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Calibrated Confidence Intervals for Intensities of a Two Stage Open Queueing Network

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Abstract: In this paper calibration technique is used to construct approximate $100(1-\alpha)\%$ confidence intervals for intensity parameters ρ_1, ρ_2 of a two-stage open queueing network with distribution-free interval and service times. Numerical simulation study is conducted to demonstrate performances of the calibrated confidence intervals. Also a measure, named relative coverage, is used to compare different calibrated confidence intervals.

Keywords: Calibration technique, Confidence intervals, Coverage percentage, Relative coverage.

1 Introduction

Consider the two-stage open queueing network shown in Fig-1.

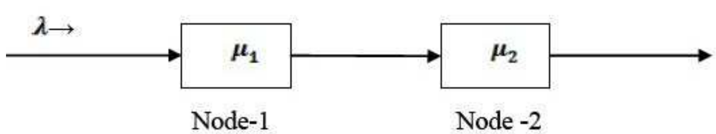

Fig-1: Two-stage open queueing network.

The system consists of two nodes with respective service rates μ_1 and μ_2. The external arrival rate is λ. The output of the node-1 is the input to the node-2. Traffic intensities are defined as the ratios

$$\rho_1 = \frac{\lambda}{\mu_1}, \quad \rho_2 = \frac{\lambda}{\mu_2} \tag{1}$$

where $1/\lambda$ represent mean inter-arrival time and $1/\mu_1, 1/\mu_2$ denotes mean service times at node-1 and node-2 respectively. The condition for stability of the system is that ρ_1 and ρ_2 must be less than unity.

Burke [2] has shown that the output of an M/M/1 queue is also Poisson with rate λ. Jackson [14] showed that the product form solution also applies to open network of Markovian queues with feedback, also Jackson's theorem states that each node behaves like an independent queue. Disney [3] introduces basic properties of queueing networks. Thiruvaiyaru, Basawa and Bhat [23] established maximum likelihood estimators of the parameters of an open Jackson network. Thiruvaiyaru and Basawa [22] considered the problem of estimation for the parameters in a Jackson's type queueing network. Ke and Chu [15] constructed various confidence intervals for intensity parameter of a queueing system.

Calibration technique is used for improving the coverage accuracy of any system of approximate confidence intervals. The idea of the bootstrap calibration technique is, first use bootstrap to estimate the true coverage of confidence intervals and the intervals is then adjusted by comparing with the target nominal level. In literature, there is no work regarding the

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calibration technique used in queueing networks. So it is tempting to use calibration technique to construct new confidence intervals called calibrated for intensity parameters of a two stage open queueing network whose true coverage probabilities come closer to desired value.

Nonparametric inference for estimating intensity parameters is discussed in Section 2. In Section 2.1, we have explained calibration technique. In Section 2.2 to 2.9 we have proposed calibrated consistent and asymptotically normal, exact t, standard bootstrap, bootstrap-t, variance-stabilized bootstrap-t, Bayesian bootstrap, percentile bootstrap, bias-corrected and accelerated bootstrap confidence interval for intensity parameters ρ_1, ρ_2. In Section 3, numerical simulation study is conducted. All simulation results are given in appropriate tables for illustrating performances of all estimation approaches. Finally some conclusions are drawn in Section 4.

2 Nonparametric Statistical Inference for Estimating Intensity Parameters

Let $(X_i, Y_i, i = 1,2)$ be nonnegative, independent random variables representing respectively inter-arrival times and service times at first and second node of a two stage queueing network. Consider $(X_{ij}, Y_{ij}, i = 1,2, j = 1,2,\cdots,n)$ is a random sample drawn from $(X_i, Y_i, i = 1,2)$ for j^{th} customer at i^{th} node. The intensities are defined as follows:

$$\rho_1 = \frac{\mu_{Y_1}}{\mu_{X_1}} \quad \text{and} \quad \rho_2 = \frac{\mu_{Y_2}}{\mu_{X_2}}, \tag{1}$$

