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Estimation on System Reliability in Generalized Lindley Stress-Strength Model

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Abstract: This paper discusses the problem of the estimation of stressed system reliability under both classical and Bayesian paradigms. It is assumed that the strength of a system and the environmental stress applied on it, follow the generalized Lindley distribution. For Bayesian calculation, we proposed the use of MCMC technique viz Metropolis-Hastings algorithm to approximate the posteriors of the stress-strength parameters. The behaviours of the maximum likelihood and Bayes estimators of stress-strength parameters and reliability have been studied through the Monte Carlo simulation study. Finally, the empirical illustration based on a real data set has been provided.

Keywords: Generalized Lindley distribution, Stress-strength reliability, Maximum likelihood estimates, Bayes estimates, Metropolis-Hastings algorithm

1 Introduction

The stress-strength reliability can be described as an assessment of reliability of a component/system in terms of random variables X representing stress experienced by a component and Y representing the strength of a component available to overcome the stress. In simple way, if the stress exceeds the strength the component would fail; and visa versa. The stress-strength reliability R is then defined as the probability of not failing i.e. $P[Y > X]$. The idea of stress strength reliability has been originally proposed by [14]. After that, [15] and [16] have discussed the procedure for obtaining the distribution free confidence interval for stress strength reliability. [21] have compared the performances of different methods of constructing the confidence interval of stress strength reliability and proposed a new method for obtaining the s-confidence intervals for the reliability in the stress-strength model. Over a last decade, the several papers demonstrating the use of different stress-strength models have been discussed in exiting literature. To name a few, [13], [19], [17], [18] and references cited therein, who have discussed the estimation of stress-strength reliability from different stress-strength models. In this paper, we have discussed the classical and Bayesian estimation procedures for estimating the stress-strength reliability when strength of a system and stress applied on a system follow the generalized Lindey distribution.

The Lindley distribution is mixture of $exponential(\theta)$ and $gamma(2,\theta)$ distributions with mixing proportions are $(1/(1+\theta))$ and $(\theta/(1+\theta))$, respectively and has been proposed by [20] as counter example of fiducial statistics in Bayesian theory. The mathematical treatments to describe the statistical properties of the Lindley distribution have been provided by [18]. After that, the Lindley distribution received the great attention of the researchers and extensively discussed in the life testing contexts such as under censored survival data [3,5,14], competing risk model [8], load sharing system models [7], and stress-strength model [13,19] etc.

There may be the situation where the Lindley distribution having only scale parameter, seems to be incompatible with real problems and so the more flexible and parsimonious generalizations of the Lindley distributions having shape parameter also are inevitable. Keeping this point in mind, [1] have introduced a two parameter generalization of Lindley distribution as an extended model for modelling of bathtub data alternative to gamma, lognormal, Weibull, and exponentiated exponential distributions. The generalized Lindley distribution has the following probability density

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function (PDF):

$$f(x)=\frac{\alpha\lambda^2}{(1+\lambda)}(1+x)\left[1-\frac{1+\lambda+\lambda x}{(1+\lambda)}e^{-\lambda x}\right]^{\alpha-1}e^{-\lambda x};x>0,\ \alpha,\lambda>0 \tag{1}$$

Where, α and λ are the shape and scale parameters, respectively. The cumulative distribution function (CDF) corresponding to (1) is given by

$$F(x)=\left[1-\frac{1+\lambda+\lambda x}{(1+\lambda)}e^{-\lambda x}\right]^{\alpha};x>0,\ \alpha,\lambda>0 \tag{2}$$

[1] have discussed the properties and the inferential procedure for this distribution under classical paradigm. They have also shown that the generalized Lindley distribution is able to model the various ageing classes of the lifetime scenario including increasing, decreasing and bathtub hazard rates with different combinations of its parameters. Therefore, this distribution can be recommended as an alternative model to the various existing lifetime models. [2] considered the problem of the Bayesian estimation of the parameters of generalized Lindley distribution under general entropy loss function based on complete sample of observations. In the next article [4], they discussed the estimation and prediction problems for this distribution under the progressive Type-II censoring with Beta-Binomial removals.

Bayesian inference requires the numerical approximation of the posteriors since the posteriors consist intractable integrals. To overcome such difficulty, the suitable approximation techniques based on three main strategies, Taylor's expansions such as Laplace and Lindey's approximations, quadratures based on classical numerical analysis and standard Monte Carlo importance sampling have been used for Bayesian calculations. Here, we proposed the use of Markov chain Monte Carlo techniques namely Gibbs sampler ([10], [22]) and Metropolis Hastings ([12], [11]) algorithm.

The rest of the paper has been organized as follows. The expression of reliability form stress-strength generalized Lindley model has been derived in section 2. The maximum likelihood and Bayes estimators of stress-strength reliability have been constructed in section 3 and 4 respectively. Section 5 consists a simulation study to compare the performances of the proposed estimators of stress-strength reliability. Finally, a real data illustration has been presented in section 6.

2 Stress-Strength reliability

Let, X and Y be independent stress and strength random variables following generalized Lindley distribution with parameters (α_1,λ_1) and (α_2,λ_2) respectively. Then, the stress-strength reliability of system is defined as follows:

Lemma 1*The stress-strength reliability of system is*

$$\begin{aligned}R&=P[X<Y]\\&=\int_0^\infty f(x,\alpha_1,\lambda_1)F(x,\alpha_2,\lambda_2)\,dx\end{aligned} \tag{3}$$

Then, we have

$$R=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1}\binom{\alpha_1-1}{i}\binom{\alpha_2}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m}\frac{(-1)^{(i+j)}\alpha_1\lambda_1^{k+2}\lambda_2^{l}}{(1+\lambda_1)^{i+1}(1+\lambda_2)^{j}}\frac{\Gamma(m+1)}{[\lambda_1+i\lambda_1+j\lambda_2]^{m+1}}$$

*Proof.*Let us consider, the integral

$$R=\int_0^\infty f(x,\alpha_1,\lambda_1)F(x,\alpha_2,\lambda_2)\,dx$$

Substituting (1) and (2) in above expression, we get

$$R = \frac{\alpha_1 \lambda_1^2}{(1+\lambda_1)} \int_0^\infty (1+x) e^{-\lambda_1 x} \left[1 - \frac{1+\lambda_1+\lambda_1 x}{(1+\lambda_1)} e^{-\lambda_1 x} \right]^{\alpha_1 - 1} \left[1 - \frac{1+\lambda_2+\lambda_2 x}{(1+\lambda_2)} e^{-\lambda_2 x} \right]^{\alpha_2} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{\alpha_1 - 1}{i} \binom{\alpha_2}{j} \frac{(-1)^{i+j} \alpha_1 \lambda_1^2}{(1+\lambda_1)^{i+1} (1+\lambda_2)^j} \int_0^\infty (1+x) exp\left(-x\left(\lambda_1 + i\lambda_1 + j\lambda\right)\right)$$

