}$ to be effective in the closed ball $\overline{B}(r)$ is that [1, Theorem 1] $\Omega(r) = r$. For effectiveness of a base $\{P_n(x)\}$ of sm-polynomials in the open ball $B(r)$, $D(r^+)$ for all entire sm-functions and at the origin, the respective necessary and sufficient conditions are (see [7]).

1. $\Omega(r) < R,$ for all $r < R,$
2. $\Omega(r^+) = r,$
3. $\Omega(r) < \infty$ for all $r < \infty,$
4. $\Omega(0^+) = 0,$ where $\Omega(\rho^+) = \lim_{r\downarrow\rho} \Omega(r).$

2 Similar Transposed Bases of SM-Polynomials

In Newns' paper [15, p.455], the definition of transposed base of complex polynomials was introduced. This definition can be adapted to the case of sm-polynomials as follows.

Definition 2.1 (The transposed base of sm-polynomials)**.** The transpose of a base $\{P_n(x)\}$ of sm-polynomials is the base whose Clifford matrix of coefficients is the transpose of that of the given base and we will denote to that transposed base by $\{\widetilde{P}_n(x)\}$.

Remark 2.1. When the given base is simple, absolutely monic and effective in the unit ball it can be shown that the transposed base is also effective in the same ball. (The proof is very similar to the complex case see [14]).

According to the definition of the inverse base (see [2]) we may have.

Definition 2.2 (The inverse transposed base)**.** The inverse transposed base denoted by $\{\widehat{P}_n(x)\}$ is defined to be the inverse of the base $\{\widetilde{P}_n(x)\}$, i.e. $\{\widehat{P}_n(x)\} = \{\overline{\widetilde{P}}_n(x)\}$ where $\widehat{P}_n(x) = \sum_i z_i(x)\overline{P}_{in}$, and $\left(\overline{P}_{in}\right)$ is the inverse matrix of the transposed matrix (P_{in}).

Also, according to the definition of the product base and the similar base of polynomials in Clifford setting [5, 6], one can define the similar transposed base of special monogenic polynomials as follows.

Definition 2.3 (Similar transposed base)**.** Let $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ be two transposed bases of sm-polynomials. The base $\{\widetilde{U}_n(x)\}$ of sm-polynomials defined by

$$\{\widetilde{U}_n(x)\} = \{\widehat{P}_n(x)\}\{\widetilde{Q}_n(x)\}\{\widetilde{P}_n(x)\} \tag{2.1}$$

where $\{\widehat{P}_n(x)\}$ is the inverse transposed base, is called similar transposed base to the transposed base $\{\widetilde{Q}_n(x)\}$ with respect to the transposed base $\{\widetilde{P}_n(x)\}$. The two bases $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ are called the constituent bases (or factors) of $\{\widetilde{U}_n(x)\}$.

In fact $\{\widetilde{U}_n(x)\}$ forms a base of $SM_{[x]}$ (or in the terminology of Whittaker [17], is a basic set of sm-polynomials). This is easy to verify as we shall see below.

Let $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{U}$ be the matrices of coefficients of the transposed bases respectively. Write

$$\begin{aligned}
\widetilde{P}_n(x) &= \sum_i z_i(x)\widetilde{P}_{ni},\\
\widetilde{Q}_n(x) &= \sum_j z_j(x)\widetilde{Q}_{nj},\\
\widetilde{U}_n(x) &= \sum_k z_k(x)\widetilde{U}_{nk},\\
\widehat{P}_n(x) &= \overline{\widetilde{P}}_n(x) = \sum_h z_h(x)\overline{\widetilde{P}}_{nh} = \sum_h z_h(x)\widehat{P}_{nh},
\end{aligned}$$

where $(\widehat{P}_{nh})$ is the inverse matrix of the transposed matrix $(\widetilde{P}_{nh})$ or is called the inverse matrix of operators of $\{\widetilde{P}_n(x)\}$. Then (2.1) gives $\widetilde{U}_{nk} = \sum_i \sum_j \widehat{P}_{ik}\widetilde{Q}_{ji}\widetilde{P}_{nj}$, and so $\widetilde{U} = \widehat{P}\widetilde{Q}\widetilde{P}$.

