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The logistic transformation of real groupoids

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Abstract: In this paper, we introduce the notion of a logistic groupoid on the real numbers $\mathbf{R}$, and show that, given a groupoid $(\mathbf{R}, \star)$ with some conditions, there exists a groupoid $(X, \circledast)$ such that $(\mathbf{R}, \star)$ is the logistic groupoid of $(X, \circledast)$.

Keywords: d-algebra, logistic groupoid, logistically (Jordan, (anti-)associative.

1. Introduction

The general study of binary operations on sets has produced a substantial literature which considers various types of structures via sets of axioms, such as BCK-algebras, BCI-algebras, pseudo-BCK-algebras, d-algebras, MV-algebras ([1, 2, 4, 5, 6, 7]) among others. For set such as the real numbers $\mathbf{R}$, it is possible to go even further and consider groupoids $(\mathbf{R}, *)$, where $x * y$ satisfies certain conditions in terms of the standard structure of $\mathbf{R}$ as a field. For example $(\mathbf{R}, *)$ is a linear groupoid if $x*y = \alpha x+\beta y+\gamma$, where α, β, γ are constants, a quadratic groupoid, a cubic groupoid, etc.. Similarly, $(\mathbf{R}, *)$ is a bounded groupoid if $L \leq x \star y \leq U$ for all $x, y \in \mathbf{R}$, the literature along these lines is more limited. It is quite clearly an area where there is much that can be discovered. The results obtained below mostly concern a type of real groupoid $(\mathbf{R}, *)$ where $L = -\frac{1}{2}$ and $U = \frac{1}{2}$ with the additional restriction that $x \star y + y \star x = 0$ for all $x, y \in \mathbf{R}$, i.e., the groupoid is anti-commutative. For reasons made clear below, we have named such real groupoids logistic groupoids. It appears that as an example of what is possible along the lines we have indicated as well as intrinsically, these groupoids are of good interest.

2. Preliminaries

A (*ordinary*) d-*algebra* ([6, 7]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(A)$x * x = 0$,
(B)$0 * x = 0$,
(C)$x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(D)$((x * y) * (x * z)) * (z * y) = 0$,
(E)$(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

An algebra $(X, *, 0)$ is said to be a *pre-d-algebra* if it satisfies the conditions (A) and (B). An algebra $(X, *, 0)$ is said to be a *quasi-d-algebra* if it satisfies the conditions (A) and (C). The notion of d-algebras is a generalization of BCK-algebras, and the notion of quasi-d-algebras is a generalization of BCI-algebras.

Example 2.1. ([3]) Let $X := \{0, 1, 2, \cdots\}$. Define $x * y := 0$ if x is even. Let $x * y := \max\{x, y\} - \min\{x, y\}$ if x and y are both odd and let $x*y := x+y+1$ if x is odd and y is even. Then $0*x = 0$, since 0 is even. Also, $x*x = 0$ if x is even, and $x*x = \max\{x, x\}-\min\{x, x\} = x-x = 0$ if x is odd. Thus $(X, *, 0)$ is a pre-d-algebra.

Note that Example 2.1 is neither a d-algebra nor a quasi-d-algebra, since $2 * 4 = 0 = 4 * 2$, but $2 \neq 4$.

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Example 2.2. ([3]) Let $X := [0, \infty)$ and $\lceil x \rceil$ be the least integer greater that x, let $x * y := 0$ if x is rational; $x * y := \lceil \max\{x, y\} - \min\{x, y\} \rceil$ if both x and y are irrational; $x * y := \lceil x + y + 1 \rceil$ if x is irrational and y is rational. Also, $x * x = 0$, either because x is rational or because it is irrational and $\lceil \max\{x, x\} - \min\{x, x\} \rceil = \lceil 0 \rceil = 0$. Of course, $0 * x = 0$ since 0 is rational. This proves that $(X, *, 0)$ is a pre-d-algebra.

Note that Example 2.2 is neither a d-algebra nor a quasi-d-algebra, since $3 * 4 = 0 = 4 * 3$, but $3 \neq 4$.

