General Solution of a Second-Order Nonhomogenous Linear Operator

Difference Equation with Noncommutative Coefficients

M. A. Jivulescu[1], A. Napoli[2], and A. Messina[2]

[1]Department of Mathematics, University Politehnica Timişoara
P-ta Victoriei Nr. 2, 300006 Timişoara, Romania
Email Address: maria.jivulescu@mat.upt.ro

[2] Dipartimento di Scienze Fisiche ed Astronomiche, Università di Palermo
via Archirafi 36, 90123 Palermo, Italy
Email Address: messina@fisica.unipa.it

The detailed construction of the general solution of a second-order nonhomogenous linear operator-difference equation is presented. The wide applicability of such an equation as well as the usefulness of its resolutive formula is shown by studying some applications belonging to different mathematical contexts.

1   Introduction

In this paper we report the explicit representation of the general solution of the second-order nonhomogenous linear operator-difference equation

$$Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1} + \phi_{p+1}, \tag{1.1}$$

where the unknown $\{Y_p\}_{p \in \mathbb{N}}$ as well as the nonhomogenous term $\{\phi_p\}_{p \in \mathbb{N}}$ are sequences in a vector space V, and the coefficients $\mathcal{L}_0$ and $\mathcal{L}_1$ are noncommutative linear operators mapping V into itself, and are independent from the discrete variable $p \in \mathbb{N}$. This equation encompasses interesting problems arising in very different scenarios [1]. If, for instance, the reference space V is the complex Euclidean space $\mathbb{C}^n$, that is Y_p and ϕ_{p+1} are n-dimensional vectors, $\mathcal{L}_0$ and $\mathcal{L}_1$ $n \times n$ complex matrices, then eq. (1.1) is the vector

representation of a system of second-order linear nonhomogenous difference equations. As another example, we identify V as the vector space of all linear bounded operators defined on a given Hilbert space H, that is $\mathcal{B}(H)$. The operators $\mathcal{L}_0$ and $\mathcal{L}_1$ act upon operators and for this reason are called superoperators. The master equations appearing in the theory of open quantum systems provide examples of equations belonging to this class [2, 3].

It is of relevance to emphasize from the very beginning that the ingredients, Y_p, ϕ_p, $\mathcal{L}_0$ and $\mathcal{L}_1$, of eq. (1.1) may be also interpreted as elements of an assigned algebra V. Consider, for example, V as the noncommutative algebra of all square matrices of order n, that is $\mathcal{M}_n[\mathbb{C}]$. Then eq. (1.1) defines a second-order nonhomogenous linear matrix-difference equation, where Y_p, ϕ_p belong to $M_n(\mathbb{C})$ and $\mathcal{L}_0$, $\mathcal{L}_1$ are multiplying operators [10, 12, 16–19]. We wish moreover to stress that, if V is the vector space of the smooth functions over an interval $I \subset \mathbb{R}$, that is $C^\infty(I)$, then eq. (1.1) represents a wide class of functional-difference equations [11], including difference-differential equations or integro-difference equations [4–6]. These few examples motivate the general interest toward the search of techniques for solving the operator eq.(1.1), with $\mathcal{L}_0$ and $\mathcal{L}_1$ noncommutative coefficients.

In this paper we cope with such a problem by giving the explicit solution of eq. (1.1), leaving unspecified the abstract "support vector space" V, wherein it is formulated. This means that we do not choose from the very beginning the mathematical nature of its ingredients, rather we only require that all the symbols and operations appearing in eq. (1.1) are meaningful. Thus we put $(\mathcal{L}_a\mathcal{L}_b)Y \equiv \mathcal{L}_a(\mathcal{L}_bY) \equiv \mathcal{L}_a\mathcal{L}_bY$ with a or $b = 0, 1$ and 0, $E : V \to V$ define the null and the identity operator in the vector space V, respectively.

The paper is organized as follows: In the following section we present the main and novel result, that is, the solution of an arbitrary Cauchy problem associated with eq. (1.1). Some interesting consequences of such a result are derived in the subsequent section. The practical usefulness of our resolutive formula is illustrated in the fourth section where we solve some nontrivial functional-difference and integral-difference equations. Some concluding remarks are presented in the last section.