If V is a matrix given by $V = \widehat{P}\widehat{Q}\widetilde{P}$ then

$$\begin{aligned}
\widetilde{U}V &= \widehat{P}\widetilde{Q}\widetilde{P} \cdot \widehat{P}\widehat{Q}\widetilde{P} = I, \\
V\widetilde{U} &= \widehat{P}\widehat{Q}\widetilde{P} \cdot \widehat{P}\widetilde{Q}\widetilde{P} = I,
\end{aligned}$$

where I is the unit infinite matrix, i.e. the matrix $\widetilde{U}$ of coefficients of the base $\{\widetilde{U}_n(x)\}$ has a unique inverse $V = \widehat{U} = \overline{\widetilde{U}}$, called the matrix of operators. This gives that the set $\{\widetilde{U}_n(x)\}$ is a basic set, i.e. it is a base for the space of sm-polynomials $SM_{[x]}$ (see [1]).

3 Effectiveness Property for the Similar Transposed Base When Its Factors Are Simple Monic

It is clear that if $\{P_n(x)\}$ is a simple monic base then its transposed base $\{\widetilde{P}_n(x)\}$ is also simple monic. Therefore we have

Theorem 3.1. *Let $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ be simple monic bases of sm-polynomials effective in the closed ball $\overline{B}(r)$. Then the similar transposed base $\{\widetilde{U}_n(x)\} = \{\widehat{P}_n(x)\}\{\widetilde{Q}_n(x)\}\{\widetilde{P}_n(x)\}$ is effective in the same domain $\overline{B}(r)$, if and only if, the base $\{\widetilde{Q}_n(x)\}$ is effective in $\overline{B}(r)$.*

Proof. Let the two bases $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ be simple monic bases and each of them be effective in the closed ball $\overline{B}(r)$. Then due to [3, 5] the product base $\{\widetilde{Q}_n(x)\}\{\widetilde{P}_n(x)\}$ and the inverse base $\{\overline{\widetilde{P}}_n(x)\}$ are simple monic and effective in $\overline{B}(r)$. Hence the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective in $\overline{B}(r)$. On the other hand, let $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{U}_n(x)\}$ be effective in $\overline{B}(r)$. Since $\{\widetilde{Q}_n(x)\} = \{\widetilde{P}_n(x)\}\{\widetilde{U}_n(x)\}\{\widehat{P}_n(x)\}$ then, by the same way as above, it follows that $\{\widetilde{Q}_n(x)\}$ is effective in $\overline{B}(r)$ and the theorem follows. $\square$

4 Effectiveness of Nonmonic Similar Transposed Base in the Closed Ball

The effectiveness of similar transposed base $\{\widetilde{U}_n(x)\}$ of sm-polynomials in closed balls will be obtained whenever the two bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ are effective in $B(R)$ and satisfy the analogue conditions of Newns' [15, Theorem 19.2, p.460] in the form

$$\begin{cases} \mu_P(r) < R, & \mu_Q(r) < R, \quad r < R \\ \nu_P(R^-) \geq R, & \nu_Q(R^-) \geq R, \end{cases} \tag{4.1}$$

where

$$\mu_P(r) = \limsup_{n\to\infty} \{A_n(r)\}^{1/n}, \qquad \mu_Q(r) = \limsup_{n\to\infty} \{B_n(r)\}^{1/n}$$
$$\nu_P(R^-) = \liminf_{n\to\infty} \{A_n(R)\}^{1/n}, \qquad \nu_Q(R^-) = \liminf_{n\to\infty} \{B_n(r)\}^{1/n}$$
$$A_n(r) = \sup_{|x|=r} |P_n(x)|, \qquad B_n(r) = \sup_{|x|=r} |Q_n(x)|.$$

Theorem 4.1. *Let $\{P_n(x)\}$and $\{Q_n(x)\}$ be two bases of sm-polynomials, each of them is effective in the open ball $B(R)$ and satisfies conditions (3.1). Then the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective in the closed ball $\overline{B}(1/R)$.*

Proof. Let $\{\overline{P}_n(x)\}$ and $\{\overline{Q}_n(x)\}$ be the inverse bases of the bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ respectively. Since each of the bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ is effective in $B(R)$ and satisfies conditions (4.1), then each of the inverse bases $\{\overline{P}_n(x)\}$ and $\{\overline{Q}_n(x)\}$ is effective in $B(R)$ and satisfies the conditions (see [7]):

$$\begin{cases} \overline{\mu}_P(r) < R, & \overline{\mu}_Q(r) < R, \quad r < R \\ \overline{\nu}_P(R^-) \geq R, & \overline{\nu}_Q(R^-) \geq R. \end{cases} \tag{4.2}$$

For our purpose in the proof, let r_m be chosen such that $r < r_m < R$, $m = 1, 2, \ldots, 11$ and

$$\begin{cases} B_n(r_7) < k_1 r_8^n \\ \overline{A}_n(r_9) < k_1 r_{10}^n, \qquad n \geq 0, \end{cases} \tag{4.3}$$

where the constant k_1 denotes positive finite numbers independent of the index n and does not retain the same values at different occurrences.