J. S. Han et. al ([3]) introduced the notion of a strong d-algebra as follow: An algebra $(X; *, 0)$ is said to be a *strong d-algebra* ([3]) if it satisfies (A), (B) and (C)* where

(C)*if $x * y = y * x$, then $x = y$.

An algebra $(X; *, 0)$ is said to be a *strong quasi-d-algebra* ([3]) if it satisfies (A) and (C)*

Obviously, every strong d-algebra is a d-algebra, but the converse need not be true in general.

Example 2.3. ([3]) If $X = [0, \infty)$ and if $x * y := \max\{0, x - y\}$, then $(X; *, 0)$ is an ordinary d-algebra, since $x * y = 0$ means $y \leq x$, and $x \leq y, y \leq x$ means $x = y$. We claim that $(X, *, 0)$ is a strong d-algebra. In fact, if $x * y = y * x$ and if $x < y$, then $y * x = y - x > 0$, so that $x - y > 0$ as well, which is an impossibility. Hence $(X; *, 0)$ is a strong d-algebra.

In the following we show an ordinary d-algebra which is not a strong d-algebra.

Example 2.4. ([3]) Let $\mathbf{R}$ be the set of all real numbers and define $x * y := (x - y) \cdot (x - e) + e$, $x, y, e \in \mathbf{R}$, where " $\cdot$ " and " $-$ " are the ordinary product and subtraction of real numbers. Then $x * x = e; e * x = e; x * y = y * x = e$ yields $(x - y) \cdot (x - e) = 0$, $(y - x) \cdot (y - e) = 0$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbf{R}, *, e)$ is a d-algebra.

However, $(\mathbf{R}, *, e)$ is not a strong d-algebra. If $x * y = y * x \Leftrightarrow (x - y) \cdot (x - e) + e = (y - x) \cdot (y - e) + e \Leftrightarrow (x - y) \cdot (x - e) = -(x - y) \cdot (y - e) \Leftrightarrow (x - y) \cdot (x - e + y - e) = 0 \Leftrightarrow (x - y) \cdot (x + y - 2e) = 0 \Leftrightarrow (x = y$ or $x + y = 2e)$, then there exist $x = e + \alpha$ and $y = e - \alpha$ such that $x + y = 2e$, i.e., $x * y = y * x$ and $x \neq y$. Hence, axiom $(C)^*$ fails and thus the d-algebra $(\mathbf{R}, *, e)$ is not a strong d-algebra.

3. The Logistic Transformation

One way to generate quasi-d-algebras over the real numbers is via the following mechanism. Given a groupoid $(\mathbf{R}, *)$, we define an algebra $(\mathbf{R}, \bigtriangledown_*)$ as follows:

$$x \bigtriangledown_* y := \frac{e^{x*y}}{e^{x*y} + e^{y*x}} - \frac{1}{2}$$

for any $x, y \in \mathbf{R}$. We call $(\mathbf{R}, \bigtriangledown_*)$ the *logistic groupoid* of $(\mathbf{R}, *)$. Obviously, we have $-\frac{1}{2} < x \bigtriangledown_* y < \frac{1}{2}$ for any $x, y \in \mathbf{R}$. We denote the notation $\bigtriangledown_*$ by $\bigtriangledown$ if there is no confusion.

Example 3.1. Let $(\mathbf{R}, *)$ be a groupoid defined by

$$x * y := \begin{cases} -1 \text{ if } x < y, \\ \ \ 0 \text{ if } x = y, \\ \ \ 1 \text{ if } x > y \end{cases}$$

for any $x, y \in \mathbf{R}$. If $x < y$, then $x * y = -1, y * x = 1$ so that $x \bigtriangledown y = \frac{e^{-1}}{e^{-1}+e^{1}} - \frac{1}{2} = \frac{1-e^2}{2(1+e^2)}$. If $x = y$, then $x * y = y * x = 0$ so that $x \bigtriangledown \bigtriangledown y = 0$. If $x > y$, then $x * y = 1$ and $y * x = -1$ so that $x \bigtriangledown y = \frac{e^2-1}{2(1+e^2)}$. Hence we obtain