2   Explicit Construction of the Resolutive Formula of Eq. (1.1)

We begin by recalling that, if $\{Y_p^*\}_{p\in\mathbb{N}}$ and $\{Y_p\}_{p\in\mathbb{N}}$ are solutions of eq. (1.1), then $\{Y_p^H\}_{p\in\mathbb{N}}$ defined as $Y_p - Y_p^* \equiv Y_p^H$ is a solution of the associated homogenous equation

$$Y_{p+2} = \mathcal{L}_0Y_p + \mathcal{L}_1Y_{p+1}. \tag{2.1}$$

Thus, as for the linear differential equations, and independently from the noncommutative nature of $\mathcal{L}_0$ and $\mathcal{L}_1$, the solution of eq. (1.1) amounts at being able to construct the general integral of eq. (2.1) and to find out a particular solution of eq. (1.1). To this end we start with the following theorem which extends a recently published result [9] concerning the

resolution of the following Cauchy problem

$$\begin{cases} Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1}, \\ \qquad\quad Y_0 = 0, \quad Y_1 = B \end{cases}. \tag{2.2}$$

Theorem 2.1. *The solution of the Cauchy problem,*

$$\begin{cases} Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1} \\ \qquad\quad Y_0 = A, \quad Y_1 = B \end{cases}, \tag{2.3}$$

can be written as

$$Y_p^{(H)} = \alpha_p A + \beta_p B, \tag{2.4}$$

where the operators α_p and β_p have the following form

$$\alpha_p = \begin{cases} \sum\limits_{t=0}^{[(p-2)/2]} \left\{ \mathcal{L}_0^{(t)} \mathcal{L}_1^{(p-2-2t)} \right\} \mathcal{L}_0 & if \quad p \geq 2 \\ 0 & if \quad p = 1 \\ E & if \quad p = 0 \end{cases}, \tag{2.5}$$

$$\beta_p = \begin{cases} \sum\limits_{t=0}^{[(p-1)/2]} \left\{ \mathcal{L}_0^{(t)} \mathcal{L}_1^{(p-1-2t)} \right\} & if \quad p \geq 2 \\ E & if \quad p = 1 \\ 0 & if \quad p = 0 \end{cases}. \tag{2.6}$$

In accordance with [9], the mathematical symbol $\{\mathcal{L}_0^{(u)} \mathcal{L}_1^{(v)}\}$ denotes the sum of all possible distinct permutations of u factors $\mathcal{L}_0$ and v factors $\mathcal{L}_1$. We omit the proof of this theorem since it is practically coincident with that given in [9]. Here instead we demonstrate the following:

Theorem 2.2. *Eq. (1.1) admits the particular solution*

$$Y_p^* = \begin{cases} \sum\limits_{r=1}^{p-1} \beta_{p-r} \phi_r & if \quad p \geq 2 \\ 0 & if \quad p = 0, 1 \end{cases}. \tag{2.7}$$

Proof. It is immediate to verify by direct substitution that the sequence given by eq. (2.7) satisfies eq. (1.1) written for $p = 0$ and $p = 1$. To this end it is enough to exploit eqs. (2.6) and (2.7) getting $Y_2^* = \beta_1 \phi_1 = \phi_1$ and $Y_3^* = \beta_2 \phi_1 + \beta_1 \phi_2 = \mathcal{L}_1 \phi_1 + \phi_2$.

For a generic $p \geq 2$ the introduction of Y_p^*, as given by eq. (2.7), in the right hand of eq. (1.1) yields

$$\mathcal{L}_0 \sum_{r=1}^{p-1} \beta_{p-r} \phi_r + \mathcal{L}_1 \sum_{r=1}^{p} \beta_{p+1-r} \phi_r + \phi_{p+1}$$

$$= \sum_{r=1}^{p-1} (\mathcal{L}_0 \beta_{p-r} + \mathcal{L}_1 \beta_{p+1-r}) \phi_r + \mathcal{L}_1 \beta_1 \phi_p + \phi_{p+1} \tag{2.8}$$

Applying theorem (2.1) to the Cauchy problem expressed by eq. (2.2), we easily deduce that for $p \geq 2$ and $r = 1, 2, \ldots, p-1$ the following operator equality

$$\mathcal{L}_0 \beta_{p-r} + \mathcal{L}_1 \beta_{p+1-r} = \beta_{p+2-r} \tag{2.9}$$

holds. Thus the expression given by eq. (2.8) may be cast as follows

$$\sum_{r=1}^{p-1} \beta_{p+2-r} \phi_r + \beta_{p+2-(p)} \phi_p + \beta_{p+2-(p+1)} \phi_{p+1} = \sum_{r=1}^{p+1} \beta_{p+2-r} \phi_r, \tag{2.10}$$

where we have exploited the equality $\beta_2 = \mathcal{L}_1 \beta_1$ based on eq. (2.6). Since the right hand of eq. (2.10) coincides with Y_{p+2}^* as given by eq. (2.7), we may conclude that $\{Y_p^*\}_{p \in \mathbb{N}}$, expressed by eq. (2.7), provides a particular solution of eq. (1.1). $\square$

On the basis of theorem (2.1) and (2.2) we hence state our main result.