where μ_{X_1}, μ_{X_2} denote the mean inter-arrival times at two nodes, μ_{Y_1}, μ_{Y_2} denote the mean service times at two nodes respectively. Define $(\overline{X}_i, \overline{Y}_i, i = 1,2)$ to be the sample means of $(X_{ij}, Y_{ij}, i = 1,2, j = 1,2,\cdots,n)$ respectively. According to the Strong Law of Large Numbers (Rousses [20]), we know that $(\overline{X}_i, \overline{Y}_i, i = 1,2)$ are strongly consistent estimator of $(\mu_{X_i}, \mu_{Y_i}, i = 1,2)$ respectively. Thus a strongly consistent estimator of intensities are given by

$$\hat{\rho}_i = \frac{\overline{Y}_i}{\overline{X}_i}, \;\; i = 1,2. \tag{3}$$

The true distributions of $(X_i, Y_i, i = 1,2)$ are not often known in practice so the exact distributions of $\hat{\rho}_i, i = 1,2$ cannot be derived. But under the assumption that X_i and Y_i being independent, the asymptotical distributions of $\hat{\rho}_i, i = 1,2$ can be developed. By Slutsky's theorem (Hogg & Craig, [13]), we have

$$\sqrt{n}(\hat{\rho}_i - \rho_i) \xrightarrow{D} N(0, \sigma_i^2), \;\; i = 1,2$$

where $\sigma_i^2 = (\mu_{X_i}^2 \sigma_{Y_i}^2 + \mu_{Y_i}^2 \sigma_{X_i}^2)/\mu_{X_i}^4, i = 1,2$ and $\xrightarrow{D}$ denotes convergence in distribution. Now we can estimate σ_i^2 as

$$\hat{\sigma}_i^2 = (\overline{X}_i^2 S_{Y_i}^2 + \overline{Y}_i^2 S_{X_i}^2)/\overline{X}_i^4, \;\; i = 1,2$$

where

$$S_{X_i}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_{ij} - \overline{X}_i)^2, \;\; S_{Y_i}^2 = \frac{1}{n}\sum_{j=1}^{n}(Y_{ij} - Y_i)^2, \;\; i = 1,2$$

Then $\hat{\sigma}_i^2, i = 1,2$ is a strongly consistent estimator of $\sigma_i^2, i = 1,2$. Again applying the Slutsky's theorem we have, $\frac{\sqrt{n}(\hat{\rho}_i - \rho_i)}{\hat{\sigma}_i} \xrightarrow{D} N(0,1), i = 1,2$. Thus $\hat{\rho}_i, i = 1,2$ is a strongly consistent and asymptotically normal (CAN) estimator with approximate variances $\hat{\sigma}_i^2/n, i = 1,2$.

2.1 Calibration Technique

The general theory of calibration is reviewed in Efron and Tibshirani [6] , following ideas of Loh [16] , Beran [1] , Hall [11] and Hall and Martin [12]. The bootstrap calibration technique was introduced by Loh [16,17] . A confidence limit $\hat{\rho}_i[\alpha]$ is supposed to have probability α of covering the true value ρ_i, that is $P_{F_i}\{\rho_i \leq \hat{\rho}_i[\alpha]\} = \alpha, i = 1,2$ where F_i is unknown continuous probability distribution. For an approximate confidence limit there is true probability β_i that ρ_i is less than $\hat{\rho}_i[\alpha]$ say, $\beta_i(\alpha) = P_{F_i}\{\rho_i \leq \hat{\rho}_i[\alpha]\}$. The actual coverage of a confidence procedure is rarely equal to the desired coverage and often it is substantially different. If we knew the function $\beta_i(\alpha)$ then we could calibrate an approximate confidence interval to give exact coverage. Suppose we know that $\beta_i(0.03) = 0.05$ and $\beta_i(0.94) = 0.95$. Then instead of $(\hat{\rho}_i[0.05], \hat{\rho}_i[0.!95])$ we would use $(\hat{\rho}_i[0.03], \hat{\rho}_i[0.94])$ to get a central 90% interval with correct coverage probabilities.

In practice usually we don't know the calibration function $\beta_i(\alpha)$. However we can use the bootstrap to estimate $\beta_i(\alpha)$. The bootstrap estimate of $\beta_i(\alpha)$ is $\hat{\beta}_i(\alpha) = P_{\hat{F}_i}\{\hat{\rho}_i \le \hat{\rho}_i[\alpha]^*\}$ where $\hat{F}_i$ and $\hat{\rho}_i$ are fixed while $\hat{\rho}_i[\alpha]^*$ is the α^{th} confidence limit based on bootstrap dataset from $\hat{F}_i$. The estimate $\hat{\beta}_i(\alpha)$ is obtained by taking B bootstrap data sets and seeing what proportion of them have $\hat{\rho}_i \le \hat{\rho}_i[\alpha]^*$.