$$x \bigtriangledown y := \begin{cases} \frac{1-e^2}{2(1+e^2)} & \text{if } x < y, \\ 0 & \text{if } x = y, \\ \frac{e^2-1}{2(1+e^2)} & \text{if } x > y \end{cases}$$

for any $x, y \in \mathbf{R}$. The groupoid $(\mathbf{R}, \bigtriangledown)$ is the logistic groupoid of $(\mathbf{R}, *)$. It follows that $-\frac{1}{2} < x \bigtriangledown y < \frac{1}{2}$ and $x \bigtriangledown y + y \bigtriangledown x = 0$ for any $x, y \in \mathbf{R}$.

The following theorem shows that a groupoid with a special condition can be the logistic groupoid of a groupoid.

Theorem 3.2. *Given a groupoid* $(\mathbf{R}, \star)$ *with* $-\frac{1}{2} < x \star y < \frac{1}{2}, \forall x, y \in \mathbf{R}$, *if we define a binary operation "$\circledast$" on* $\mathbf{R}$ *which satisfies*

$$x \circledast y - y \circledast x := ln\left[\frac{1 - 2(x \star y)}{1 + 2(x \star y)}\right],$$

then $(\mathbf{R}, \star)$ *is a logistic groupoid of* $(\mathbf{R}, \circledast)$, *i.e.,* $\star = \bigtriangledown_{\circledast}$.

Proof. If we let $\alpha := x \circledast y - y \circledast x$, then $e^\alpha = e^{x \circledast y - y \circledast x} = \frac{1-2(x \star y)}{1+2(x \star y)}$ and $2(e^\alpha + 1)(x \star y) = 1 - e^\alpha$. It follows that $x \star y = \frac{1-e^\alpha}{2(1+e^\alpha)} = \frac{e^{x \circledast y} - e^{y \circledast x}}{2(e^{x \circledast y} + e^{y \circledast x})} = \frac{e^{x \circledast y}}{e^{x \circledast y} + e^{y \circledast x}} - \frac{1}{2} = x \bigtriangledown_{\circledast} y$, for any $x, y \in \mathbf{R}$. This proves that $(\mathbf{R}, \star)$ is the logistic groupoid of $(\mathbf{R}, \circledast)$. $\square$

Remark. 1. The groupoid $(\mathbf{R}, \bigtriangledown)$ in Example 3.1 is the logistic groupoid of $(\mathbf{R}, *)$ satisfying the condition: $-\frac{1}{2} < x \bigtriangledown y < \frac{1}{2}$ for any $x, y \in \mathbf{R}$. If we define a binary operation "$\circledast$" on $\mathbf{R}$ which satisfies the condition:

$$x \circledast y - y \circledast x := ln\left[\frac{1 - 2(x \star y)}{1 + 2(x \star y)}\right]$$

as in Theorem 3.2, then it satisfies the following condition:

$$x \circledast y - y \circledast x = \begin{cases} \ \ 2 \text{ if } x < y, \\ \ \ 0 \text{ if } x = y, \\ -2 \text{ if } x > y \end{cases}$$

for any $x, y \in \mathbf{R}$. In fact, if $x < y$, then $x \bigtriangledown y = \frac{1-e^2}{2(1+e^2)}$ and hence $x \circledast y - y \circledast x = 2$. If $x = y$, then $x \bigtriangledown y = 0$ and hence $x \circledast y - y \circledast x = 0$. Similarly, if $x > y$, then $x \bigtriangledown y = \frac{e^2-1}{2(1+e^2)}$ and $x \circledast y - y \circledast x = -2$.

2. Define a binary operation "$*_\alpha$" on $\mathbf{R}$ by

$$x *_\alpha y := \begin{cases} -1+\alpha \text{ if } x < y, \\ \alpha \text{ if } x = y, \\ 1+\alpha \text{ if } x > y \end{cases}$$

for any $x, y \in \mathbf{R}$ where α is a non-zero element of $\mathbf{R}$. By routine calculations we obtain that the logistic groupoid $(\mathbf{R}, \triangledown_{*_\alpha})$ is equal to $(\mathbf{R}, \triangledown_*)$ for any non-zero $\alpha \in \mathbf{R}$, i.e., $(\mathbf{R}, \triangledown_{*_\alpha})$ is the logistic groupoid of any groupoid $(\mathbf{R}, *_\alpha)$ by Theorem 3.2 and Example 3.1.