Theorem 2.3. *The solution of the Cauchy problem*

$$\begin{cases} Y_{p+2} = \mathcal{L}_0 Y_p + \mathcal{L}_1 Y_{p+1} + \phi_{p+1} \\ Y_0 = A, \quad Y_1 = B \end{cases} \tag{2.11}$$

is

$$Y_p = \alpha_p A + \beta_p B + \sum_{r=1}^{p-1} \beta_{p-r} \phi_r, \tag{2.12}$$

where A and B are generic admissible initial conditions and the α_p and β_p are defined by eqs. (2.5) *and* (2.6)*, respectively.*

We emphasize that eq. (2.12) furnishes a recipe to solve explicitly, that is in terms of all ingredients $\mathcal{L}_0, \mathcal{L}_1$ and $\{\phi_{p+1}\}_{p \in \mathbb{N}}$ appearing in eq. (1.1), the general Cauchy problem expressed by eq. (2.11) and from this point of view it represents a generalization to the non-commutative case of the results presented in [16]. In the subsequent sections we highlight that our result is effectively exploitable, providing indeed a useful approach to solve problems belonging to very different mathematical contexts. This circumstance adds interest towards our result and a posteriori makes even more robust the motivations to investigate eq. (1.1).

We conclude this section by presenting the structural form assumed by eq. (2.12) in the space $\mathcal{B}(H)$ by solely relaxing the noncommutativity between the two operator coefficients $\mathcal{L}_0$ and $\mathcal{L}_1$. To this end it is useful to recall the definition of the Chebyshev polynomials of the second kind [14] denoted by $\mathcal{U}_p(x), x \in \mathbb{C}$,

$$\mathcal{U}_p(x) = \sum_{m=0}^{[p/2]} (-1)^m \frac{(p-m)!}{m!(p-2m)!} (2x)^{p-2m}. \tag{2.13}$$

Indeed, taking into consideration that the number of all the different terms appearing in the operator symbol $\{\mathcal{L}_0^{(u)}\mathcal{L}_1^{(v)}\}$ coincides with the binomial coefficient $\begin{pmatrix} u+v \\ m \end{pmatrix}$, with $m = \min(u, v)$ as well as assuming the existence and the uniqueness of the operator $(-\mathcal{L}_0)^{-1/2}$ (if, for example, $-\mathcal{L}_0$ is positive definite, that is $< -\mathcal{L}_0 x, x >\geq 0, \forall x \in H$), then the operators α_p and β_p for $p \geq 2$ may be cast as follows:

$$\alpha_p = -(-\mathcal{L}_0)^{p/2}\mathcal{U}_{p-2}\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right) \tag{2.14}$$

and

$$\beta_p = (-\mathcal{L}_0)^{(p-1)/2}\mathcal{U}_{p-1}\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right), \tag{2.15}$$

where $\mathcal{U}_p\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right)$ means the operator value of the polynomial $\mathcal{U}_p$ defined in accordance with eq. (2.13) for $x = \left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right)$. Thus, when $[\mathcal{L}_0, \mathcal{L}_1] = 0$, the solution of Cauchy problem (2.11) may be rewritten for $p \geq 2$ as

$$Y_p = (-\mathcal{L}_0)^{(p-1)/2}\mathcal{U}_{p-1}\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right)B - (-\mathcal{L}_0)^{p/2}\mathcal{U}_{p-2}\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right)A$$

$$+ \sum_{r=1}^{p-1}(-\mathcal{L}_0)^{(p-r-1)/2}\mathcal{U}_{p-r-1}\left(\frac{1}{2}\mathcal{L}_1(-\mathcal{L}_0)^{-1/2}\right)\phi_r, \tag{2.16}$$

where $Y_0 = A$ and $Y_1 = B$ are the prescribed initial conditions.

3   Some Consequences of the Resolutive Formula

The mathematical literature offers several ways to solve scalar linear-second difference equations such as, for example, the matrix method or the generating function method [7, 8]. In the following we heuristically generalize these methods to the operator case. The novelty of our method enables one to deduce, by comparison with these approaches, some interesting consequent identities. The second-order operator difference equation (1.1) may be indeed traced back to the first-order vector representation

$$\mathbf{Y}_{p+1} = C_1\mathbf{Y}_p + \Phi_{p+1}, \tag{3.1}$$

where

$$\mathbf{Y}_p = \begin{pmatrix} Y_p \\ Y_{p+1} \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & E \\ \mathcal{L}_0 & \mathcal{L}_1 \end{pmatrix}, \quad \Phi_{p+1} = \begin{pmatrix} 0 \\ \phi_{p+1} \end{pmatrix}, \quad \mathbf{Y}_0 = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Successive iterations easily lead us to the formal solution of eq. (3.1) in the form