$$\times (1+\lambda_1+\lambda_1 x)^i (1+\lambda_2+\lambda_2 x)^j dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \sum_{l=0}^{j} \binom{\alpha_1 - 1}{i} \binom{\alpha_2}{j} \binom{i}{k} \binom{j}{l} \frac{(-1)^{i+j} \alpha_1 \lambda_1^{k+2} \lambda_2^l}{(1+\lambda_1)^{i+1} (1+\lambda_2)^j} \int_0^\infty (1+x)^{k+l+1} e^{-x(\lambda_1 + i\lambda_1 + j\lambda)} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \sum_{l=0}^{j} \sum_{m=0}^{k+l+1} \binom{\alpha_1 - 1}{i} \binom{\alpha_2}{j} \binom{i}{k} \binom{j}{l} \binom{k+l+1}{m} \frac{(-1)^{i+j} \alpha_1 \lambda_1^{k+2} \lambda_2^l}{(1+\lambda_1)^{i+1} (1+\lambda_2)^j} \int_0^\infty x^m e^{-x(\lambda_1 + i\lambda_1 + j\lambda)} dx$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \sum_{l=0}^{j} \sum_{m=0}^{k+l+1} \binom{\alpha_1 - 1}{i} \binom{\alpha_2}{j} \binom{i}{k} \binom{j}{l} \binom{k+l+1}{m} \frac{(-1)^{(i+j)} \alpha_1 \lambda_1^{k+2} \lambda_2^l}{(1+\lambda_1)^{i+1} (1+\lambda_2)^j} \frac{\Gamma(m+1)}{[\lambda_1 + i\lambda_1 + j\lambda_2]^{m+1}}$$

Quit Easily Done.

3 Maximum likelihood estimation

In this section, we have discussed maximum likelihood estimation (MLE) procedure for the model parameters as well as the stress-strength system reliability in case of complete sample of observations. Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ be two independent random samples from GLD(α_1, λ_1) and GLD(α_2, λ_2), respectively. Then, the likelihood function is given by

$$L\left(\underset{\sim}{x}, \underset{\sim}{y} | \alpha_1, \lambda_1, \alpha_2, \lambda_2\right) = \left[\frac{\alpha_1 \lambda_1^2}{(1+\lambda_1)}\right]^n \left[\frac{\alpha_2 \lambda_2^2}{(1+\lambda_2)}\right]^m e^{-\lambda_1 \sum\limits_{i=1}^{n} x_i} e^{-\lambda_2 \sum\limits_{j=1}^{m} y_j} \prod_{i=1}^{n} (1+x_i) \prod_{j=1}^{m} (1+y_j)$$
$$\prod_{i=1}^{n} \left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]^{\alpha_1 - 1} \prod_{j=1}^{m} \left[1 - \frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)} e^{-\lambda_2 y_j}\right]^{\alpha_2 - 1} \tag{4}$$

The Log-likelihood function can be written as

$$LogL = n \, \ln\left[\frac{\alpha_1 \lambda_1^2}{(1+\lambda_1)}\right] - \lambda_1 \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(1+x_i) + (\alpha_1 - 1) \sum_{i=1}^{n} \ln\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right] + m \, \ln\left[\frac{\alpha_2 \lambda_2^2}{(1+\lambda_2)}\right]$$
$$- \lambda_2 \sum_{j=1}^{m} y_j + \sum_{j=1}^{m} \ln(1+y_j) + (\alpha_2 - 1) \sum_{j=1}^{m} \ln\left[1 - \frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)} e^{-\lambda_2 y_j}\right] \tag{5}$$

For the estimation of stress-strength reliability, we have considered the following cases:

3.1 *Case I:* When scale parameters (λ_1, λ_2) are known

The MLEs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ of α_1 and α_2, respectively can be obtained by solving the non-linear equations independently as follows:

$$\frac{n}{\alpha_1} + \sum_{i=1}^{n} \ln\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right] = 0 \tag{6}$$

$$\frac{m}{\alpha_2} + \sum_{j=1}^{m} \ln\left[1 - \frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)} e^{-\lambda_2 y_j}\right] = 0 \tag{7}$$

From equations (6) and (7), the estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$ for known values of λ_1 and λ_2 are given by

$$\hat{\alpha}_1 = -\frac{n}{\sum\limits_{i=1}^{n} \ln\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]}$$

$$\hat{\alpha}_2 = -\frac{m}{\sum\limits_{j=1}^{m} \ln\left[1 - \frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)} e^{-\lambda_2 y_j}\right]}$$

Once, we have the MLEs of the model parameters, we can also obtained the maximum likelihood estimates of stress-strength reliability by using the invariance property of maximum likelihood estimator. By the invariance property of MLE, the MLE $\hat{R}$ of R is given by

$$\hat{R} = [R]_{\alpha_1=\hat{\alpha}_1, \alpha_2=\hat{\alpha}_2, \lambda_1, \lambda_2}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1} \binom{\hat{\alpha}_1 - 1}{i}\binom{\hat{\alpha}_2}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m} \frac{(-1)^{(i+j)}\hat{\alpha}_1 \lambda_1^{k+2}\lambda_2^{l}}{(1+\lambda_1)^{i+1}(1+\lambda_2)^{j}} \frac{\Gamma(m+1)}{[\lambda_1 + i\lambda_1 + j\lambda_2]^{m+1}}$$

3.2 ***Case -II:*** *When shape parameters (α_1, α_1) are known*

The MLEs $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of λ_1 and λ_2, respectively can be obtained by solving the non-linear equations independently as follows:

$$\frac{2n}{\lambda_1} - \frac{n}{(1+\lambda_1)} + \frac{\lambda_1(\alpha_1 - 1)}{(1+\lambda_1)^2} \sum_{i=1}^{n} \frac{(2 + x_i + \lambda_1(1+x_i))}{\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]} x_i e^{\lambda_1 x_i} = 0 \tag{8}$$

$$\frac{2m}{\lambda_2} - \frac{m}{(1+\lambda_2)} + \frac{\lambda_2(\alpha_2 - 1)}{(1+\lambda_2)^2} \sum_{j=1}^{m} \frac{(2 + y_j + \lambda_2(1+y_j))}{\left[1 - \frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)} e^{-\lambda_2 y_j}\right]} y_j e^{\lambda_2 y_j} = 0 \tag{9}$$

One can easily solve the above equations using any iterative procedure. In this case, the MLE $\hat{R}$ of R becomes

$$\hat{R} = [R]_{\alpha_1, \alpha_2, \lambda_1=\hat{\lambda}_1, \lambda_2=\hat{\lambda}_2}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1} \binom{\alpha_1 - 1}{i}\binom{\alpha_2}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m} \frac{(-1)^{(i+j)}\alpha_1 \hat{\lambda}_1^{k+2}\hat{\lambda}_2^{l}}{\left(1+\hat{\lambda}_1\right)^{i+1}\left(1+\hat{\lambda}_2\right)^{j}} \frac{\Gamma(m+1)}{\left[\hat{\lambda}_1 + i\hat{\lambda}_1 + j\hat{\lambda}_2\right]^{m+1}}$$

3.3 ***Case -III:*** *When shape and scale parameters are unknown*

The MLEs $\hat{\alpha}_1$ and $\hat{\lambda}_1$ of α_1 and λ_1, respectively can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{n}{\alpha_1} + \sum_{i=1}^{n} \ln\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right] = 0 \tag{10}$$

$$\frac{2n}{\lambda_1} - \frac{n}{(1+\lambda_1)} + \frac{\lambda_1(\alpha_1 - 1)}{(1+\lambda_1)^2} \sum_{i=1}^{n} \frac{(2 + x_i + \lambda_1(1+x_i))}{\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]} x_i e^{\lambda_1 x_i} = 0 \tag{11}$$

From equation (10), the MLE $\hat{\alpha}_1$ of α_1 can be obtained in terms of λ_1 as follows

$$\hat{\alpha}_1(\lambda_1) = -\frac{n}{\sum\limits_{i=1}^{n} \ln\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]}$$

Then, the MLE $\hat{\lambda}_1$ of λ_1 can be uniquely determined by solving the following non-linear equation

$$\frac{2n}{\lambda_1} - \frac{n}{(1+\lambda_1)} + \frac{\lambda_1(\hat{\alpha}_1(\lambda_1) - 1)}{(1+\lambda_1)^2} \sum_{i=1}^{n} \frac{(2 + x_i + \lambda_1(1+x_i))}{\left[1 - \frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)} e^{-\lambda_1 x_i}\right]} x_i e^{\lambda_1 x_i} = 0 \tag{12}$$

In order to solve the above equation (12), we can apply any suitable iterative procedure such as Newton Raphson?s method.