Relying on Cauchy's inequality [2] for $\overline{P}_n(x)$ and $Q_n(x)$ and using (4.3) one can get

$$\begin{aligned} C_n\left(\frac{1}{r_{11}}\right) &= \sup_{|x|=1/r_{11}} \left|\widetilde{U}_n(x)\right| \leq \sum_k \sup_{|x|=1/r_{11}} \left|z_k(x)\widetilde{U}_{nk}\right| \\ &< 2^{3m/2} \sum_{i,j,k} \frac{(m)_k}{k!} \left(\frac{1}{r_{11}}\right)^k \left|\overline{P}_{kj}\right| |Q_{ji}| \, |P_{in}| \\ &< k 2^{3m/2} \sum_{i,j,k} \frac{(m)_k}{k!} \sqrt{\frac{k!j!n!}{(m)_k (m)_j (m)_n}} \left(\frac{r_{10}}{r_{11}}\right)^k \left(\frac{r_8}{r_9}\right)^j \frac{1}{r_7^i} \frac{\widetilde{A}_n(1/r_7)}{(1/r_7)^i} \\ &< k 2^{3m/2} \cdot \widetilde{A}_n\left(\frac{1}{r_7}\right) \end{aligned} \tag{4.4}$$

Now, since the bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ and their inverses satisfy (4.1) and (4.2), respectively, we have

$$\begin{cases} A_n(r_1) < k_1 r_2^n \\ B_n(r_3) < k_1 r_4^n. \end{cases} \tag{4.5}$$

As the base $\{P_n(x)\}$ is effective in $B(R)$, then due to the results given in [7], its inverse is also effective in the same ball, from which we obtain

$$\overline{\Omega}_j^{(P)}(r_5) < k_1 r_6^j, \quad j \geq 0. \tag{4.6}$$

Moreover, from (4.6) and applying Cauchy's inequality (see [2]) with respect to the relative bases, we have the Cannon sum of the transposed base $\{\widetilde{P}_n(x)\}$ as follows:

$$\begin{aligned}\widetilde{\Omega}_n^{(P)}\left(\frac{1}{r_7}\right) &= \sum_k \sup_{|x|=1/r_7} \left|\widetilde{P}_k(x) \cdot \overline{\widetilde{P}}_{nk}\right| \leq 2^{m/2} \sum_k \sup_{|x|=1/r_7} \left|\widetilde{P}_k(x)\right| \left|\overline{P}_{kn}\right| \\ &\leq 2^m \sum_{k,j} \frac{(m)_j}{j!} \left(\frac{1}{r_7}\right)^j |P_{jk}| \sqrt{\frac{k!}{(m)_k}} \frac{\overline{A}_k(r_5)}{r_5^n}.\end{aligned}$$

Using the relation $|P_{jk}|\, \overline{A}_k(r_5) \leq 2^{-m/2} \overline{\Omega}_j(r_5)$ (see [6]) we get

$$\begin{aligned}\Omega_n^{(P)}\left(\frac{1}{r_7}\right) &\leq 2^{m/2} \frac{1}{r_5^n} \sum_{k,j} \frac{(m)_j}{j!} \sqrt{\frac{k!}{(m)_k}} \left(\frac{1}{r_7}\right)^j \cdot \overline{\Omega}_j(r_5) \\ &\leq 2^{m/2} \frac{k_1}{r_5^n} \sum_{k,j} \frac{(m)_j}{j!} \left(\frac{r_6}{r_7}\right)^j \\ &\leq 2^{m/2} k \frac{1}{(r_5)^n}.\end{aligned} \tag{4.7}$$

Combining (4.4), (4.5) with (4.7) and using Cauchy's inequality one obtains the Cannon sum of the similar transposed base $\{\widetilde{U}_n(x)\}$ in the form

$$\Omega_n^{(\widetilde{U})}\left(\frac{1}{r_{11}}\right) \leq 2^{m/2} \sum_k \sup_{|x|=1/r_{11}} \left|\widetilde{U}_k(x)\right| \left|\overline{\widetilde{U}}_{nk}\right|.$$