$$\mathbf{Y}_p = C_1^p\mathbf{Y}_0 + \sum_{r=1}^{p}C_1^{p-r}\Phi_r. \tag{3.2}$$

When we follow this route, the solution of eq. (1.1) may be written as [7]

$$Y_p = P_1^t C_1^p \mathbf{Y}_0 + P_1^t \sum_{r=1}^{p} C_1^{p-r} P_2 \Phi_r, \tag{3.3}$$

where

$$P_1^t = \begin{pmatrix} E \\ 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 \\ E \end{pmatrix}.$$

This solution is of practical use only if we are able to evaluate the general integer power of the companion formal matrix C_1. If we exploit our main result expressed by eq. (2.12), the vector $\mathbf{Y}_p$ may be expressed in terms of the operator sequences α_p and β_p as follows

$$
\begin{aligned}
\mathbf{Y}_p &= \begin{pmatrix} \alpha_p A + \beta_p B + \sum_{r=1}^{p-1} \beta_{p-r} \phi_r \\ \alpha_{p+1} A + \beta_{p+1} B + \sum_{r=1}^{p} \beta_{p+1-r} \phi_r \end{pmatrix} \\
&= \begin{pmatrix} \alpha_p & \beta_p \\ \alpha_{p+1} & \beta_{p+1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \sum_{r=1}^{p-1} \beta_{p-r} \phi_r \\ \sum_{r=1}^{p} \beta_{p+1-r} \phi_r \end{pmatrix}.
\end{aligned} \tag{3.4}
$$

Taking into account the first term in the right-hand side of eq. (3.4) represents the general solution of the nonhomogenous equation associated to eq. (3.1), we easily get the formula for the p-th power of the companion matrix C_1 as follows

$$C_1^p = \begin{pmatrix} 0 & E \\ \mathcal{L}_0 & \mathcal{L}_1 \end{pmatrix}^p = \begin{pmatrix} \alpha_p & \beta_p \\ \alpha_{p+1} & \beta_{p+1} \end{pmatrix}. \tag{3.5}$$

We mention that this result generalizes to the noncommutative case the formula for the p-th power of the companion matrix associated to a scalar second-order difference equation [13, 17].

Another possible way to treat eq. (1.1) is via the method of generating functions [7, 8]. We recall that, given the sequence $\{Y_p\}_{p \in \mathbb{N}}$, the associated generating function $Y(s), s \in \mathbb{C}$, is defined as

$$\mathcal{G} Y_p \equiv Y(s) \equiv \sum_{p=0}^{\infty} Y_p s^p. \tag{3.6}$$

To make sure that the eq. (3.6) is meaningful we assume that Y_p belongs to an Hilbert space H and that the series converges when $|s| \leq \xi$ for some positive number ξ. The advantage of this method consists in the systematical possibility of transforming a difference equation in an algebraic one in the unknown $Y(s)$. In order to apply such an approach to the operator-difference equation given by eq. (1.1), we stipulate that $\mathcal{G}(\mathcal{L}_i Y_p)$, $\mathcal{L}_i(\mathcal{G} Y_p)$ are both defined

and that $\mathcal{G}(\mathcal{L}_i Y_p) = \mathcal{L}_i(\mathcal{G}Y_p), i = 0, 1$. Accordingly, heuristically, we transform both sides of eq. (1.1) getting

$$\frac{Y(s) - A - Bs}{s^2} = \mathcal{L}_1 \frac{Y(s) - A}{s} + \mathcal{L}_0 Y(s) + \Phi(s).$$

Thus, assuming the existence of $(E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}$ within the disk of convergence $|s| < \xi$, we have

$$Y(s) = (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1} A + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}(B - \mathcal{L}_1 A)s$$
$$+ (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}\Phi(s)s^2, \tag{3.7}$$

or equivalently

$$Y(s) = (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}(E - \mathcal{L}_1 s)A + (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1} Bs$$
$$+ (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}\Phi(s)s^2. \tag{3.8}$$

On the other hand, accordingly with eqs. (3.6) and (2.12), it holds that

$$Y(s) = \sum_{p=0}^{\infty} \left(\alpha_p A + \beta_p B + \sum_{r=1}^{p-1} \beta_{p-r} \phi_r \right) s^p. \tag{3.9}$$