Similarly, the MLEs $\hat{\alpha}_2$ and $\hat{\lambda}_2$ of α_2 and λ_2, respectively can be obtained as the simultaneous solution of the following non-linear equations:

$$\frac{m}{\alpha_2}+\sum_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)}e^{-\lambda_2 y_i}\right]=0 \tag{13}$$

$$\frac{2m}{\lambda_2}-\frac{m}{(1+\lambda_2)}+\frac{\lambda_2(\alpha_2-1)}{(1+\lambda_2)^2}\sum_{j=1}^{m}\frac{(2+y_j+\lambda_2(1+y_j))}{\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)}e^{-\lambda_2 y_j}\right]}y_j e^{\lambda_2 y_j}=0 \tag{14}$$

From equation (13), the MLE $\hat{\alpha}_2$ of α_2 can be obtained in terms of λ_2 as follows

$$\hat{\alpha}_2(\lambda_2)=-\frac{m}{\sum\limits_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)}e^{-\lambda_2 y_j}\right]}$$

Therefore, the MLE $\hat{\lambda}_2$ of λ_2 can be uniquely determined by solving the following non-linear equation

$$\frac{2m}{\lambda_2}-\frac{m}{(1+\lambda_2)}+\frac{\lambda_2(\hat{\alpha}_2(\lambda_2)-1)}{(1+\lambda_2)^2}\sum_{j=1}^{m}\frac{(2+y_j+\lambda_2(1+y_j))}{\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)}e^{-\lambda_2 y_j}\right]}y_j e^{\lambda_2 y_j}=0 \tag{15}$$

We apply Newton Raphson?s method to solve the above equation (15) as it can not be solved analytically. In this case, the MLE $\hat{R}$ of R becomes

$$\begin{aligned}\hat{R}&=[R]_{\alpha_1=\hat{\alpha}_1,\alpha_2=\hat{\alpha}_2,\lambda_1=\hat{\lambda}_1,\lambda_2=\hat{\lambda}_2}\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1}\binom{\hat{\alpha}_1-1}{i}\binom{\hat{\alpha}_2}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m}\frac{(-1)^{(i+j)}\hat{\alpha}_1\hat{\lambda}_1^{k+2}\hat{\lambda}_2^{l}}{\left(1+\hat{\lambda}_1\right)^{i+1}\left(1+\hat{\lambda}_2\right)^{j}}\frac{\Gamma(m+1)}{\left[\hat{\lambda}_1+i\hat{\lambda}_1+j\hat{\lambda}_2\right]^{m+1}}\end{aligned}$$

4 Bayes estimation

In this section, we have developed the Bayesian estimation procedure for the parameters estimation of stress-strength reliability from generlized Lindley distribution assuming independent gamma priors for the unknown model parameters. Further, it is assumed that all the parameters of stress and strength models are independent. The following cases are considered as follows:

*4.1 **Case-I:** When scale parameters (λ_1, λ_2) are known*

The prior distributions of α_1, α_2 are given as follows

$$\alpha_1\sim Gamma(b_1,a_1)$$

$$\alpha_2\sim Gamma(b_2,a_2)$$

Therefore, the joint posterior of α_1, α_2 is given by

$$\begin{aligned}\pi_1\left(\alpha_1,\alpha_2|\lambda_1,\lambda_2,\underset{\sim}{x},\underset{\sim}{y}\right)&=K_1\alpha_1^{n+b_1-1}exp\left(-\alpha_1\left(a_1-\sum_{i=1}^{n}\ln\left[1-\frac{1+\lambda_1+\lambda_1 x_i}{1+\lambda_1}e^{-\lambda_1 x_i}\right]\right)\right)\\&\quad\times K_2\alpha_2^{m+b_2-1}exp\left(-\alpha_2\left(a_2-\sum_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{1+\lambda_2}e^{-\lambda_2 y_j}\right]\right)\right)\\&=\pi_{11}\left(\alpha_1|\underset{\sim}{x},\lambda_1\right)\times\pi_{12}\left(\alpha_2|\underset{\sim}{y},\lambda_2\right)\end{aligned} \tag{16}$$

Where, K_1 and K_2 are the normalizing constants. The marginal posteriors of α_1 and α_2 are given by

$$\pi_{11}\left(\alpha_1|\underset{\sim}{x},\lambda_1\right)\propto Gamma\left(n+b_1,\left(a_1-\sum_{i=1}^{n}\ln\left[1-\frac{1+\lambda_1+\lambda_1x_i}{1+\lambda_1}e^{-\lambda_1x_i}\right]\right)\right) \tag{17}$$

$$\pi_{12}\left(\alpha_2|\underset{\sim}{y},\lambda_2\right)\propto Gamma\left(m+b_2,\left(a_2-\sum_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2y_j}{1+\lambda_2}e^{-\lambda_2y_j}\right]\right)\right) \tag{18}$$

The Bayes estimate $\hat{R}_B$ of R under squared error loss is defined as

$$\hat{R}_B=\int_0^\infty\int_0^\infty R\,\pi_1\left(\alpha_1,\alpha_2|\lambda_1,\lambda_2,\underset{\sim}{x},\underset{\sim}{y}\right)d\alpha_1 d\alpha_2$$

It is to be noticed here that the above expression is not easy to calculate and one needs numerical approximation techniques. Therefore, we proposed to use the importance sampling procedure to obtain the Bayes estimates of R. By this sampling procedure, Bayes estimate $\hat{R}_B$ of R is given by

$$\begin{aligned}\hat{R}_B&=\frac{1}{M}\sum_{p=1}^{M}[R]_{\alpha_1=\alpha_{p:1},\alpha_2=\alpha_{p:2},\lambda_1,\lambda_2}\\&=\frac{1}{M}\sum_{p=1}^{M}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1}\binom{\alpha_{p:1}-1}{i}\binom{\alpha_{p:2}}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m}\frac{(-1)^{(i+j)}\alpha_{p:1}\lambda_1^{k+2}\lambda_2^{l}}{(1+\lambda_1)^{i+1}(1+\lambda_2)^{j}}\frac{\Gamma(m+1)}{[\lambda_1+i\lambda_1+j\lambda_2]^{m+1}}\end{aligned}$$

Where, $\alpha_{p:1}, p=1,2,\ldots,M$ and $\alpha_{p:2}, p=1,2,\ldots,M$ be two independent random samples of size M, and drawn from (17) and (18) respectively.