From $\widetilde{U}_{nk} = \sum_{i,j} \widehat{P}_{ik} \widetilde{Q}_{ji} \widetilde{P}_{nj}$, and the definition of the inverse similar matrix we shall have $\overline{\widetilde{U}}_{nk} = \sum_{i,j} \widehat{P}_{jk} \overline{\widetilde{Q}}_{ij} \widetilde{P}_{ni}$, and so

$$\begin{aligned}\Omega_n^{(\widetilde{U})}\left(\frac{1}{r_{11}}\right) &\leq 2^{3m/2} \sum_{i,j,k} C_k\left(\frac{1}{r_{11}}\right) \left|\widehat{P}_{jk}\right| \left|\widehat{Q}_{ij}\right| \left|\widetilde{P}_{ni}\right| \\ &= 2^{3m/2} \sum_{i,j,k} C_k\left(\frac{1}{r_{11}}\right) \left|\overline{P}_{kj}\right| \left|\overline{Q}_{ji}\right| |P_{in}| \\ &\leq 2^{3m} k \sum_{i,j,k} \widetilde{\Omega}_j\left(\frac{1}{r_7}\right) \sqrt{\frac{j!}{(m)_j}} \frac{\overline{B}_j(r_3)}{r_3^i} \sqrt{\frac{i!}{(m)_i}} \frac{A_i(r_1)}{r_1^n} \\ &\leq 2^{3m} k \frac{1}{r_1^n} \sum_{i,j,k} \left(\frac{r_4}{r_5}\right)^j \left(\frac{r_2}{r_3}\right)^i \leq k \left(\frac{1}{r_1}\right)^n.\end{aligned}$$

From the choice of the numbers $(r)_1^{11}$ when $r_1 \downarrow R$ and $r_{11} \downarrow R$ we get

$$\Omega_n^{(\widetilde{U})}\left(\frac{1}{R}\right) \leq k\left(\frac{1}{R}\right)^n.$$

From which the Cannon function yields $\widetilde{\Omega}^{(U)}(1/R) \leq 1/R$, and thus the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective in $\overline{B}(1/R)$, as required. Therefore the theorem is completely established. □

5 Effectiveness of the Similar Transposed Base in an Open Ball

To justify the effectiveness property for the similar transposed base $\{\widetilde{U}_n(x)\}$ of sm-polynomials, let its factors $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ be effective in $B(r)$ and satisfy the following conditions:

$$\begin{cases} \mu_P(r^+) < r, & \mu_Q(r^+) < r, \\ \nu_P(R) > r, & \nu_Q(R) > r, \qquad\qquad \forall R > r. \end{cases} \tag{5.1}$$

Then we have the following result:

Theorem 5.1. *Let $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_n(x)\}$ be two bases of sm-polynomials, each of which is effective in $B(r)$ and satisfy conditions (5.1). Then the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective in $B(1/r)$.*

Proof. Let $\{\overline{P}_n(x)\}$ and $\{\overline{Q}_n(x)\}$ be the inverse bases of the bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ respectively, satisfying the following conditions:

$$\begin{cases} \overline{\mu}_P(r^+) \leq r, & \overline{\mu}_Q(r^+) \leq r, \\ \overline{\nu}_P(R) \geq r, & \overline{\nu}_Q(R) \geq r, \qquad\qquad \forall R > r. \end{cases} \tag{5.2}$$

If the bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ are effective in $\overline{B}(r)$ and satisfy (5.1), then we have.

$$\begin{cases} A_n(r_6) < k_1 r_7^n \\ B_n(r_8) < k_1 r_9^n \\ \overline{A}_n(r_{10}) < k_1 r_{11}, \qquad\qquad n \geq 0. \end{cases} \tag{5.3}$$

Using (5.3) and Cauchy's inequality, we obtain

$$\begin{aligned} C_n\left(\frac{1}{r_{12}}\right) &= \sup_{|x|=1/r_{12}} \left|\widetilde{U}_n(x)\right| \\ &\leq 2^{3m/2} \sum_{i,j,k} \sup_{|x|=1/r_{12}} |z_k(x)| \left|\widehat{P}_{jk}\right| \left|\widetilde{Q}_{ij}\right| \left|\widetilde{P}_{ni}\right| \\ &< 2^{3m/2} k \sum_{i,j,k} \frac{(m)_k}{k!} \sqrt{\frac{k!j!i!}{(m)_k(m)_j(m)_i}} \left(\frac{r_{11}}{r_{12}}\right)^k \left(\frac{r_9}{r_{10}}\right)^j \left(\frac{r_7}{r_8}\right)^i \frac{1}{r_6^n} \end{aligned} \tag{5.4}$$