Proposition 3.3. *Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. Then $(\mathbf{R}, *)$ is commutative, i.e., $x * y = y * x$ if and only if $x \triangledown y = 0$ for any $x, y \in \mathbf{R}$.*

Proof. If we assume that $x*y = y*x$ where $x, y \in \mathbf{R}$. Then $x \triangledown y = \frac{e^{x*y}}{e^{x*y}+e^{y*x}} - \frac{1}{2} = \frac{e^{x*y}}{2e^{x*y}} - \frac{1}{2} = 0$. Conversely, if we assume that $x \triangledown y = 0$, then $2e^{x*y} = e^{x*y} + e^{y*x}$ and hence $e^{x*y} = e^{y*x}$. Since the function e^x is a one-one function, we obtain $x * y = y * x$. □

Corollary 3.4. *Let $(\mathbf{R}, *)$ be a groupoid. Then $x \triangledown x = 0$ for any $x \in \mathbf{R}$.*

Proof. Since $x * x = x * x$ for any $x \in \mathbf{R}$, it follows from Proposition 3.3 that $x \triangledown x = 0$. □

Proposition 3.5. *Let $(\mathbf{R}, *)$ be a groupoid. Then $x \triangledown y + y \triangledown x = 0$ for any $x \in \mathbf{R}$.*

Proof. Straightforward. □

Proposition 3.6. *Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. Then $(\mathbf{R}, *)$ satisfies the condition (C)* if and only if $(\mathbf{R}, \triangledown)$ satisfies the condition (C)*.*

Proof. ($\Longrightarrow$) If $x * y = y * x$ where $x, y \in \mathbf{R}$, then $x \triangledown y = 0 = y \triangledown x$ by Proposition 3.3. By assumption, we obtain $x = y$, proving that $(\mathbf{R}, *)$ satisfies the condition (C)*.

($\Longleftarrow$) If $x \triangledown y = y \triangledown x$ where $x, y \in \mathbf{R}$, then $0 = x \triangledown y + y \triangledown x = 2x \triangledown y$ by Proposition 3.5, and hence $x \triangledown y = 0 = y \triangledown x$. By Proposition 3.3, we obtain $x * y = y * x$. By assumption, we obtain $x = y$, proving that $(\mathbf{R}, \triangledown)$ satisfies the condition (C)*. □

Proposition 3.7. *Let $(\mathbf{R}, \triangledown)$ be the logistic groupoid of a groupoid $(\mathbf{R}, *)$ and let $x, y \in \mathbf{R}$. If $x * y = 0$, then $y * x = ln[\frac{1-2(x\triangledown y)}{1+2(x\triangledown y)}]$.*

Proof. Let $x * y = 0$. Then $x \triangledown y = \frac{1}{1+e^{y*x}} - \frac{1}{2}$ and $1 + e^{y*x} = (x \triangledown y + \frac{1}{2})^{-1}$, whence $y * x = ln[(x \triangledown y + \frac{1}{2})^{-1} - 1] = ln[\frac{1-2(x\triangledown y)}{1+2(x\triangledown y)}]$. □

Proposition 3.8. *If $(\mathbf{R}, *)$ be a strong quasi-d-algebra, then $(\mathbf{R}, \triangledown)$ is a strong quasi-d-algebra.*

Proof. It follows immediately from Corollary 3.4 and Proposition 3.6. □

Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. The groupoid $(\mathbf{R}, *)$ is said to be *logistically associative* if $(\mathbf{R}, \triangledown)$ is associative.