4.2 *Case-II: When shape parameters (α_1, α_2) are known*

The prior distributions of λ_1, λ_2 are given as follows

$$\lambda_1\sim Gamma(d_1,c_1)$$

$$\lambda_2\sim Gamma(d_2,c_2)$$

Therefore, the joint posterior of λ_1, λ_2 is given by

$$\begin{aligned}\pi_2\left(\lambda_1,\lambda_2\mid\alpha_1,\alpha_2,\underset{\sim}{x},\underset{\sim}{y}\right)&=C_1\frac{\lambda_1^{2n+d_1-1}}{(1+\lambda_1)^n}e^{-\lambda_1(c_1+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\frac{1+\lambda_1+\lambda_1x_i}{(1+\lambda_1)}e^{-\lambda_1x_i}\right]^{\alpha_1-1}\\&\times C_2\frac{\lambda_2^{2m+d_2-1}}{(1+\lambda_2)^m}e^{-\lambda_2(c_2+\sum_{j=1}^{m}y_j)}\prod_{j=1}^{m}\left[1-\frac{1+\lambda_2+\lambda_2y_j}{(1+\lambda_2)}e^{-\lambda_2y_j}\right]^{\alpha_2-1}\\&=\pi_{23}\left(\lambda_1|\alpha_1,\underset{\sim}{x}\right)\times\pi_{24}\left(\lambda_2|\alpha_2,\underset{\sim}{y}\right)\end{aligned} \tag{19}$$

Thus, the marginal posteriors of λ_1 and λ_2 are given by

$$\pi_{23}\left(\lambda_1|\alpha_1,\underset{\sim}{x}\right)=C_1\frac{\lambda_1^{2n+d_1-1}}{(1+\lambda_1)^n}e^{-\lambda_1(c_1+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}\left[1-\frac{1+\lambda_1+\lambda_1x_i}{(1+\lambda_1)}e^{-\lambda_1x_i}\right]^{\alpha_1-1} \tag{20}$$

$$\pi_{24}\left(\lambda_2|\alpha_2,\underset{\sim}{y}\right)=C_2\frac{\lambda_2^{2m+d_2-1}}{(1+\lambda_2)^m}e^{-\lambda_2(c_2+\sum_{j=1}^{m}y_j)}\prod_{j=1}^{m}\left[1-\frac{1+\lambda_2+\lambda_2y_j}{(1+\lambda_2)}e^{-\lambda_2y_j}\right]^{\alpha_2-1} \tag{21}$$

Where, C_1 and C_2 are the normalizing constants. The Bayes estimate $\hat{R}_B$ of R is defined as

$$\hat{R}_B = \int_0^\infty \int_0^\infty R\, \pi_2 \left(\lambda_1, \lambda_2 \mid \alpha_1, \alpha_2, \underset{\sim}{x}, \underset{\sim}{y} \right) d\lambda_1 d\lambda_2$$

Again, the above expression of Bayes estimates of R can not be solved analytically. But it is to be noticed here that the marginal posteriors are also not in any standard distributional form. In such a situation, the MCMC technique viz Metropolis-Hastings algorithm can be effectively utilized to draw the samples from any arbitrary posterior. To generate a random sample of size M from $f(x)$, the Metropolis-Hastings algorithm consists the following steps:

Step 1.Start with j=1 and set initial value of $x^{(0)}$ such that $f\left(x^{(0)}\right) > 0$.

Step 2.Using initial value $x^{(0)}$, generate a candidate point $x^{[cand]}$ from a proposal density $q\left(x^{[cand]}, x^{(0)}\right)$. Where, $q\left(x^{[cand]}, x^{(0)}\right) = q\left(x^{[cand]} \leftarrow x^{(0)}\right)\}$ is the probability of returning a value of $x^{[cand]}$ given a previous value of $x^{(0)}$.

Step 3.Generate U uniform variate on range 0 and 1 i.e. $u \sim U(0,1)$.

Step 4.Calculate the ratios at the candidate point $x^{[cand]}$ and previous point $x^{(0)}$

$$Ratio = \left[\frac{f\left(x^{[cand]}\right) q\left(x^{(0)}, x^{[cand]}\right)}{f\left(x^{(0)}\right) q\left(x^{[cand]}, x^{(0)}\right)} \right]$$

Step 5.If, $u \leq min(1, Ratio)$ accept candidate point with probability $min(1, Ratio)$ i.e. $x_{[j]} = x^{[cand]}$, $x^{(0)} = x^{[cand]}$, else $x_{[j]} = x^{(0)}$.

Step 6.Repeat steps 2-5 for all j= 1, 2, ..., M and obtained $x_{[1]}, x_{[2]}, \ldots, x_{[M]}$.

By using above algorithm, one can easily obtain the samples from π_{23} (20) and π_{24} (21). Then, Bayes estimate $\hat{R}_B$ of R is given by

$$\begin{aligned}\hat{R}_B &= \frac{1}{M} \sum_{p=1}^{M} [R]_{\alpha_1, \alpha_2, \lambda_1 = \lambda_{1:p}, \lambda_2 = \lambda_{2:p}} \\ &= \frac{1}{M} \sum_{p=1}^{M} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i} \sum_{l=0}^{j} \sum_{m=0}^{k+l+1} \binom{\alpha_1 - 1}{i} \binom{\alpha_2}{j} \binom{i}{k} \binom{j}{l} \binom{k+l+1}{m} \frac{(-1)^{(i+j)} \alpha_1 \lambda_{1:p}^{k+2} \lambda_{2:p}^{l}}{(1+\lambda_{1:p})^{i+1} (1+\lambda_{2:p})^{j}} \frac{\Gamma(m+1)}{[\lambda_{1:p} + i\lambda_{1:p} + j\lambda_{2:p}]^{m+1}}\end{aligned}$$

Where, $\lambda_{p:1}, p = 1, 2, \ldots, M$ and $\lambda_{p:2}, p = 1, 2, \ldots, M$ be two independent random samples of size M, and drawn from (20) and (21) respectively.

4.3 ***Case-III:*** *When shape and scale parameters are unknown*

The prior distributions of α_1, α_2 λ_1 and λ_2 are given as follows

$$\alpha_1 \sim Gamma(b_1, a_1)$$

$$\alpha_2 \sim Gamma(b_2, a_2)$$

$$\lambda_1 \sim Gamma(d_1, c_1)$$

$$\lambda_2 \sim Gamma(d_2, c_2)$$

Then, the joint posterior PDF of $\alpha_1, \lambda_1, \alpha_2$ and λ_2 is defined as

$$\pi_3 \left(\alpha_1, \lambda_1, \alpha_2, \lambda_2 \mid \underset{\sim}{x}, \underset{\sim}{y} \right) = \frac{L\left(\underset{\sim}{x}, \underset{\sim}{y} | \alpha_1, \lambda_1, \alpha_2, \lambda_2\right) g_1(\alpha_1) g_3(\lambda_1) g_2(\alpha_2) g_4(\lambda_2)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty L\left(\underset{\sim}{x}, \underset{\sim}{y} | \alpha_1, \lambda_1, \alpha_2, \lambda_2\right) g_1(\alpha_1) g_3(\lambda_1) g_2(\alpha_2) g_4(\lambda_2) d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2} \tag{22}$$