2.2 Consistent and Asymptotically Normal (CAN) Calibrated Confidence Interval

With the CAN estimators $\hat{\rho}_i, i = 1,2$ and its associated approximate variances $\hat{\sigma}_i^2/n, i = 1,2$, we construct calibrated confidence intervals for intensities $\rho_i, i = 1,2$. Let z_α be the upper α^{th} quantile of the standard normal distribution. Compute $\hat{\beta}(\alpha_1) = P\{\rho_i \le (\hat{\rho}_i - z_{\alpha/2}\hat{\sigma}_i/\sqrt{n})\}$ and $\hat{\beta}(\alpha_2) = P\{\rho_i \le (\hat{\rho}_i + z_{\alpha/2}\hat{\sigma}_i/\sqrt{n})\}$ where $\alpha_2 = 1-\alpha_1$ and $0 \le \alpha_1 \le 1$. Then approximate $100(1-\alpha)\%$ calibrated confidence intervals for $\rho_i, i = 1,2$ are given by,

$$\left(\hat{\rho}_i' - z_{\hat{\beta}(\alpha_1)/2}\hat{\sigma}_i'/\sqrt{n},\ \ \hat{\rho}_i' + z_{\hat{\beta}(\alpha_2)/2}\hat{\sigma}_i'/\sqrt{n}\right),\ \ i = 1,2. \tag{4}$$

2.3 Exact- t Calibrated Confidence Intervals

Let t_α be the upper α^{th} quantile of the Student's t-distribution. Compute $\hat{\beta}(\alpha_3) = P\{\rho_i \le (\hat{\rho}_i - t_{(n-1),\alpha/2}\hat{\sigma}_i/\sqrt{n}\}$ and $\hat{\beta}(\alpha_4) = P\{\rho_i \le (\hat{\rho}_i + t_{(n-1),\alpha/2}\hat{\sigma}_i/\sqrt{n}\}$, where $\alpha_4 = 1-\alpha_3$ and $0 \le \alpha_3 \le 1$. Then approximate $100(1-\alpha)\%$ exact-t calibrated confidence intervals of $\rho_i, i = 1,2$ are given by,

$$\left(\hat{\rho}_i' - t_{(n-1),\hat{\beta}(\alpha_3)/2}\hat{\sigma}_i'/\sqrt{n},\ \ \hat{\rho}_i' + t_{(n-1)\hat{\beta}(\alpha_4)/2}\hat{\sigma}_i'/\sqrt{n}\right),\ \ i = 1,2. \tag{5}$$

2.4 Standard Bootstrap (SB) Calibrated Confidence Intervals

Efron [7,8,9] originally developed and proposed the bootstrap, which is a resampling technique that can be effectively applied to estimate the sampling distribution of any statistic. According to the bootstrap procedure, a simple random sample $(X_{ij}^*, Y_{ij}^*, i = 1,2, j = 1,2,\cdots,n)$ is taken from the empirical distribution function of $(X_{ij}, Y_{ij}, i = 1,2; j = 1,2,\cdots,n)$. Thus bootstrap estimate for intensity $\rho_i, i = 1,2$ can be calculated as $\hat{\rho}_i^* = \frac{\bar{y}_i^*}{\bar{x}_i^*}, i = 1,2$. The resampling process is repeated N times and $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i = 1,2$ are computed from the bootstrap resample. Averaging the N bootstrap estimates we get bootstrap estimate of $\hat{\rho}_i, i = 1,2$ as $\hat{\rho}_N(i) = \frac{1}{N}\sum_{j=1}^{N}\hat{\rho}_{ij}^*, i = 1,2$ and standard deviation of $\hat{\rho}_i, i = 1,2$ is estimated by

$$sd(\hat{\rho}_N(i)) = \left\{\frac{1}{N-1}\sum_{j=1}^{N}(\hat{\rho}_{ij}^* - \hat{\rho}_N(i))^2\right\}^{1/2},\ \ i = 1,2.$$

For necessary background on bootstrap technique, we refer to Efron and Gong [4] , Efron and Tibshirani [5] , Guntur [10] , Mooney and Duval [19] , Young [24], Rubin [21], Miller [18].