Proposition 3.9. *If the groupoid $(\mathbf{R}, *)$ is commutative, then it is logistically associative.*

Proof. If the groupoid $(\mathbf{R}, *)$ is commutative, then $x \triangledown y = 0$ for any $x, y \in \mathbf{R}$ by Proposition 3.3. This means that $(x \triangledown y) \triangledown z = 0 = x \triangledown (y \triangledown z)$ for all $x, y, z \in \mathbf{R}$. □

Note that the converse of Proposition 3.9 need not be true in general.

Example 3.10. Given $x \in \mathbf{R}$, if we define a map $q : \mathbf{R} \to \mathbf{R}$ by

$$q(x) := \begin{cases} x - \lfloor x \rfloor \text{ if } x - \lfloor x \rfloor \leq \frac{1}{2}, \\ x - \lceil x \rceil \text{ otherwise} \end{cases}$$

then $q(3.75) = 3.75 - \lceil 3.75 \rceil = -0.25$ and $q(2.25) = 2.25 - \lfloor 2.25 \rfloor = 0.25$. If n is an integer, then $q(n) = n - \lfloor n \rfloor = 0$, and $n - \lceil n \rceil = 0$ as well. If $-\frac{1}{2} \leq x < \frac{1}{2}$, then $x - \lfloor x \rfloor = x$, i.e., $q(x) = x$. Hence $q(q(x)) = q(x) = x$. It is easy to see that if $(\mathbf{R}, \triangledown_\bullet)$ is a logistic groupoid of any groupoid $(\mathbf{R}, \bullet)$, then $-\frac{1}{2} < x \triangledown_\bullet y < \frac{1}{2}, \forall x, y \in \mathbf{R}$ and $q(x \triangledown_\bullet y) = x \triangledown_\bullet y$.

Define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := \begin{cases} x(y - q(y)) & \text{if } |x| > \frac{1}{2}, \\ (x - q(x))(y - q(y)) & \text{otherwise} \end{cases}$$

Let $x, y, z \in \mathbf{R}$ with $|x| > \frac{1}{2}$. Then $x*(y \triangledown z) = x(y \triangledown z - q(y \triangledown z)) = x(y \triangledown z - y \triangledown z) = 0$ and $(y \triangledown z)*x = (y \triangledown z - q(y \triangledown z))(x - q(x)) = (y \triangledown z - y \triangledown z)(x - q(x)) = 0$. Hence $x*(y \triangledown z) = 0 = (y \triangledown z)*x$. By applying Proposition 3.3, we obtain $x \triangledown (y \triangledown z) = 0$. Let $x, y, z \in \mathbf{R}$ with $|x| \leq \frac{1}{2}$. Then $x * (y \triangledown z) = (x - q(x))(y \triangledown z - q(y \triangledown z)) = (x - q(x))(y \triangledown z - y \triangledown z) = 0$. By applying Proposition 3.3, we obtain $x \triangledown (y \triangledown z) = 0$. Hence $x \triangledown (y \triangledown z) = 0$ for any $x, y, z \in \mathbf{R}$. Also $(x \triangledown y) \triangledown z = 0$ whence $(x \triangledown y) \triangledown z = x \triangledown (y \triangledown z)$ and $(\mathbf{R}, *)$ is logistically associative. But $(\mathbf{R}, *)$ is not commutative, since $3.6 * 2.9 = -0.36$ and $2.9 * 3.6 = -1.16$.

Theorem 3.11. *If the groupoid $(\mathbf{R}, *)$ is logistically associative, then*

$$(x \triangledown y) \triangledown z = 0 = x \triangledown (y \triangledown z)$$

for all $x, y, z \in \mathbf{R}$

Proof. For any $x, y, z \in \mathbf{R}$, by applying Proposition 3.5, we obtain $(x \triangledown y) \triangledown z = x \triangledown (y \triangledown z) = -[(y \triangledown z) \triangledown x]$ for all $x, y, z \in \mathbf{R}$. Moreover, we obtain $(x \triangledown y) \triangledown z = -[z \triangledown (x \triangledown y)] = -[(z \triangledown x) \triangledown y] = y \triangledown (z \triangledown x) = (y \triangledown z) \triangledown x$. Hence $(y \triangledown z) \triangledown x = -[(y \triangledown z) \triangledown x]$ and $(y \triangledown z) \triangledown x = 0$ for all $x, y, z \in \mathbf{R}$, proving the theorem. □

Note that, if $(\mathbf{R}, *)$ is logistically associative, by applying Proposition 3.3 and Theorem 3.11, we obtain $x * (y \triangledown z) = (y \triangledown z) * x$ for all $x, y, z \in \mathbf{R}$.

Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. The groupoid $(\mathbf{R}, *)$ is said to be *logistically Jordan* if

$$(x \triangledown y) \triangledown z + (z \triangledown x) \triangledown y + (y \triangledown z) \triangledown x = 0$$

for any $x, y, z \in \mathbf{R}$.

Corollary 3.12. *If the groupoid $(\mathbf{R}, *)$ is logistically associative, then it is logistically Jordan.*

Proof. It follows immediately from Theorem 3.11. □

Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. The groupoid $(\mathbf{R}, *)$ is said to be *logistically anti-associative* if

$$(x \triangledown y) \triangledown z = -[x \triangledown (y \triangledown z)]$$

for any $x, y, z \in \mathbf{R}$.

Corollary 3.13. *If $(\mathbf{R}, *)$ is logistically associative, then it is logistically anti-associative as well.*

Proof. It follows immediately from Theorem 3.11. □

Let $(\mathbf{R}, \triangledown)$ be a logistic groupoid of a groupoid $(\mathbf{R}, *)$. The groupoid $(\mathbf{R}, *)$ is said to be *logistically medial* if

$$(x \triangledown y) \triangledown (y \triangledown z) = 0$$

for any $x, y, z \in \mathbf{R}$.

Corollary 3.14. *If $(\mathbf{R}, *)$ is logistically associative, then it is logistically medial.*

Proof. By applying Theorem 3.11, we have $x \triangledown (y \triangledown z) = 0$ for all $x, y, z \in \mathbf{R}$. If we replace x by $x \triangledown y$ then $(x \triangledown y) \triangledown (y \triangledown z) = 0$. □

Theorem 3.15. *If $(\mathbf{R}, *)$ is logistically anti-associative, then*

$$(x \triangledown y) \triangledown z = (z \triangledown x) \triangledown y = (y \triangledown z) \triangledown x$$

for any $x, y, z \in \mathbf{R}$.

Proof. Let $(\mathbf{R}, *)$ be a logistically anti-associative groupoid. Given $x, y, z \in \mathbf{R}$, by Proposition 3.5, we have $(x \triangledown y) \triangledown z = -[z \triangledown (x \triangledown y)] = (z \triangledown x) \triangledown y$ and $(x \triangledown y) \triangledown z = -[x \triangledown (y \triangledown z)] = (y \triangledown z) \triangledown x$. □

Corollary 3.16. *If $(\mathbf{R}, *)$ is both logistically Jordan and logistically anti-associative, then it is logistically associative.*

Proof. If it is logistically Jordan, then $(x \triangledown y) \triangledown z + (z \triangledown x) \triangledown y + (y \triangledown z) \triangledown x = 3[(x \triangledown y) \triangledown z] = 0$, and $(x \triangledown y) \triangledown z = 0$ for all $x, y, z \in \mathbf{R}$. Since $(\mathbf{R}, *)$ is logistically anti-associative, we have $0 = (x \triangledown y) \triangledown z = -[z \triangledown (x \triangledown y)]$ and hence $z \triangledown (x \triangledown y) = 0$ for any $x, y, z \in \mathbf{R}$. It follows that $(\mathbf{R}, *)$ is logistically associative. □

Proposition 3.17. *If $(\mathbf{R}, *)$ is logistically anti-associative and $x \triangledown y = y \triangledown x$ for any $x, y \in \mathbf{R}$, then it is logistically medial.*