$$< k^3 \frac{1}{r_6^n}.$$

Since the two bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ satisfy (5.1) respectively and the two inverse bases $\{\overline{P}_n(x)\}$ and $\{\overline{Q}_n(x)\}$ satisfy (5.2) respectively, then

$$\begin{cases} A_n(r) < k_1 r_1^n \\ \overline{A}_n(r_4) < k_1 r_5^n \\ B_n(r_2) < k_1 r_3^n, \qquad n \geq 0. \end{cases} \tag{5.5}$$

Applying Cauchy's inequality to the polynomials $P_i(x)$, $\overline{Q}_j(x)$ and $\overline{P}_k(x)$, in view of (5.4) and (5.5), we obtain

$$\begin{aligned} \Omega_n^{(\widetilde{U})}\left(\frac{1}{r_{12}}\right) &\leq 2^{3m/2} \sum_{i,j,k} C_k\left(\frac{1}{r_{12}}\right) \left|\widehat{P}_{jk}\right| \left|\widehat{Q}_{ij}\right| \left|\widetilde{P}_{ni}\right| \\ &< 2^{3m/2} k \sum_{i,j,k} \sqrt{\frac{k!j!i!}{(m)_k (m)_j (m)i}} \cdot \frac{1}{r_6^k} \frac{\overline{A}_k(r_4)}{r_4^j} \cdot \frac{\overline{B}_j(r_2)}{r_2^i} \cdot \frac{A_i(r)}{r^n} \\ &< 2^{3m/2} k \left(\frac{1}{r}\right)^n. \end{aligned}$$

As r_{12} can be chosen arbitrarily close to R, we conclude that

$$\Omega_n^{(\widetilde{U})}\left(\frac{1}{R}\right) < 2^{3m/2} k \left(\frac{1}{r}\right)^n.$$

Taking the n-th root and making n tends to infinity we obtain the Cannon function for the similar transposed base $\{\widetilde{U}_n(x)\}$ such that

$$\Omega^{(\widetilde{U})}\left(\frac{1}{R}\right) < \frac{1}{r}, \quad \forall r < R$$

which shows that the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective in $B(1/r)$ as required. □

6 Effectiveness of Similar Transposed Bases for All Entire SM-Functions

In order for the similar transposed base $\{\widetilde{U}_n(x)\}$ to be effective for all entire sm-functions, some additional conditions should be imposed on the associated infinite matrices, related essentially to the so-called algebraicness of these matrices. Since the notion of algebraic property [11] (which always offers some new results) has been extended to Clifford setting [4] one can deduce the effectiveness of the similar transposed base $\{\widetilde{U}_n(x)\}$ for all entire sm-functions and then effectiveness at the origin.

For this purpose, we first recall the following definitions in Clifford setting (see [4]).

Definition 6.1. A base of sm-polynomials $\{P_n(x)\}$ is said to be *algebraic* [4] when its Clifford matrix of coefficients P satisfies an algebraic matrix equation of finite degree, i.e. P is an algebraic matrix .

Definition 6.2. When N is the degree of the equation of least degree satisfied by P, the base is said to be *algebraic of degree* N.

Proposition 6.1. *In analogy with the complex case, if $\{P_n(x)\}$ is algebraic base of sm-polynomials, then the following holds (see [4]).*

$$\overline{G}_{nk} = \delta_{nk}\alpha_0 + \sum_{t=1}^{s}\alpha_t \left(G_{nk}\right)^{(t)}$$

where α_t are constants and $\{G_n(x)\}^{(t)}$, $1 \le t \le s < \infty$, is the (t)-th power of the base $\{G_n(x)\}$.