By central limit theorem the distribution of $\hat{\rho}_i, i = 1,2$ is approximately normal. After Computing $\hat{\beta}(\alpha_5) = P\{\rho_i \le (\hat{\rho}_i - z_{\alpha/2}sd(\hat{\rho}_N(i)))\}$ and $\hat{\beta}(\alpha_6) = P\{\rho_i \le (\hat{\rho}_i + z_{\alpha/2}sd(\hat{\rho}_N(i)))\}$ where $\alpha_6 = 1-\alpha_5$ and $0 \le \alpha_5 \le 1$. We get $100(1-\alpha)\%$ SB calibrated confidence interval for ρ_i is

$$\left(\hat{\rho}_i' - z_{\hat{\beta}(\alpha_5)/2}sd(\hat{\rho}_N(i))',\ \ \hat{\rho}_i' + z_{\hat{\beta}(\alpha_6)/2}sd(\hat{\rho}_N(i))'\right) \tag{6}$$

2.5 Bootstrap-t Calibrated Confidence Intervals

Consider N bootstrap estimate $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i = 1,2$ computed from the bootstrap resample. We compute $Z_{ij}^* = \frac{(\hat{\rho}_{ij}^* - \hat{\rho}_N(i))}{sd(\hat{\rho}_N(i))}, i = 1,2, j = 1,2,\cdots,N$ and $Z_{ij}^*, i = 1,2, j = 1,2,\cdots,N$ follow an approximate t distribution. Also compute $\hat{\beta}(\alpha_7) = P\{\rho_i \le (\hat{\rho}_i - t_{\alpha/2}sd(\hat{\rho}_N(i)))\}$ and $\hat{\beta}(\alpha_8) = P\{\rho_i \le (\hat{\rho}_i + t_{\alpha/2}sd(\hat{\rho}_N(i)))\}$ where $\alpha_8 = 1-\alpha_7$ and $0 \le \alpha_7 \le 1$. Then $100(1-\alpha)\%$ bootstrap-t calibrated confidence interval for ρ_i is

$$\left(\hat{\rho}_i' - \hat{t}_{\hat{\beta}(\alpha_7)/2}sd(\hat{\rho}_N(i))',\ \hat{\rho}_i' + \hat{t}_{\hat{\beta}(\alpha_8)/2}sd(\hat{\rho}_N(i))'\right),\ i=1,2 \tag{7}$$

where $\hat{t}_{\hat{\beta}_7(\alpha)/2}$ and $\hat{t}_{\hat{\beta}_8(\alpha)/2}$ equals the $\alpha/2$ percentile of the random sample $Z_{i1}^*, Z_{i2}^*, \cdots, Z_{iN}^*$, $i=1,2$.

2.6 Variance-stabilized Bootstrap-t (VST) Calibrated Confidence Intervals

Let $\hat{\rho}_i, i=1,2$ be a strongly consistent and asymptotically normal estimator with approximate variances $\hat{\sigma}_i^2/n, i=1,2$. Let $\hat{\sigma}_i = \phi(\hat{\rho}_i)$. To find a transformation $f(\hat{\rho}_i)$ such that $Var(f(\hat{\rho}_i)) \approx$ constant, we use the first order Taylor series expansion and taking expectations on both sides, we get

$$Var[f(\hat{\rho}_i)] \approx Var(\hat{\rho}_i)(f'(\rho_i))^2 = (\phi(\rho_i))^2(f'(\rho_i))^2,\ \ i=1,2.$$

Now consider $f(\hat{\rho}_i) = \sqrt{n}\log(\phi(\hat{\rho}_i)), i=1,2$ is the variance-stabilizing transformation. Then we have,

$$V[f(\hat{\rho}_i)] \approx \left(\frac{\sqrt{n}}{\phi(\hat{\rho}_i)}\right)^2 Var[\hat{\rho}_i] = \left(\frac{\sqrt{n}}{\hat{\sigma}_i}\right)^2 Var[\hat{\rho}_i] = \frac{n}{\hat{\sigma}_i^2}\frac{\hat{\sigma}_i^2}{n} = 1, i=1,2.$$