Proof. Given $x, y, z \in \mathbf{R}$, $(x \triangledown y) \triangledown z = (y \triangledown x) \triangledown z = -[y \triangledown (x \triangledown z)]$ and $(x \triangledown y) \triangledown z = z \triangledown (x \triangledown y) = z \triangledown (y \triangledown x) = -[(z \triangledown y) \triangledown x] = -[(y \triangledown z) \triangledown x)] = y \triangledown (x \triangledown z)$. Hence $y \triangledown (x \triangledown z) = -[y \triangledown (x \triangledown z)]$ and $y \triangledown (x \triangledown z) = 0$ for all $x, y, z \in \mathbf{R}$, proving that $(\mathbf{R}, *)$ is logistically associative. By Corollary 3.14 , we obtain that $(\mathbf{R}, *)$ is logistically medial. □

Proposition 3.18. *Let $(\mathbf{R}, *)$ and $(\mathbf{R}, \bullet)$ be two groupoids with the same logistic groupoid $(\mathbf{R}, \star)$. If we define a binary operation $\Box$ on $\mathbf{R}$ by*

$$x \Box y := x * y - x \bullet y, \quad \forall x, y \in \mathbf{R},$$

then $(\mathbf{R}, \Box)$ is commutative and hence it is logistically associative.

Proof. Since $(\mathbf{R}, *)$ and $(\mathbf{R}, \bullet)$ have the same logistic groupoid $(\mathbf{R}, \star)$, we obtain

$$x * y - y * x = x \bullet y - y \bullet x = ln\left[\frac{1 + 2(x \star y)}{1 - 2(x \star y)}\right]$$

Hence $x \Box y - y \Box x = (x * y - x \bullet y) - (y * x - y \bullet x) = (x * y - y * x) - (x \bullet y - y \bullet x) = 0$, proving that $x \Box y = y \Box x$, i.e., $(\mathbf{R}, \Box)$ is commutative. It follows from Proposition 3.9 that $(\mathbf{R}, \Box)$ is logistically associative. □

Remark. 3. Note that $(\mathbf{R}, \Box)$ does not have $(\mathbf{R}, \star)$ as its logistic groupoid in Proposition 3.18. In Example 3.1 and Remark 2, we see that $(\mathbf{R}, *)$ and $(\mathbf{R}, *_1)$ have the same logistic groupoid $(\mathbf{R}, \triangledown_*)$. Let $x \Box y := x * y - x *_1 y, \forall x, y \in \mathbf{R}$. If $x < y$, then $x \Box y = -1 - 0 = -1$. If $x = y$, then $x \Box y = 0 - 1 = -1$. If $x > y$, then $x \Box y = 1 - 2 = -1$, i.e., $x \Box y = -1$ for any $x, y \in \mathbf{R}$. Hence $x \triangledown_\Box y = \frac{e^{x \Box y}}{e^{x \Box y} + e^{y \Box x}} - \frac{1}{2} = 0$ for any $x, y \in \mathbf{R}$. This proves that $(\mathbf{R}, \Box)$ does not have $(\mathbf{R}, \triangledown_*)$ as its logistic groupoid.

Of course, Proposition 3.18 does not provide a solution $(\mathbf{R}, *)$ for a given groupoid $(\mathbf{R}, \star)$ which is its prescribed logistic groupoid. The answer to this problem is contained in Theorem 3.19.

Theorem 3.19. *Given a groupoid $(\mathbf{R}, \star)$ with $-\frac{1}{2} < x \star y < \frac{1}{2}$ and $x \star y + y \star x = 0$, if we define a groupoid $(\mathbf{R}, \circledcirc)$ by*

$$x \circledcirc y := \begin{cases} \frac{x}{x-y} ln\left[\frac{1-2(x\star y)}{1+2(x\star y)}\right] & \text{if } x \neq y, \\ a & \text{if } x = y \end{cases}$$

where $a \in \mathbf{R}$, then $(\mathbf{R}, \star)$ is the logistic groupoid of $(\mathbf{R}, \circledcirc)$.