Thus one notes that imposing $\phi_{p+1} \equiv 0$ and $B = 0$, respectively $A = 0$, we heuristically find the generating function of the operator sequences α_p and β_p in closed form as

$$\mathcal{G}\alpha_p \equiv \sum_{p=0}^{\infty} \alpha_p s^p := (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1}(E - \mathcal{L}_1 s), \tag{3.10}$$

respectively

$$\mathcal{G}\beta_p \equiv \sum_{p=0}^{\infty} \beta_p s^p := (E - \mathcal{L}_1 s - \mathcal{L}_0 s^2)^{-1} s. \tag{3.11}$$

The particular case $\mathcal{L}_0 = -E$ reproduces the generating functions of the Chebyshev polynomials of second kind [14]. Extracting indeed, for the sake of convenience, the first two terms of the series, that is, writing

$$\sum_{p=0}^{\infty} \alpha_p s^p = \alpha_0 + \alpha_1 s + \sum_{p=2}^{\infty} \alpha_p s^p$$

with the help of eq. (2.14) and $\alpha_0 = E, \alpha_1 = 0$ we get

$$(E - \mathcal{L}_1 s + s^2)^{-1}(E - \mathcal{L}_1 s) = E - \sum_{p=2}^{\infty} \mathcal{U}_{p-2}\left(\frac{\mathcal{L}_1}{2}\right) s^p. \tag{3.12}$$

Eq. (3.12) easily determines the generating function of the sequence $\{\mathcal{U}_p(\mathcal{L}_1/2)\}_{p\in\mathbb{N}}$ in the form

$$\sum_{p=2}^{\infty}\mathcal{U}_{p-2}\left(\frac{\mathcal{L}_1}{2}\right)s^p = (E - \mathcal{L}_1 s + s^2)^{-1}(E - \mathcal{L}_1 s + s^2 - E + \mathcal{L}_1 s)$$

$$\Leftrightarrow \sum_{p=0}^{\infty}\mathcal{U}_p\left(\frac{\mathcal{L}_1}{2}\right)s^p = (E - \mathcal{L}_1 s + s^2)^{-1}E. \tag{3.13}$$

The novel results obtained in this paper exploiting our resolutive formula (eqs. (3.5), (3.10), (3.11)) clearly evidence that our recipe to manage eq. (1.1) successfully integrates with other resolutive methods. Thus we may claim that our resolutive formula does not possess a formal character only since it helps to provide new interesting identities.

4   Applications of Our Resolutive Formula

4.1   A matrix-difference equation treated with our formula

Consider the second-order matrix-difference equation

$$Y_{p+2} = M_0 Y_p + M_1 Y_{p+1} + \Phi_{p+1}, \tag{4.1}$$

where $M_0, M_1 \in \mathcal{M}_n(\mathbb{C})$ are noncommutative nilpotent matrices of index 2, that is $M_i^2 = 0$, $i = 0, 1$. If one prescribes the initial conditions $Y_0 = A, Y_1 = B$, then the solution of this equation is given by eq. (2.12). The analysis of the matrix term $\{M_0^{(u)}M_1^{(v)}\}$, which appears in the composition of the matrix-operators α_p and β_p, brings to light interesting peculiarities due to the specific nature of the coefficients M_0 and M_1. By definition the term $\{M_0^{(u)}M_1^{(v)}\}$ represents the sum of all possible terms of u factors M_0 and v factors M_1. Thus it is quite simple to deduce that now the matrix-term of the form $M_0^{\nu_1}M_1^{\nu_2}M_0^{\nu_3}M_1^{\nu_4}\ldots$ is equal to zero, if $\nu_i > 1, \forall i$. Hence we deduce that the operator $\{M_0^{(u)}M_1^{(v)}\}$ does not vanish when $u = v$ or $v = u \pm 1$. Indeed, when $u = v, u \geq 2$, only the terms $\underbrace{[M_0 M_1][M_0 M_1]\ldots[M_0 M_1]}_{u-times}$

and $\underbrace{[M_1 M_0][M_1 M_0]\ldots[M_1 M_0]}_{u-times}$ survive in the sum $\{M_0^{(u)}M_1^{(u)}\}$. In addition the only nonvanishing matrix-terms of $\{M_0^{(u)}M_1^{(u+1)}\},\{M_0^{(u)}M_1^{(u-1)}\}$ are $M_1\underbrace{[M_0 M_1][M_0 M_1]\ldots[M_0 M_1]}_{u-times}$, $\underbrace{[M_0 M_1][M_0 M_1]\ldots[M_0 M_1]}_{(u-1)-times}M_0$, respectively. Exploiting these arguments, we easily establish that