Here we consider N bootstrap estimates $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i=1,2$ computed from the bootstrap resample. We obtain

$$\theta_{ij}^* = (\sqrt{n}\log(\hat{\rho}_{ij}^*) - \sqrt{n}\log(\hat{\rho}_i)), i=1,2, j=1,2,\cdots,N.$$

Also Compute $\hat{\beta}(\alpha_9) = P\{\rho_i \le e^{\log(\hat{\rho}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{1-\alpha/2}}\}$ and $\hat{\beta}(\alpha_{10}) = P\{\rho_i \le e^{\log(\hat{\rho}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\alpha/2}}\}$ where $\alpha_{10} = 1-\alpha_9$ and $0 \le \alpha_9 \le 1$. A $100(1-\alpha)\%$ VST calibrated confidence interval for $\rho_i, i=1,2$ is

$$\left(e^{\log(\hat{\rho}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\hat{\beta}(\alpha_9)}},\ e^{\log(\hat{\rho}_i) - \frac{1}{\sqrt{n}}\hat{v}_i t_{\hat{\beta}(\alpha_{10})}}\right) \tag{8}$$

where $\hat{v}_i t_{\hat{\beta}_9(\alpha)}$ and $\hat{v}_i t_{\hat{\beta}_{10}(\alpha)}$ are $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ percentile of the random sample $\theta_{i1}^*, \theta_{i2}^*, \cdots, \theta_{iN}^*, i=1,2$.

2.7 Bayesian Bootstrap (BB) Calibrated Confidence Intervals

Each BB replication generates a posterior probability for each $X_{ij}, i=1,2, j=1,2,\cdots,n$. One BB replication is generated by drawing $n-1$ uniform $(0,1)$ random numbers $r_1, r_2, \cdots, r_{n-1}$, ordering them, and calculating the gaps $w_j = r_{(j)} - r_{(j-1)}, j=1,2,\cdots,n$, where $r_{(0)} = 0$ and $r_{(n)} = 1$. Then $w_i = (w_{i1}, w_{i2}, \cdots, w_{in}), i=1,2$ is the vector of probabilities attached to the inter-arrival data $X_{ij}, i=1,2, j=1,2,\cdots,n$. Considering all BB replications gives the BB distribution of the distribution of X_i. Hence for $\mu_{x_i}, i=1,2$ (the mean of X_i) in each BB replication we calculate $\overline{X}_i^{**} = \sum_{j=1}^{n} w_{ij}x_{ij}, i=1,2$. The distribution of the values of $\overline{X}_i^{**}$ over all BB replications is the BB distribution of μ_{X_i}. Also, generating a vector of probabilities $v_i = (v_{i1}, v_{i2}, \cdots, v_{in}), i=1,2$ attached to the service time data values $Y_{ij}, i=1,2. j=1,2\cdots n$ in a BB replication, and we calculate $\overline{Y}_i^{**} = \sum_{j=1}^{n} v_{ij}y_{ij}$ for μ_{Y_j} (the mean of Y_i). An estimate of intensity ρ_i can be calculated from BB replications as $\hat{\rho}_i^{**} = \frac{\overline{Y}_i^{**}}{\overline{X}_i^{**}}, i=1,2$, where $\hat{\rho}_i^{**}, i=1,2$ is called a Bayesian bootstrap estimate of $\rho_i, i=1,2$. The above BB process can be repeated N times. The N BB estimates $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i=1,2$ can be computed from the *BB* replications. Averaging the N BB estimates, we obtain that $\hat{\rho}_{BB}'(i) = \frac{1}{N}\sum_{j=1}^{N}\hat{\rho}_{ij}^{**}, i=1,2$ is the *BB* estimate of $\rho_i, i=1,2$. Also the standard deviation of $\hat{\rho}_i$ can be estimated by

$$sd(\hat{\rho}_{BB}'(i)) = \left\{\frac{1}{N-1}\sum_{j=1}^{N}(\hat{\rho}_{ij}^{**} - \hat{\rho}_{BB}'(i))^2\right\}^{1/2},\ \ i=1,2.$$