Thus, the marginal posteriors of α_1, α_2 λ_1 and λ_2 are given by

$$\pi_{31}\left(\alpha_1|\underset{\sim}{x},\lambda_1\right)\propto Gamma\left(n+b_1,\left(a_1-\sum_{i=1}^{n}\ln\left[1-\frac{1+\lambda_1+\lambda_1 x_i}{1+\lambda_1}e^{-\lambda_1 x_i}\right]\right)\right) \tag{23}$$

$$\pi_{32}\left(\alpha_2|\underset{\sim}{y},\lambda_2\right)\propto Gamma\left(m+b_2,\left(a_2-\sum_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{1+\lambda_2}e^{-\lambda_2 y_j}\right]\right)\right) \tag{24}$$

$$\pi_{33}\left(\lambda_1|\underset{\sim}{x}\right)=A_1\frac{\lambda_1^{2n+d_1-1}}{(1+\lambda_1)^n}e^{-\lambda_1(c_1+\sum\limits_{i=1}^{n}x_i)}e^{-\sum\limits_{i=1}^{n}\ln\left[1-\frac{1+\lambda_1+\lambda_1 x_i}{(1+\lambda_1)}e^{-\lambda_1 x_i}\right]} \tag{25}$$

$$\pi_{34}\left(\lambda_2|\underset{\sim}{y}\right)=A_2\frac{\lambda_2^{2m+d_2-1}}{(1+\lambda_2)^m}e^{-\lambda_2(c_2+\sum\limits_{j=1}^{m}y_j)}e^{-\sum\limits_{j=1}^{m}\ln\left[1-\frac{1+\lambda_2+\lambda_2 y_j}{(1+\lambda_2)}e^{-\lambda_2 y_j}\right]} \tag{26}$$

Where, A_1 and A_2 are the normalizing constants. To obtain a posterior samples from the above posteriors, we follow the following steps:

Step 1.Start with j=1.
Step 2.Set starting point $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$.
Step 3.Using MH algorithm, generate $\lambda_{1:j}\sim\pi_{33}\left(.|\underset{\sim}{x}\right)$.
Step 4.Using MH algorithm, generate $\lambda_{2:j}\sim\pi_{34}\left(.|\underset{\sim}{y}\right)$.
Step 5.Generate $\alpha_{1:j}\sim\pi_{31}\left(.|\underset{\sim}{x},\lambda_{1:j}\right)$.
Step 6.Generate $\alpha_{2:j}\sim\pi_{32}\left(.|\underset{\sim}{y},\lambda_{2:j}\right)$.
Step 7.Repeat steps 3-6, for all j= 1, 2,..., M and obtained $(\alpha_{1:1},\lambda_{1:1}),(\alpha_{1:2},\lambda_{1:2}),\ldots,(\alpha_{1:M},\lambda_{1:M})$ and $(\alpha_{2:1},\ \lambda_{2:1})$, $(\alpha_{2:2},\lambda_{2:2})$, ..., $(\alpha_{2:M},\ \lambda_{2:M})$.

The Bayes estimate $\hat{R}_B$ of R under squared error loss is defined as

$$\begin{aligned}\hat{R}_B&=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R\,\pi_3\left(\alpha_1,\lambda_1,\alpha_2,\lambda_2\mid\underset{\sim}{x},\underset{\sim}{y}\right)d\alpha_1 d\alpha_2 d\lambda_1 d\lambda_2\\&\approx\frac{1}{M}\sum_{p=1}^{M}[R]_{\alpha_1=\alpha_{1:p},\alpha_2=\alpha_{2:p},\lambda_1=\lambda_{1:p},\lambda_2=\lambda_{2:p}}\\&\approx\frac{1}{M}\sum_{p=1}^{M}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\sum_{m=0}^{k+l+1}\binom{\alpha_{1:p}-1}{i}\binom{\alpha_{2:p}}{j}\binom{i}{k}\binom{j}{l}\binom{k+l+1}{m}\frac{(-1)^{(i+j)}\alpha_{1:p}\lambda_{1:p}^{k+2}\lambda_{2:p}^{l}}{(1+\lambda_{1:p})^{i+1}(1+\lambda_{2:p})^{j}}\frac{\Gamma(m+1)}{[\lambda_{1:p}+i\lambda_{1:p}+j\lambda_{2:p}]^{m+1}}\end{aligned}$$

Where, $\alpha_{p:1};p=1,2,\ldots,M$, $\alpha_{p:2};p=1,2,\ldots,M$, $\lambda_{p:1};p=1,2,\ldots,M$ and $\lambda_{p:2};p=1,2,\ldots,M$ be four independent random samples of size M, and drawn from the densities (23-26) respectively.

5 Simulation Study

In this section, we studied the behaviour of the stress-strength reliability on the basis of simulated sample with varying sample sizes and different combinations of the stress-strength parameters. For this purpose, we simulated the random sample of different sizes from generalized Lindley distribution using inversion method (probability integral transformation). Since the probability integral transformation can not be applied explicitly, we, therefore need to follow the following steps for generating a sample of size n from GLD(α,λ):

Step 1.Set n, α and λ.
Step 2.Set initial value x^0.
Step 3.Set $j=1$.
Step 4.Generate $U\sim Unifrom(0,1)$.

Step 5.Update x^0 by using the Newton's formula for solving the non-linear equation $F(x) = u$ such as

$$x^{\star} = x^0 - \frac{F\left(x^0,\alpha,\lambda\right)-U}{f\left(x^0,\alpha,\lambda\right)}$$

Step 6.If $| x^0 - x^{\star} | \le \varepsilon$, (very small, $\varepsilon > 0$ tolerance limit). Then, $x^{\star}$ will be the desired sample from $F(x)$.
Step 7.If $| x^0 - x^{\star} | > \varepsilon$, then, set $x^0 = x^{\star}$ and go to step 5.
Step 8.Repeat steps 4-7, for $j = 1,2,\ldots,n$ and obtained $x_1,x_2,\ldots,x_n$.

The performances of the maximum likelihood and Bayes estimators have been studied for varying samples sizes with fixed values of stress-strength parameters and for different combinations of the model parameters with fixed samples sizes. For this purpose, first, we generated the samples of different sizes (n,m) from stress-strength set-up for fixed values of $\alpha_1 = \lambda_1 = \alpha_2 = \lambda_2 = 2$. The maximum likelihood and Bayes estimates of the stress-strength parameters and reliability, have been obtained for each generated samples from stress and strength generalized Lindley models.

We repeated the whole procedure 1000 of times and reported the average estimates and the corresponding mean square errors of the MLEs and Bayes estimators of the model parameters and stress-strength reliability. For non-informative prior, we take hyper-parameters equating to zero i.e. $a_1 = b_1 = c_1 = d_1 = 0$ and $a_2 = b_2 = c_2 = d_2 = 0$ and Bayes estimates then obtained called as Bayes-1. For gamma priors, the hyper-parameters are chosen such that the prior mean are equal to the true values of the parameters. To run the Metropolis-Hastings algorithm, the asymptotic normal distributions of the respective posteriors are considered as the proposal distributions. For obtaining the Bayes estimates, ten thousand random samples from the posteriors are drawn. The simulation results obtained for the considered cases, (i) scale parameters of stress-strength model are known, (ii) shape parameters of stress-strength model are known, and (iii) shape and scale parameters of stress-strength model are unknown, are presented in Tables 1, 2 and 3 respectively.