$$\beta_p = \begin{cases} \{M_0^{(\frac{p-1}{3})}M_1^{(\frac{p-1}{3})}\} = \underbrace{[M_0M_1]\ldots[M_0M_1]}_{k-times} + \underbrace{[M_1M_0]\ldots[M_1M_0]}_{k-times}, \\ \hspace{6cm} p = 3k+1 \\ \{M_0^{(\frac{p-2}{3})}M_1^{(\frac{p+1}{3})}\} = M_1 \underbrace{[M_0M_1]\ldots[M_0M_1]}_{k-times}, \hspace{1cm} p = 3k+2 \\ \{M_0^{(\frac{p}{3})}M_1^{(\frac{p-3}{3})}\} = \underbrace{[M_0M_1]\ldots[M_0M_1]}_{(k-1)-times} M_0, \hspace{1.3cm} p = 3k, \end{cases}$$

(4.2)

where $k = 1, 2, \ldots$. Similarly we have that

$$\alpha_p = \begin{cases} \{M_0^{(\frac{p-2}{3})}M_1^{(\frac{p-2}{3})}\}M_0 = \underbrace{[M_0M_1]\ldots[M_0M_1]}_{k-times} M_0, \quad p = 3k+2 \\ \{M_0^{(\frac{p-3}{3})}M_1^{(\frac{p}{3})}\}M_0 = M_1 \underbrace{[M_0M_1]\ldots[M_0M_1]}_{(k)-times} M_0, \quad p = 3k+3 \\ \{M_0^{(\frac{p-1}{3})}M_1^{(\frac{p-4}{3})}\}M_0 = (\underbrace{[M_0M_1]\ldots[M_0M_1]}_{(k-1)-times} M_0)M_0, \quad p = 3k+1 \end{cases}$$

. (4.3)

The solution of eq. (4.1), associated to the initial conditions $Y_0 = A$ and $Y_1 = B$, may be cast in the following closed form

$$Y_p = \left[\delta_{\frac{p-2}{3},[\frac{p-2}{3}]}(M_0M_1)^{[\frac{p-2}{3}]}M_0 + \delta_{\frac{p-3}{3},[\frac{p-3}{3}]}M_1(M_0M_1)^{[\frac{p-3}{3}]}M_0 \right] A$$
$$+ \left[\delta_{\frac{p-1}{3},[\frac{p-1}{3}]}(M_0M_1)^{[\frac{p-1}{3}]} + \delta_{\frac{p-1}{3},[\frac{p-1}{3}]}(M_1M_0)^{[\frac{p-1}{3}]} + \delta_{\frac{p-2}{3},[\frac{p-2}{3}]}M_1(M_0M_1)^{[\frac{p-2}{3}]} \right.$$
$$\left. + \delta_{\frac{p-3}{3},[\frac{p-3}{3}]}(M_0M_1)^{[\frac{p-3}{3}]}M_0 \right]B + \sum_{r=1}^{p-1} [\delta_{\frac{p-r-1}{3},[\frac{p-r-1}{3}]} \left((M_0M_1)^{[\frac{p-r-1}{3}]} + (M_1M_0)^{[\frac{p-r-1}{3}]} \right)$$
$$+ \delta_{\frac{p-r-2}{3},[\frac{p-r-2}{3}]}M_1(M_0M_1)^{[\frac{p-r-2}{3}]} + \delta_{\frac{p-r-3}{3},[\frac{p-r-3}{3}]}(M_0M_1)^{[\frac{p-r-3}{3}]}M_0]\Phi_r$$

(4.4)

4.2 A functional-difference equation treated with our method

The three-term recurrence relation

$$f_{p+2}(t) = -f_p(t - \tau_0) + f_{p+1}(t + \tau_1) \tag{4.5}$$

with the initial conditions $A = f_0(t)$ and $B = f_1(t)$ is an example of a functional difference equation, traceable back to eq. (1.1). It is indeed well-known that, if $f(t)$ is a function of class C^∞, then the translation of its independent variable from t to $t + \tau$ can be represented as the effect on the same function of the operator $\exp(\tau \frac{d}{dt})$. This operator appears

in a natural way when one studies problems characterized by translational invariance in a physical context [15]. Thus, by putting

$$\mathcal{L}_i = (-1)^{i+1} \exp\left((-1)^{i+1}\tau_i \frac{d.}{dt}\right), \quad i = 0, 1,$$

the commutativity property of the two operator coefficients $\mathcal{L}_0$ and $\mathcal{L}_1$ allows us to write the solution of eq. (4.5) as follows:

$$f_p(t) = \exp\left(-(\frac{p-1}{2})\tau_0 \frac{d}{dt}\right)\mathcal{U}_{p-1}\left(\frac{1}{2}\exp\left((\tau_1 + \frac{\tau_0}{2})\frac{d}{dt}\right)\right)f_1(t)$$
$$- \exp\left(-(\frac{p}{2})\tau_0 \frac{d}{dt}\right)\mathcal{U}_{p-2}\left(\frac{1}{2}\exp\left((\tau_1 + \frac{\tau_0}{2})\frac{d}{dt}\right)\right)f_0(t). \tag{4.6}$$