Therefore, if $\{P_n(x)\}$ and $\{Q_n(x)\}$ are two algebraic bases of sm-polynomials, then

$$\begin{cases} \overline{P}_{nk} = \delta_{nk}\alpha_0 + \sum\limits_{t=1}^{s_1}\alpha_t \left(P_{nk}\right)^{(t)}, \\ \overline{Q}_{nk} = \delta_{nk}\grave{\alpha}_0 + \sum\limits_{t=1}^{s_2}\grave{\alpha}_t \left(Q_{nk}\right)^{(t)}. \end{cases} \tag{6.1}$$

Now, to deduce the effectiveness of the base $\{\widetilde{U}_n(x)\}$, let us consider the following normalizing conditions

$$\mu_P(0^+) = 0, \qquad \mu_Q(0^+) = 0 \tag{6.2}$$

where

$$\mu_P(r) = \limsup_{n\to\infty}\{A_n(r)\}^{1/n}, \qquad \mu_Q(r) = \limsup_{n\to\infty}\{B_n(r)\}^{1/n},$$
$$A_n(r) = \sup_{|x|=r}|P_n(x)|, \qquad B_n(r) = \sup_{|x|=r}|Q_n(x)|.$$

Theorem 6.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two algebraic bases of sm-polynomials satisfying conditions (6.2). Then the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective for all entire sm-functions.*

Proof. Choose numbers $r_i > 0$, $i = 1, 2, \ldots, 12$, such that $r < r_i < r_{i+1}$. By (6.2) one obtains the inequalities

$$\begin{cases} A_n(r_6) < k_1 r_7^n \\ \overline{B}_n(r_8) < k_1 r_9^n, \qquad n \ge 0. \end{cases} \tag{6.3}$$

Let $\{P_n(x)\}^{(M)}$ be the M-power of the base $\{P_n(x)\}$. Set

$$[A_n(r)]^{(M)} = \sup_{|x|=r}|P_n(x)|^{(M)} \quad \text{and} \quad \mu_P^{(M)}(r) = \limsup_{n\to\infty}\left\{[A_n(r)]^{(M)}\right\}^{1/n}.$$

Then, in analogy with the complex case [15, lemma 2] it can easily be seen that if $\mu_P(0^+) = 0$, then $\mu_P^{(M)}(0^+) = 0,\ M \geq 1$. It follows that

$$[A_n(r)]^{(M)} < k_1 r_{11}^n, \quad n \geq 0,\ 1 \leq M \leq s_1. \tag{6.4}$$

Using (6.4), (6.1) and applying Cauchy's inequality we obtain

$$\left|\overline{P}_{nk}\right| < k_1\beta_1(s_1+1)\Big(\frac{r_{11}^n}{r_{10}^k}\Big), \quad \text{for } n = k,\ \beta_1 = \sup_{1\leq t\leq s_1} |\alpha_t|\,. \tag{6.5}$$

Combining (6.3) and (6.5), and relying on Cauchy's inequality we get

$$\begin{aligned} C_n\Big(\frac{1}{r_{12}}\Big) &= \sup_{|x|=1/r_{12}} \left|\widetilde{U}_n(x)\right| \leq 2^{3m/2} \sum_{i,j,k} \sup_{|x|=1/r_{12}} |z_k(x)| \left|\widehat{P}_{jk}\right| \left|\widetilde{Q}_{ij}\right| \left|\widetilde{P}_{ni}\right| \\ &\leq 2^{3m/2} k_1\beta_1(s_1+1) \sum_{i,j,k} \frac{(m)_k}{k!} \left(\frac{1}{r_{12}}\right)^k \frac{r_{11}^k}{r_{10}^j} |Q_{ji}|\,|P_{in}| \\ &< 2^{3m/2} k_1^3\beta_1(s_1+1) \frac{1}{r_6^n} \sum_{i,j,k} \sqrt{\frac{j!i!}{(m)_j(m)i}} \frac{(m)_k}{k!} \left(\frac{1}{r_6}\right)^n \\ &< k\frac{1}{r_6^n}. \end{aligned} \tag{6.6}$$

Again, making use of (6.1) and (6.2) and in view of (5.4) and applying Cauchy's inequality we obtain

$$\left\{ \begin{array}{ll} \left|\overline{P}_{nk}\right| < k_1\beta_1(s_1+1)\Big(\frac{r_5^n}{r_4^k}\Big) & \\ \left|\overline{Q}_{nk}\right| < k_1\beta_2(s_2+1)\Big(\frac{r_3^n}{r_2^k}\Big), & n = k \\ A_n(r) < k_1 r_1^n, \quad \&n \geq 0, & \end{array} \right. \tag{6.7}$$

where

$$\beta_2 = \sup_{1\leq t\leq s_2} |\grave{\alpha}_t|\,.$$

Substituting (6.6) and (6.7) into the Cannon sum $\Omega_n^{(\widetilde{U})}(1/r_{12})$ of the similar transposed base $\{\widetilde{U}_n(x)\}$ then