Also find $\hat{\beta}(\alpha_{11}) = P\{\rho_i \leq (\hat{\rho}_i - z_{\alpha/2} sd(\hat{\rho}'_{BB}(i)))\}$ and and $\hat{\beta}(\alpha_{12}) = P\{\rho_i \leq (\hat{\rho}_i - z_{\alpha/2} sd(\hat{\rho}'_{BB}(i)))\}$, where $\alpha_{12} = 1 - \alpha_{11}, 0 \leq \alpha_{11} \leq 1$. Applying the asymptotical normality of $\hat{\rho}_i, i = 1,2$, $100(1-\alpha)\%BB$ calibrated confidence interval for $\rho_i, i = 1,2$ is

$$\left(\hat{\rho}'_i - z_{\hat{\beta}(\alpha_{11})/2} sd(\hat{\rho}'_{BB}(i))', \hat{\rho}'_i + z_{\hat{\beta}(\alpha_{12})/2} sd(\hat{\rho}'_{BB}(i))'\right), \quad i = 1,2. \tag{9}$$

2.8 Percentile Bootstrap (PB) Calibrated Confidence Intervals

Now call $\hat{\rho}_i^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i = 1,2$ the bootstrap distribution of $\hat{\rho}_i, i = 1,2$. Let $\hat{\rho}_i^*(1), \hat{\rho}_i^*(2), \cdots, \hat{\rho}_i^*(N), i = 1,2$ be the order statistics of $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i = 1,2$. Compute $\hat{\beta}(\alpha_{13}) = P\{\rho_i \leq \hat{\rho}_i^*([N(\frac{\alpha}{2})]\}$ and $\hat{\beta}(\alpha_{14}) = P\{\rho_i \leq \hat{\rho}_i^*([N(1-\frac{\alpha}{2})]\}$. Then utilizing the $100(\alpha/2)^{th}$ and $100(1-\alpha/2)^{th}$ percentage points of the bootstrap distribution, Then $100(1-\alpha)\%$ PB calibrated confidence interval for $\rho_i, i = 1,2$ is given by,

$$\left(\hat{\rho}_i^*\left(\left[N\left(\frac{\hat{\beta}(\alpha_{13})}{2}\right)\right]\right),\ \hat{\rho}_i^*\left(\left[N\left(\frac{\hat{\beta}(\alpha_{14})}{2}\right)\right]\right)\right), \quad i = 1,2 \tag{10}$$

where $[x]$ denotes the greatest integer less than or equal to x.

2.9 Bias-Corrected and Accelerated Bootstrap(BCaB) Calibrated Confidence Intervals

The bootstrap distribution $\hat{\rho}_{i1}^*, \hat{\rho}_{i2}^*, \cdots, \hat{\rho}_{iN}^*, i = 1,2$ may be biased, consequently the PB calibrated confidence interval of intensity method is designed to correct this potential bias of the bootstrap designed. Set $p_i = \frac{1}{N}\sum_{j=1}^{N} I(\hat{\rho}_{ij}^* < \hat{\rho}_i), i = 1,2$ where $I(\cdot)$ is the indicator function. Define $\hat{z}_i = \phi^{-1}(p_i), i = 1,2$ where ϕ^{-1} denotes the inverse function of the standard normal distribution ϕ. Except for correcting the potential bias of the bootstrap distribution, we can accelerate convergence of bootstrap distribution. Let $(\tilde{X}_i(k), \tilde{Y}_i(k), i = 1,2, k = 1,2,\cdots,n)$ denote the original samples with the k^{th} observation $(X_{ik}, Y_{ik}, i = 1,2)$ deleted, also $\hat{\rho}_{ik}, i = 1,2$ be the estimator of $\rho_i, i = 1,2$ calculated by using $(\tilde{X}_i(k), \tilde{Y}_i(k), i = 1,2)$.

Define $\tilde{\rho}_i = \frac{1}{n}\sum_{k=1}^{n} \hat{\rho}_{ik}, i = 1,2$ and

$$\hat{a}_i = \frac{\sum_{k=1}^{n} (\tilde{\rho}_i - \hat{\rho}_{ik})^3}{\left\{6\left(\sum_{k=1}^{n} (\tilde{\rho}_i - \hat{\rho}_{ik})^2\right)^{\left(\frac{3}{2}\right)}\right\}}, i = 1,2$$

$\hat{z}_i$ and $\hat{a}_i, i = 1,2$ are named bias-correction and acceleration respectively.