Exploiting eq. (2.13) we may then write that

$$f_p(t) = \sum_{k=0}^{[p-1/2]} (-1)^k \binom{p-1-k}{k} f_1\left(t + (p-1-2k)\tau_1 - k\tau_0\right)$$
$$- \sum_{k=0}^{[p-2/2]} (-1)^k \binom{p-2-k}{k} f_0\left(t + (p-2-2k)\tau_1 - (k+1)\tau_0\right). \tag{4.7}$$

Imposing, for example, the initial conditions $f_0(t) = e^{-t}$ and $f_1(t) = e^t$ we get

$$f_p(t) = \exp\left(-(\frac{p-1}{2})\tau_0\right)\mathcal{U}_{p-1}\left(\frac{1}{2}\exp\left(\tau_1 + \frac{\tau_0}{2}\right)\right)e^t$$
$$- \exp\left(\frac{p}{2}\tau_0\right)\mathcal{U}_{p-2}\left(\frac{1}{2}\exp\left(-(\tau_1 + \frac{\tau_0}{2})\right)\right)e^{-t}. \tag{4.8}$$

4.3   An integrodifference equation treated with our method

Consider the difference-differential equation

$$f'_{p+2}(t) = \beta f_{p+1}(t) + \alpha f_p(t), \quad \alpha, \beta \in \mathbb{R}, \quad p = 0, 1, \dots, \tag{4.9}$$

where $f_p(t)$ belongs to $C^\infty(I)$ and $f_0(t)$, $f_1(t)$ are prescribed functions. Moreover we suppose that we know $f_p(0)$ for any $p \geq 2$. The above equation may be rewritten in the equivalent form

$$f_{p+2}(t) = \mathcal{L}_1 f_{p+1}(t) + \mathcal{L}_0 f_p(t) + f_{p+2}(0), \tag{4.10}$$

where $\mathcal{L}_0 = \alpha\mathcal{L}$, $\mathcal{L}_1 = \beta\mathcal{L}$ and $\mathcal{L}(\cdot) = \int_0^t \cdot d\tau$. Thus eq. (4.10) appears as a particular case of eq.(1.1).

The explicit solution of this equation requires the construction of the operator terms α_p and β_p. To this end we begin remarking that β_p is the sum of $\left[\frac{p-1}{2}\right] + 1$ operators of the

form $\{\mathcal{L}_0^t \mathcal{L}_1^{p-1-2t}\}$. Because $\mathcal{L}_0 = \alpha\mathcal{L}$ and $\mathcal{L}_1 = \beta\mathcal{L}$ then, for any finite p,

$$\{\mathcal{L}_0^k \mathcal{L}_1^{p-1-2k}\} = \alpha^k \beta^{p-1-2k} \binom{p-1-k}{k} \mathcal{L}^{p-1-k} \tag{4.11}$$

holds. Therefore by direct substitution into eq. (2.6) it follows that

$$\beta_p = \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^k \beta^{p-1-2k} \binom{p-1-k}{k} \mathcal{L}^{p-1-k}. \tag{4.12}$$

Similarly we have that

$$\alpha_p = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1} \beta^{p-2-2k} \binom{p-2-k}{k} \mathcal{L}^{p-1-k}. \tag{4.13}$$

In accordance with the prescribed initial conditions, the solution of the corresponding homogenous equation is then

$$f_p^{(H)}(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1} \beta^{p-2-2k} \binom{p-2-k}{k} \mathcal{L}^{p-1-k}(f_0(t))$$

$$+ \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^k \beta^{p-1-2k} \binom{p-1-k}{k} \mathcal{L}^{p-1-k}(f_1(t)). \tag{4.14}$$

Exploiting our central theorem (2), we easily find that

$$f_p^* = \sum_{m=1}^{p-1} \beta_{p-m} f_{m+1}(0) = \sum_{m=1}^{p-2} \beta_{p-m} f_{m+1}(0) + f_p(0) \tag{4.15}$$

is a particular solution of the nonhomogenous equation (4.10), namely that for which $f_0(t) = f_1(t) = 0, \forall t$. Substituting eq. (4.12) into eq. (4.15) yields

$$f_p^*(t) = \sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^k \beta^{p-m-1-2k} \binom{p-m-1-k}{k}$$

$$\cdot \mathcal{L}^{p-m-1-k}(f_{m+1}(0)) + f_p(0). \tag{4.16}$$

These expressions may be further simplified. Indeed, observing that

$$\mathcal{L}^n(f(t)) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1, \tag{4.17}$$

it is not difficult to prove that

$$\mathcal{L}^n(f(t)) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{4.18}$$