After that, we studied the behaviour of the discussed estimators with varying stress-strength parameters for fixed values of n and m. For this propose, we fixed $n = 25$ and $m = 25$. For the different combinations of the stress-strength parameters, the average estimates (AV) and the corresponding mean square error (MSE) of the estimators of the stress-strength parameters and reliability have been summarised in Table 4 and 5. Table 6 shows the behaviour of the maximum likelihood estimator of stress-strength reliability (R) with varying values of R. The same pattern can be easily seen in the MSEs of Bayes estimators of R. From the above study, the following conclusions can be stated as follows:

- ⋆The MSEs of all the estimators decrease as sample size increases in all the considered cases.
- ⋆The MLEs and Bayes-I (obtained under non-informative case) estimators behave more or less same in nature whatever be the values of stress-strength parameters.
- ⋆Bayes estimators obtained under the assumption of gamma prior, are superior than MLEs and Bayes-I estimators in all the considered cases.
- ⋆When scale parameters (λ_1,λ_2) are known, the MSE of the estimator of α_1 increases as α_1 increases for fixed values of α_2, n and m.
- ⋆When scale parameters (λ_1,λ_2) are known, the MSE of the estimator of α_2 increases as α_2 increases for fixed values of α_1, n and m.
- ⋆When scale parameters (α_1,α_2) are known, the MSE of the estimator of λ_1 increases as λ_1 increases for fixed values of λ_2, n and m.
- ⋆When scale parameters (α_1,α_2) are known, the MSE of the estimator of λ_2 increases as λ_2 increases for fixed values of λ_1, n and m.
- ⋆The MSE of the estimator of stress-strength reliability for small (nearer to 0) and large (nearer to 1) values of R, are smaller than that of moderate value (nearer to 0.5) of R. That is, there is a more chance to commit an error (risk) for estimating the stress-strength reliability when R is nearer to 0.5.

6 Real data analysis

In this section, we used the real data sets of the waiting times before service of the customers of two banks A and B, respectively. These data sets simultaneously have been reported by [13,19] for estimating the stress-strength reliability in case of the Lindley distribution. The use of Lindley distribution for the waiting times (bank A) data has been originally discussed by [20]. Since then, many authors have used the data under different set-up for Lindley distribution. Among them, [14] checked the suitability of waiting times data for Lindley distribution over exponential, gamma, Weibull and log-normal distributions, and found that the Lindley distribution reasonably fits better than other considered distributions. The data are as follows:

Waiting time (in minutes) before customer service in Bank A:
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8,

Table 1: Average estimates (AV) and mean square errors (MSE) of the estimators of α_1, α_2 and R for fixed values of $\alpha_1 = \alpha_2 = 2, a_1 = a_2 = 2, b_1 = b_2 = 4$ with varying n and m, when $\lambda_1 = \lambda_2 = 2$ are known

n, m		MLE		Bayes-1		Bayes-2	
		AV	MSE	AV	MSE	AV	MSE
15, 20	α_1	2.135330	0.395312	2.135463	0.395518	2.077225	0.204545
	α_2	2.095270	0.253344	2.095282	0.253584	2.062774	0.158393
	R	0.501753	0.007199	0.499762	0.006827	0.499273	0.004469
20, 15	α_1	2.111397	0.282092	2.111570	0.282394	2.074634	0.173325
	α_2	2.109995	0.338941	2.110006	0.339093	2.060837	0.182325
	R	0.501671	0.007026	0.503621	0.006677	0.503345	0.004392
20, 20	α_1	2.111397	0.282092	2.111568	0.282395	2.074626	0.173320
	α_2	2.092936	0.254233	2.092903	0.254158	2.060712	0.159178
	R	0.501719	0.006205	0.501701	0.005927	0.501389	0.004119
30, 20	α_1	2.062341	0.167604	2.062468	0.167796	2.046699	0.123520
	α_2	2.101093	0.256309	2.100979	0.255889	2.067350	0.159583
	R	0.497413	0.004986	0.499482	0.004785	0.499946	0.003490
20,30	α_1	2.111397	0.282092	2.111566	0.282387	2.074630	0.173318
	α_2	2.057712	0.160256	2.057688	0.160163	2.042792	0.118511
	R	0.503759	0.005118	0.501705	0.004912	0.501024	0.003566
50,50	α_1	2.036390	0.097795	2.036512	0.097991	2.030501	0.082099
	α_2	2.048413	0.089990	2.048456	0.090193	2.041814	0.075637
	R	0.498273	0.002721	0.498298	0.002674	0.498409	0.002294

Table 2: Average estimates (AV) and mean square errors (MSE) of the estimators of λ_1, λ_2 and R for fixed values of $\lambda_1 = \lambda_2 = 2, c_1 = c_2 = 2, d_1 = d_2 = 4$ with varying n and m, when $\alpha_1 = \alpha_2 = 2$ are known

n, m		MLE		Bayes-1		Bayes-2	
		AV	MSE	AV	MSE	AV	MSE
15, 20	λ_1	2.053481	0.118669	2.063821	0.123515	2.051944	0.098841
	λ_2	2.054132	0.086784	2.061909	0.090378	2.054271	0.076188
	R	0.501941	0.008884	0.502777	0.008553	0.502966	0.007312
20, 15	λ_1	2.041152	0.085211	2.048868	0.087548	2.041742	0.074403
	λ_2	2.069224	0.115262	2.079584	0.121797	2.066457	0.096737
	R	0.504717	0.008740	0.503631	0.008380	0.503102	0.007156
20, 20	λ_1	2.041152	0.085211	2.047213	0.086938	2.040340	0.073801
	λ_2	2.052862	0.081769	2.059108	0.083140	2.051727	0.070449
	R	0.502841	0.007567	0.502815	0.007227	0.502687	0.006315
30, 20	λ_1	2.033896	0.054409	2.039309	0.055895	0.050269	0.050269
	λ_2	2.042711	0.087153	2.050683	0.091559	0.077531	0.077531
	R	0.500168	0.006295	0.499190	0.006173	0.498935	0.005481
20,30	λ_1	2.041152	0.085211	2.048620	0.086654	2.041907	0.073679
	λ_2	2.034889	0.054716	2.040021	0.057210	2.036585	0.051386
	R	0.500320	0.006322	0.501130	0.006140	0.501328	0.005455
50,50	λ_1	2.021764	0.031556	2.024272	0.031857	2.023027	0.029955
	λ_2	2.014223	0.031890	2.016788	0.032076	2.015717	0.030171
	R	0.498268	0.003160	0.498298	0.003096	0.498337	0.002926

8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1 and 38.5

Waiting time (in minutes) before customer service in Bank B:

0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5 and 28.0

As, we wants to use this data under the assumption of generalized Lindley distribution, we have to restrict our self to check the suitability of generalized Lindley distribution for the considered real data sets. We, therefore, have applied Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (KS) Statistics to test the goodness-of-fit of above data sets to the generalized Lindley distribution. The fitting summary of Lindley and generalized Lindely distributions has been presented in Table 7. The fitted and empirical distribution functions are plotted