Also compute $\hat{\beta}(\alpha_{15}) = P\{\rho_i \leq \hat{\rho}_i^*([N\alpha_{i1}])\}$ and $\hat{\beta}(\alpha_{16}) = P\{\rho_i \leq \hat{\rho}_i^*([N\alpha_{i2}])\}$, where

$$\alpha_{i1} = \phi\left\{\hat{z}_i + \frac{(\hat{z}_i - z_{\alpha/2})}{1 - \hat{a}_i(\hat{z}_i - z_{\alpha/2})}\right\} \quad \alpha_{i2} = \phi\left\{\hat{z}_i + \frac{(\hat{z}_i + z_{\alpha/2})}{1 - \hat{a}_i(\hat{z}_i + z_{\alpha/2})}\right\}, \quad i = 1,2.$$

Thus $100(1-\alpha)\%$ BCaB calibrated confidence interval of intensities $\rho_i, i = 1,2$ is given by

$$(\hat{\rho}_i^*([N\alpha'_{i1}]), \hat{\rho}_i^*([N\alpha'_{i2}])). \quad i = 1,2. \tag{11}$$

where,

$$\alpha'_{i1} = \phi\left\{\hat{z}'_i + \frac{(\hat{z}'_i - z_{\hat{\beta}_{15}(\alpha)/2})}{1 - \hat{a}_i(\hat{z}'_i - z_{\hat{\beta}_{15}(\alpha)/2})}\right\} \quad \alpha'_{i2} = \phi\left\{\hat{z}'_i + \frac{(\hat{z}'_i + z_{\hat{\beta}_{16}(\alpha)/2})}{1 - \hat{a}_i(\hat{z}'_i + z_{\hat{\beta}_{16}(\alpha)/2})}\right\}, \quad i = 1,2.$$

3 Simulation Study

To evaluate performances of calibrated confidence intervals numerical simulation study was undertaken. It is observed that most statisticians assess performances of interval estimations in terms of coverage percentages or average lengths of confidence intervals. However, through simulation study in the research work, we find that larger coverage percentages of confidence interval may often be due to wider standard deviation of interval estimation methods. Moreover, narrower confidence intervals may often lead to smaller coverage percentages. Hence, both coverage percentage and average length are not efficient for appraising interval estimation methods. In order to overcome above two shortcomings, we consider a measure, named relative coverage, to evaluate performances of interval estimation methods.

Relative coverage is defined as the ratio of coverage percentage to average length of confidence interval. Larger relative coverage implies the better performances of the corresponding confidence intervals. Here we set a continuous distribution with mean $1/\lambda$ on inter-arrival time of X_1 and X_2 and a continuous distribution with mean $1/\mu_1$ on the service time Y_1 at node-1 and that of $1/\mu_2$ on Y_2 at node-2. Different levels of intensity parameters considered in the simulation study are shown in Table-1.

Table-1 : Different levels of intensity parameters considered in the simulation study

$\rho_1 < \rho_2$	$\rho_1 > \rho_2$
(1) Low=0.1 and Moderate=0.5	(1) Moderate=0.5 and Low=0.1
(2) Low=0.1 and High=0.9	(2) High=0.9 and Low=0.1
(3) Moderate=0.5 and High=0.9	(3) High=0.9 and Moderate=0.5

For each level of $\rho_i, i = 1,2$ random samples of inter-arrival times and service times $(X_{ij}, Y_{ij}, i = 1,2, j = 1,2,\cdots,n$ are drawn from $(X_i, Y_i, i = 1,2)$. Next $N = 1000$ bootstrap resamples each of size $n = 10$ and 29 are drawn from the original samples, as well as N=1000 BB replications are simulated for the original samples. According to equations (4) to (11) we obtain calibrated CAN, Exact-t, Boot-t, VST, SB, BB, PB and BCaB confidence intervals of intensities ρ_1 and ρ_2 with confidence level 90%. The above simulation process is replicated $N = 1000$ times and we compute coverage percentages, average lengths and relative coverage of the above mentioned confidence intervals. We utilize a PC Dual Core and apply Matlab 7.0.1 to accomplish all simulations.