Thus the general solution of the homogenous equation is

$$f_p^{(H)}(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1}\beta^{p-2-2k} \begin{pmatrix} p-2-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-k} f_0(\tau)d\tau$$

$$+ \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^k \beta^{p-1-2k} \begin{pmatrix} p-1-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-r} f_1(\tau)d\tau$$

while the particular solution of the nonhomogeneous equation is

$$f_p^* = \sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^k \beta^{p-m-1-2k} \begin{pmatrix} p-m-1-k \\ k \end{pmatrix} \frac{1}{(p-m-2-k)!}$$

$$\cdot \int_0^t (t-\tau)^{p-m-2-k} f_{m+1}(0)d\tau + f_p(0).$$

The general solution of the proposed differential-difference Cauchy problem associated to eq. (4.9) assumes the form

$$f_p(t) = \sum_{k=0}^{\left[\frac{p-2}{2}\right]} \alpha^{k+1}\beta^{p-2-2k} \begin{pmatrix} p-2-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-k} f_0(\tau)d\tau$$

$$+ \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \alpha^k \beta^{p-1-2k} \begin{pmatrix} p-1-k \\ k \end{pmatrix} \frac{1}{(p-2-k)!} \int_0^t (t-\tau)^{p-2-k} f_1(\tau)d\tau$$

$$+ \sum_{m=1}^{p-2} \sum_{k=0}^{\left[\frac{p-m-1}{2}\right]} \alpha^k \beta^{p-m-1-2k} \begin{pmatrix} p-m-1-k \\ k \end{pmatrix}$$

$$\cdot \frac{t^{p-m-1-k}}{(p-m-2-k)!(p-m-1-k)!} f_{m+1}(0) + f_p(0).$$

5   Conclusive Remarks

The novel and main theoretical result of this paper is expressed by Theorem 2.2 by which we demonstrate that eq. (2.7) provides a particular solution of the second-order operator difference equation with noncommutative coefficients of form (1.1). This result, together with Theorem 2.1, completes the resolution of this equation enabling us to write formula (2.12) for the solution of the Cauchy problem defined by eq. (2.11). The operator character of eq. (1.1) and, as a consequence, the presence of generally noncommuting coefficients is the key to understand why such an equation may represent the canonical form of equations seemingly not related to each other. The applications of eq. (2.12) and reported in this paper, besides being interesting in their own, demonstrate indeed both the wide applicability of eq. (1.1) as well as the practical usefulness of this resolutive formula.

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M. A. Jivulescu is an assistant professor in the Department of Mathematics at the University Politehnica Timisoara, Romania. Her research field is the theory of operator difference equations and their applications in quantum mechanics and the quantum dynamics of spin systems. From 2006 to 2008, M. A. Jivulescu joined the Quantum Optics Group of Professor Antonino Messina in the Department of Physics at the University of Palermo, Italy. In January 2008 M. A. Jivulescu earned an Ph.D. in Mathematics from West University of Timisoara.

Anna Napoli is an assistant professor in Theoretical Physics at the University of Palermo. She earned an MD in Physics with the maximum rank in 1994 and Ph.D. in Physics in 1999. She is author of more than 70 articles published in international ISI journals. She is a member of the Quantum Optics Group of Palermo University and her research activity has been developed in the framework of quantum optics and condensed matter systems. Current research interests of A. Napoli are centered on spin systems and cavity quantum electrodynamics.

Antonino Messina, head of the Department of Physical and Astronomical Sciences, is a full professor of condensed-matter physics and is the leader of the quantum Optics group of Palermo University. He has been the coordinator of many international research projects, chairman and/or member of the advisory board of a number of international physics conferences. He is author of more than one hundred of articles published in international ISI journals. His research activity has been developed in the framework of quantum optics and condensed matter systems. In particular his scientific interests includes: dynamical properties of paraelectric defects in Halides crystals; Dicke models; spin-phonon interaction; solitonic excitations; ground state of the Lee model; sub-radiance in confined atomic system; symmetry and diagonalization of matter-radiation interaction models; cavity quantum electrodynamics; trapped ions; mesoscopic Josephson junctions; quantum Zeno effect; generation of matter and radiation nonclassical states; generation, manipulation and control of entanglement; open quantum systems, spin systems.