$$\begin{aligned} \Omega_n^{(\widetilde{U})}\Big(\frac{1}{r_{12}}\Big) &\leq 2^{3m/2} \sum_{i,j,k} C_k\Big(\frac{1}{r_{12}}\Big) \left|\widehat{P}_{jk}\right| \left|\widehat{Q}_{ij}\right| \left|\widetilde{P}_{ni}\right| \\ &< C^* 2^{3m/2} k_1^2\beta_1\beta_2(s_1+1)(s_2+1) \sum_{i,j,k} \left(\frac{r_5}{r_6}\right)^k \left(\frac{r_3}{r_4}\right)^j \frac{1}{r_2^i} \sqrt{\frac{i!}{(m)_i}} \frac{r_1^i}{r^n} \\ &< C^{**}\frac{1}{r^n}, \qquad C^{**} = C^* 2^{3m/2} k_1\beta_1\beta_2(s_1+1)(s_2+1)\sqrt{\frac{n!}{(m)_n}}. \end{aligned}$$

Taking the n-th root and allowing n to tend to infinity we obtain the Cannon function $\Omega^{(\widetilde{U})}$ such that

$$\Omega^{(\widetilde{U})}\Big(\frac{1}{r_{12}}\Big) \leq \frac{1}{r}.$$

Since r_{12} can be arbitrarily chosen near to r we conclude that

$$\Omega^{(\widetilde{U})}\Big(\frac{1}{r}\Big) \leq \frac{1}{r} < \infty, \qquad \forall r \geq 0,$$

from which we can deduce that the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective for all entire sm-functions, as required and the theorem is therefore established. □

7 Effectiveness of the Similar Transposed Base $\{\widetilde{U}_n(x)\}$ at the Origin

The effectiveness property at the origin of the similar transposed base $\{\widetilde{U}_n(x)\}$ can easily be deduced by modifying the normalizing conditions considered in the previous section to be

$$\mu_P(r) < \infty, \;\; \mu_Q(r) < \infty \quad \text{for } r \to \infty. \tag{7.1}$$

Proposition 7.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two algebraic bases of sm-polynomials satisfying the conditions (7.1), respectively. Then the similar transposed base $\{\widetilde{U}_n(x)\}$ is effective at the origin.*

Proof. The same arguments as in the proof of the theorem (5.1) in the previous section can be applied to derive the result. Then the proposition follows. □

8 Order of the Similar Transposed Base of SM-Polynomials

We consider in what follows, the representation of classes of entire functions. This representation is governed by order of the considered base. So, we estimate the order $\rho_{\widetilde{U}}$ of the similar transposed base in relation of the orders ρ_P and ρ_Q of the respective simple monic bases $\{P_n(x)\}$ and $\{Q_n(x)\}$. According to the definition of the order of the Cannon base, given in [2], we have

$$\rho_i = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\log \Omega_n^{(i)}(r)}{n \log n}, \;\; (i = P \text{ or } Q), \tag{8.1}$$

where $\Omega_n^{(P)}(r)$ and $\Omega_n^{(Q)}(r)$ stand as usual for the Cannon sums of $\{P_n(x)\}$ and $\{Q_n(x)\}$ respectively.

The following theorem relates the order of the similar transposed base with those of its constituent bases.

Theorem 8.1. *Let $\{P_n(x)\}$ and $\{Q_n(x)\}$ be two simple monic bases of sm-polynomials of respective orders ρ_P and ρ_Q. Then the similar transposed base $\{\widetilde{U}_n(x)\}$ will be of order $\rho_{\widetilde{U}}$ given by*

$$\rho_{\widetilde{U}} \leq 4\rho_P + 2\rho_Q. \tag{8.2}$$

Proof. Let σ_1, σ_2 be positive finite numbers where $\sigma_1 > \rho_P,\ \ \sigma_2 > \rho_Q$, then (8.1) implies that

$$\Omega_n^{(P)}(r) < k_1 n^{n\sigma_1}, \quad \Omega_n^{(Q)}(r) < k_1 n^{n\sigma_2}, \quad n \geq 1. \tag{8.3}$$

Since the orders ρ_P and ρ_Q of the two bases $\{P_n(x)\}$ and $\{Q_n(x)\}$ are given respectively by (8.1) and r is assuemed to be large, for $r > 1$ and applying Cauchy's inequality for $P_n(x)$ and $Q_n(x)$, it follows in view of (8.3) that

$$\begin{cases} 1 < \dfrac{(m)_n}{n!} r^n < A_n(r) < \Omega_n^{(P)}(r) < k_1 n^{n\sigma_1} \\[2em] 1 < \dfrac{(m)_n}{n!} r^n < B_n(r) < \Omega_n^{(Q)}(r) < k_1 n^{n\sigma_2}, \quad n \geq 1, \end{cases} \tag{8.4}$$

where $A_n(r)$ and $B_n(r)$ stand, as usual, for the supremum of the $P_n(x)$ and $Q_n(x)$ respectively.