Table 3: Average estimates (AV) and mean square errors (MSE) of the estimators of λ_1, λ_2, α_1, α_2 and R for fixed values of $\alpha_1 = \alpha_2 = \lambda_1 = \lambda_2 = 2$, $c_1 = c_2 = 2, d_1 = d_2 = 4$ and $a_1 = a_2 = 2, b_1 = b_2 = 4$ with varying n and m

n,m	Method		α_1	λ_1	α_2	λ_2	R
15,20	MLE	AV	2.649568	2.233226	2.475681	2.195548	0.501503
		MSE	2.489945	0.396993	1.297604	0.262153	0.010366
	Bayes-1	AV	2.593909	2.167505	2.455873	2.151541	0.500317
		MSE	2.322830	0.353065	1.306493	0.241690	0.008934
	Bayes-2	AV	2.138216	2.063415	2.144954	2.071570	0.500876
		MSE	0.250394	0.121570	0.228121	0.101647	0.007537
20,15	MLE	AV	2.487213	2.184622	2.715299	2.248417	0.503762
		MSE	1.505999	0.272250	5.913247	0.397358	0.010076
	Bayes-1	AV	2.475639	2.143241	2.634817	2.179331	0.504618
		MSE	1.557766	0.258260	4.522195	0.341732	0.008688
	Bayes-2	AV	2.151378	2.063730	2.129649	2.071643	0.503939
		MSE	0.245916	0.105633	0.239102	0.113959	0.007357
20,20	MLE	AV	2.487213	2.184622	2.478138	2.188234	0.501664
		MSE	1.505999	0.272250	1.484963	0.262448	0.008657
	Bayes-1	AV	2.462898	2.139406	2.467122	2.145232	0.501512
		MSE	1.520167	0.255348	1.812107	0.242754	0.007615
	Bayes-2	AV	2.145486	2.061789	2.140129	2.066624	0.501702
		MSE	0.239206	0.104811	0.232111	0.098110	0.006601
30,20	MLE	AV	2.298233	2.126103	2.440431	2.170630	0.498604
		MSE	0.691741	0.158908	1.260329	0.262379	0.006976
	Bayes-1	AV	2.300459	2.101299	2.426905	2.128495	0.499889
		MSE	0.779284	0.152426	1.346564	0.247535	0.006273
	Bayes-2	AV	2.127729	2.055971	2.137147	2.057251	0.499412
		MSE	0.216428	0.080926	0.247912	0.107777	0.005566
20,30	MLE	AV	2.487213	2.184622	2.280634	2.119467	0.501184
		MSE	1.505999	0.272250	0.634042	0.153855	0.007087
	Bayes-1	AV	2.474795	2.280711	2.142581	2.095127	0.500111
		MSE	1.558604	0.252442	0.682874	0.151579	0.006387
	Bayes-2	AV	2.147924	2.062137	2.125029	2.053988	0.500547
		MSE	0.240872	0.104787	0.228427	0.082761	0.005621
50,50	MLE	AV	2.168423	2.075632	2.173288	2.070759	0.498155
		MSE	0.309115	0.086185	0.296139	0.087374	0.003495
	Bayes-1	AV	2.166100	2.061514	2.173070	2.056704	0.498154
		MSE	0.314042	0.084100	0.309525	0.085570	0.003324
	Bayes-2	AV	2.100627	2.042498	2.106529	2.037184	0.498145
		MSE	0.163895	0.057196	0.160611	0.057856	0.003146

in Figures 1 and 2 for data 1 and 2 respectively. Which indicates that the generalized Lindley distribution is quit compatible to the waiting times data sets.

For obtaining the Bayes estimates, we generated ten thousand posterior samples for each stress-strength parameters using algorithms discussed in section 4. The posterior plots of the simulated stress-strength parameters are sketched in Figure 3. For the choice of hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. Such that

$$E(\alpha_1) = \frac{b_1}{a_1}$$
$$Variance(\alpha_1) = \frac{b_1}{a_1^2}$$

The prior variance shows the confidence of the experimenter on his prior guess i.e. the small variance the heigh confidence, the large variance the low confidence. In real data application, we have nothing other than few observations follow the certain probability distribution function. In such a situations, one can use the maximum likelihood or/and Bayes estimates obtained under non-informative prior assumption as the prior guesses with certain choice of the prior variance. Here, we have taken Bayes estimates as the prior guesses for the expected values of the parameters with variance one. For real data sets, the maximum likelihood and Bayes estimates of the stress-strength parameters and reliability are summarised in Table 8.

Table 4: Average estimates (AV) and mean square errors (MSE) of the estimators of α_1, α_2 and R for fixed values of $n = m = 25$ with varying α_1 and α_2, when $\lambda_1 = \lambda_2 = 1$ are known

α_1, α_2	a_1, b_1 / a_2, b_2		MLE		Bayes-1		Bayes-2	
			AV	MSE	AV	MSE	AV	MSE
1,1	1,1	α_1	1.043328	0.052517	1.043407	0.052575	1.039818	0.047525
	1,1	α_2	1.034653	0.047355	1.034645	0.047310	1.031574	0.042881
		R	0.501677	0.004856	0.501663	0.004674	0.501589	0.004324
1,2	1,1	α_1	1.043328	0.052517	1.043407	0.052575	1.039818	0.047525
	2,4	α_2	2.069307	0.189419	2.069289	0.189242	2.048782	0.131851
		R	0.337736	0.003953	0.340472	0.003876	0.339949	0.003251
1,3	1,1	α_1	1.043328	0.052517	1.043407	0.052575	1.039818	0.047525
	2,6	α_2	3.103960	0.426192	3.103934	0.425794	2.781144	0.217569
		R	0.254983	0.002897	0.258534	0.002905	0.275342	0.002920
2,1	2,4	α_1	2.086657	0.210067	2.086813	0.210301	2.062813	0.145060
	1,1	α_2	1.034653	0.047355	1.034645	0.047310	1.031563	0.042878
		R	0.665280	0.003860	0.662532	0.003774	0.662777	0.003144
2,2	2,4	α_1	2.086657	0.210067	2.086813	0.210301	2.062813	0.145060
	2,4	α_2	2.069307	0.189419	2.069289	0.189242	2.048773	0.131850
		R	0.501677	0.004856	0.501663	0.004674	0.501408	0.003483
2,3	2,4	α_1	2.086657	0.210067	2.086813	0.210301	2.062813	0.145060
	2,6	α_2	3.103960	0.426192	3.103934	0.425794	3.062543	0.254300
		R	0.403486	0.004536	0.405231	0.004397	0.404253	0.003075
3,1	2,6	α_1	3.129985	0.472649	3.130220	0.473178	3.081663	0.278454
	1,1	α_2	1.034653	0.047355	1.034645	0.047310	1.031558	0.042875
		R	0.747599	0.002794	0.744053	0.002790	0.744690	0.002189
3,2	2,6	α_1	3.129985	0.472649	3.130220	0.473178	3.081663	0.278454
	2,4	α_2	2.069307	0.189419	2.069289	0.189242	2.048758	0.131830
		R	0.599748	0.004472	0.597980	0.004327	0.598374	0.003018
3,3	2,6	α_1	3.129985	0.472649	3.130220	0.473178	3.081663	0.278454
	2,6	α_2	3.103960	0.426192	3.103934	0.425794	3.062539	0.254278
		R	0.501677	0.004856	0.501663	0.004674	0.501308	0.003054