Proof. Given $x, y \in \mathbf{R}$, if $x \neq y$, since $x \star y + y \star x = 0$, then we have

$$\begin{aligned} y \circledcirc x &= \frac{y}{y - x} ln\left[\frac{1 - 2(y \star x)}{1 + 2(y \star x)}\right] \\ &= \frac{-y}{x - y} ln\left[\frac{1 + 2(x \star y)}{1 - 2(x \star y)}\right] \\ &= \frac{-y}{x - y} ln\left[\frac{1 - 2(x \star y)}{1 + 2(x \star y)}\right]. \end{aligned}$$

Hence

$$x \circledcirc y - y \circledcirc x = \frac{x}{x-y} ln\left[\frac{1-2(x \star y)}{1+2(x \star y)}\right] - \frac{y}{x-y} ln\left[\frac{1-2(x \star y)}{1+2(x \star y)}\right] = ln\left[\frac{1-2(x \star y)}{1+2(x \star y)}\right].$$

If $x = y$, since $x \star y + y \star x = 0$, we have $x \star x = 0$ and hence $ln\left[\frac{1-2(x\star x)}{1+2(x\star x)}\right] = ln1 = 0 = a - a = x \circledast x - x \circledast x$. By applying Theorem 3.2 we prove that $(\mathbf{R}, \star)$ is the logistic groupoid of $(\mathbf{R}, \circledcirc)$. □

Corollary 3.20. *Given a groupoid* $(\mathbf{R}, \star)$ *with* $-\frac{1}{2} < x \star y < \frac{1}{2}$ *and* $x \star y + y \star x = 0$*, if* $(\mathbf{R}, \star)$ *is the logistic groupoid of* $(\mathbf{R}, \circledcirc)$ *described in Theorem 3.19, then* $(\mathbf{R}, \star)$ *is the logistic groupoid of a groupoid* $(\mathbf{R}, \bullet)$*, where* $x \bullet y := x \circledcirc y + x \Box y$ *and* $x \Box y = y \Box x$ *for any* $x, y \in \mathbf{R}$.

Proof. Since $x \Box y = y \Box x$ for any $x, y \in \mathbf{R}$, we have

$$x \bullet y - y \bullet x = (x \circledcirc y + x \Box y) - (y \circledcirc x + y \Box x) = (x \circledcirc y - y \circledcirc x) + (x \Box y - y \Box x) = x \circledcirc y - y \circledcirc x.$$

By applying Theorem 3.19, we obtain that $(\mathbf{R}, \star)$ is the logistic groupoid of a groupoid $(\mathbf{R}, \bullet)$. □

Note that this is an analogue to the problem of solving a first order linear differential equation.

Theorem 3.21. *Let* $(\mathbf{R}, *)$ *and* $(\mathbf{R}, \bullet)$ *be groupoids. If we define a binary operation "*$\Box$*" on* $\mathbf{R}$ *by* $x \Box y := x * y - x \bullet y, \forall x, y \in \mathbf{R}$ *and if this operation is commutative, then the logistic groupoid of* $(\mathbf{R}, *)$ *is equal to the logistic groupoid of* $(\mathbf{R}, \bullet)$.

Proof. Let $(\mathbf{R}, \triangledown_*)$ and $(\mathbf{R}, \triangledown_\bullet)$ be logistic groupoids of groupoids $(\mathbf{R}, *)$ and $(\mathbf{R}, \bullet)$, respectively. Then we have $x \triangledown_* y = \frac{e^{x*y}}{e^{x*y}+e^{y*x}} - \frac{1}{2}$ and $x \triangledown_\bullet y = \frac{e^{x\bullet y}}{e^{x\bullet y}+e^{y\bullet x}} - \frac{1}{2}$ for any $x, y \in \mathbf{R}$. Since $x \Box y = y \Box x$, we obtain $x * y - y * x = x \bullet y - y \bullet x$. Hence $x \triangledown_* y = \frac{e^{x*y}}{e^{x*y}+e^{y*x}} - \frac{1}{2} = \frac{1}{1+e^{x*y-y*x}} - \frac{1}{2} = \frac{1}{1+e^{x\bullet y-y\bullet x}} - \frac{1}{2} = x \triangledown_\bullet y$. Hence $(\mathbf{R}, \triangledown_*)$ and $(\mathbf{R}, \triangledown_\bullet)$ have the same logistic groupoid. □

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