Making use of (8.4) and Cauchy's inequality we get

$$C_n\left(\frac{1}{r}\right) = \sup_{|x|=1/r} \left|\widetilde{U}_n(x)\right| \leq 2^{3m/2} \sum_{i,j,k} \frac{(m)_k}{k!} \left(\frac{1}{r}\right)^k \left|\overline{P}_{kj}\right| |Q_{ji}| \, |P_{in}| \, .$$

Since the base $\{P_n(x)\}$ is monic, i.e.

$$P_n(x) = z_n(x) + \sum_{k=0}^{n-1} z_k(x) P_{nk},$$

the Cannon sum

$$\Omega_n^{(P)}(r) \geq 2^{m/2} \frac{(m)_k}{k!} r^k \left|\overline{P}_{nk}\right| .$$

Thus

$$\begin{aligned} C_n\left(\frac{1}{r}\right) &\leq 2^{3m/2} \sum_{i,j,k} \frac{(m)_k}{k!} \left(\frac{1}{r}\right)^k 2^{-\frac{m}{2}} \frac{j!}{(m)_j} \frac{\Omega_k^{(P)}(r)}{r^j} |Q_{ji}| \, |P_{in}| && (8.5) \\ &< 2^m k_1^5 \sum_{i,j,k} n^{n\sigma_1} n^{n\sigma_2} n^{n\sigma_1} \left(\frac{1}{r}\right)^n \\ &= k \left(\frac{1}{r}\right)^n n^{n(2\sigma_1+\sigma_2)} (n+1)^3 && (8.6) \\ &< k \, (n+1)^3 n^{n(2\sigma_1+\sigma_2)}. \end{aligned}$$

Introducing (8.4) and (8.5) in the Cannon sum $\Omega^{(\widetilde{U})}(1/r)$ of the similar transposed base $\left\{\widetilde{U}_n(x)\right\}$ we get

$$\begin{aligned}
\Omega_n^{(\widetilde{U})}\left(\frac{1}{r}\right) &\leq 2^{m/2}\sum_{i,j,k} C_k\left(\frac{1}{r}\right)\left|\widehat{P}_{jk}\right|\left|\widehat{Q}_{ij}\right|\left|\widetilde{P}_{ni}\right| \\
&\leq 2^{m/2}\sum_{i,j,k} C_k\left(\frac{1}{r}\right)\left|\overline{P}_{kj}\right|\left|\overline{Q}_{ji}\right|\left|P_{in}\right| \\
&\leq 2^{m/2}\sum_{i,j,k} C_k\left(\frac{1}{r}\right)\frac{j!}{(m)_j}\frac{\Omega_k^{(P)}}{r^j}\frac{i!}{(m)i}\frac{\Omega_j^{(Q)}}{r^i}\sqrt{\frac{i!}{(m)i}}\frac{A_i(r)}{r^n} \\
&\leq 2^{m/2}k\sum_{i,j,k} C_k\left(\frac{1}{r}\right)n^{n\ \sigma_1}n^{n\ \sigma_2}n^{n\ \sigma_1}\frac{1}{r^n} \\
&< k\left(\frac{1}{r}\right)^{2n} n^{n\ (4\sigma_1+2\sigma_2)}(n+1)^3 \\
&< k(n+1)^3 n^{n\ (4\sigma_1+2\sigma_2)}.
\end{aligned}$$

Making n tend to infinity it follows, in view of the definition of the order of the similar transposed base $\{\widetilde{U}_n(x)\}$, that

$$\rho_{\widetilde{U}} = \lim_{r\to\infty}\limsup_{n\to\infty}\frac{\log\Omega^{(\widetilde{U})}(r)}{n\log n} < 4\sigma_1 + 2\sigma_2.$$

Since σ_1 and σ_2 are chosen as near as we please from ρ_P and ρ_Q respectively we conclude that

$$\rho_{\widetilde{U}} \leq 4\rho_P + 2\rho_Q,$$

and the theorem is therefore established. $\square$

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