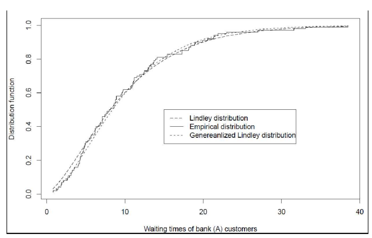

Figure 1. Empirical and fitted distribution functions for waiting times of bank (A) customers

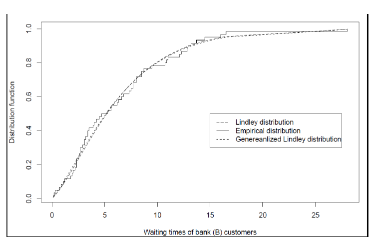

Figure 2. Empirical and fitted distribution functions for waiting times of bank (B) customers

Table 5: Average estimates (AV) and mean square errors (MSE) of the estimators of λ_1, λ_2 and R for fixed values of $n = m = 25$ with varying λ_1 and λ_2, when $\alpha_1 = \alpha_2 = 1$ are known

λ_2, λ_2	c_1, d_1 / c_2, d_2		MLE AV	MLE MSE	Bayes-1 AV	Bayes-1 MSE	Bayes-2 AV	Bayes-2 MSE
1,1	1,1	λ_1	1.029460	0.028888	1.031487	0.029258	1.030063	0.027653
	1,1	λ_2	1.032590	0.029503	1.035120	0.031046	1.033447	0.029294
		R	0.500986	0.005730	0.500967	0.005614	0.500910	0.005365
1,2	1,1	λ_1	1.029460	0.028888	1.031487	0.029258	1.030063	0.027653
	2,4	λ_2	2.073869	0.136237	2.080533	0.143713	2.065959	0.111892
		R	0.713162	0.003948	0.709719	0.003953	0.709818	0.003527
1,3	1,1	λ_1	1.029460	0.028888	1.031487	0.029258	3.094864	0.027653
	2,6	λ_2	3.119288	0.336331	3.130330	0.355302	0.805021	0.239289
		R	0.805021	0.002413	0.801026	0.002488	0.801349	0.002116
2,1	2,4	λ_1	2.067667	0.133761	2.073384	0.135815	2.059445	0.105676
	1,2	λ_2	1.032590	0.029503	1.035120	0.031046	1.033447	0.029294
		R	0.288422	0.003986	0.291846	0.004027	0.291643	0.003593
2,2	2,4	λ_1	2.067667	0.133761	2.073384	0.135815	2.059445	0.105676
	2,4	λ_2	2.073869	0.136237	2.080533	0.143713	2.065959	0.111892
		R	0.500907	0.005586	0.500870	0.005473	0.500811	0.004493
2,3	2,4	λ_1	2.067667	0.133761	2.073384	0.135815	2.059445	0.105676
	2,6	λ_2	3.119288	0.336331	3.130330	0.355302	3.094864	0.239289
		R	0.622482	0.004953	0.620230	0.004886	0.620575	0.003795
3,1	2,6	λ_1	3.109990	0.330514	3.119212	0.335123	3.085870	0.226223
	1,1	λ_2	1.032590	0.029503	1.035120	0.031046	1.033447	0.029294
		R	0.196191	0.002445	0.200204	0.002553	0.199757	0.002176
3,2	2,6	λ_1	3.109990	0.330514	3.119212	0.335123	3.085870	0.226223
	2,4	λ_2	2.073869	0.136237	2.080533	0.143713	2.065959	0.111892
		R	0.379191	0.004975	0.381392	0.004926	0.380874	0.003830
3,3	2,6	λ_1	3.109990	0.330514	3.119212	0.335123	3.085870	0.226223
	2.6	λ_2	3.119288	0.336331	3.130330	0.355302	3.094864	0.239289
		R	0.500863	0.005508	0.500839	0.005395	0.500714	0.003952

Table 6: Mean square error of maximum likelihood estimator of R with varying R

From Table 4	R	Lowest $\rightarrow$	0.250 $\rightarrow$	0.333 $\rightarrow$	0.400 $\rightarrow$	0.500 $\rightarrow$	0.600 $\rightarrow$	0.667 $\rightarrow$	0.750 $\rightarrow$	Largest
	MSE	Lowest $\rightarrow$	0.0029 $\rightarrow$	0.0040 $\rightarrow$	0.0045 $\rightarrow$	0.0049 $\leftarrow$	0.0045 $\leftarrow$	0.0039 $\leftarrow$	0.0028 $\leftarrow$	Lowest
From Table 5	R	Lowest $\rightarrow$	0.191 $\rightarrow$	0.284 $\rightarrow$	0.376 $\rightarrow$	0.500 $\rightarrow$	0.624 $\rightarrow$	0.716 $\rightarrow$	0.809 $\rightarrow$	Largest
	MSE	Lowest $\rightarrow$	0.0024 $\rightarrow$	0.0040 $\rightarrow$	0.0050 $\rightarrow$	0.0055 $\leftarrow$	0.0050 $\leftarrow$	0.0039 $\leftarrow$	0.0024 $\leftarrow$	Lowest

Table 7: Fitting summary for real data sets

Data	Model	MLE	K-S	P-Value	$-$LogL	AIC	BIC
1	LD(θ_1)	$\hat{\theta}_1$=0.18657	0.0677	0.7495	319.0374	640.0748	642.0748
	GLD(α_1, λ_1)	$\hat{\alpha}_1$=1.27728,$\hat{\lambda}_1$=0.21078	0.0503	0.9620	317.8028	639.6056	639.6056
2	LD(θ_2)	$\hat{\theta}_2$=0.27973	0.0797	0.8401	169.1014	340.2028	341.7591
	GLD(α_2, λ_2)	$\hat{\alpha}_2$=0.92672,$\hat{\lambda}_2$=0.26891	0.0683	0.9420	169.0131	342.0262	341.5825

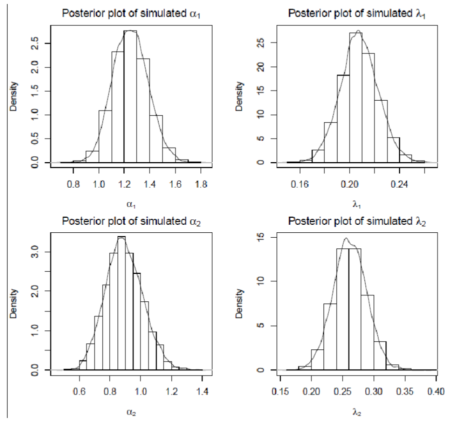

Figure 3. Posterior distributions (Bayes-2) of stress-strength parameters based on waiting times of bank customers data sets

Table 8: The MLEs and Bayes estimates of stress-strength parameters and reliability from real data sets

Method	Data 1		Data 2		R
	α_1	λ_1	α_2	λ_2	
MLE	1.277276	0.210777	0.926714	0.268914	0.669135
Bayes-1	1.271755	0.209334	0.929646	0.268612	0.667583
Bayes-2	1.240187	0.206789	0.897954	0.262332	0.664933

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