Table-2 : Different queueing network models simulated for study

Queueing Networks type	Models simulated	C.V. of inter-arrival time for node-1	C.V. of inter- arrival time for node-2	C.V. of service time for node-1	C.V. of service time for node-2
M/G/1 to G/M/1	$M/E_4/1$ to $E_4/M/1$	1	1/2	1/2	1
	$M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$	1	>1	>1	1
G/G/1 to G/G/1	$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	1/2	>1	>1	1/2
	$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	1/2	<1	<1	1/2

Here C.V. represents coefficient of variation corresponding to the inter-arrival/service time distribution, M represents an exponential distribution, E_4 a 4-stage Erlang distribution, H_4^{Pe} a 4-stage hyper-exponential distribution and H_4^{P0} a 4-stage hypo-exponential distribution. The coverage percentages, average lengths and relative coverage's of intensities ρ_1 and ρ_2 based on simulation study for queuing network models (presented in Table 2) with short run for different calibrated interval estimation approaches are shown in Tables 3 to 6.

Table-3 : Simulation results: 90% calibrated confidence intervals of $M/E_4/1$ to $E_4/M/1$

Intensity Parameters	Estimation Approches	$n=10$		$n=29$		Coverage Percentages		Average Lengths		Relative Coverage	
		$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$n=10$	$n=29$	$n=10$	$n=29$	$n=10$	$n=29$
	CAN1	0.040	0.913	0.030	0.919	0.842	0.882	0.113	0.069	7.428	12.792
	CAN2	0.017	0.847	0.030	0.911	0.782	0.840	0.546	0.329	1.432	2.551
	tCAN1	0.018	0.950	0.022	0.936	0.941	0.921	0.194	0.082	4.858	11.239
	tCAN2	0.009	0.867	0.025	0.916	0.807	0.856	0.700	0.352	1.152	2.433
	Boott1	0.016	0.891	0.016	0.900	0.832	0.869	0.113	0.067	7.338	12.965
	Boott2	0.014	0.816	0.022	0.893	0.719	0.827	0.450	0.311	1.599	2.661
	VST1	0.083	0.934	0.052	0.937	0.797	0.864	0.101	0.066	**7.902**	**13.052**
$\rho_1 = 0.1$	VST2	0.077	0.915	0.062	0.946	0.780	0.847	0.541	0.329	1.442	2.578
&	SB1	0.023	0.934	0.025	0.930	0.907	0.911	0.158	0.077	5.744	11.900
$\rho_2 = 0.5$	SB2	0.019	0.846	0.028	0.910	0.780	0.845	0.532	0.330	1.466	2.558
	BB1	0.039	0.915	0.029	0.918	0.849	0.882	0.118	0.069	7.221	12.762
	BB2	0.026	0.834	0.031	0.906	0.746	0.830	0.473	0.317	1.579	2.618
	PB1	0.129	0.963	0.076	0.955	0.779	0.842	0.138	0.070	5.658	12.011
	PB2	0.045	0.870	0.042	0.926	0.764	0.840	0.445	0.313	1.718	**2.685**
	BCaB1	0.118	0.956	0.076	0.953	0.773	0.841	0.126	0.068	6.148	12.310
	BCaB2	0.048	0.870	0.041	0.930	0.757	0.842	0.433	0.315	**1.750**	2.669
	CAN1	0.043	0.918	0.044	0.922	0.855	0.857	0.114	0.065	7.473	13.087
	CAN2	0.017	0.867	0.026	0.897	0.759	0.837	0.979	0.586	0.776	1.429
	tCAN1	0.018	0.948	0.029	0.937	0.936	0.916	0.194	0.079	4.836	11.603
	tCAN2	0.011	0.881	0.024	0.901	0.781	0.844	1.205	0.616	0.648	1.370
	Boott1	0.019	0.895	0.027	0.898	0.843	0.856	0.115	0.064	7.337	13.454
	Boott2	0.017	0.845	0.019	0.869	0.739	0.828	0.812	0.539	0.910	1.537
$\rho_1 = 01.$	VST1	0.081	0.932	0.069	0.937	0.794	0.846	0.102	0.062	**7.783**	**13.578**
&	VST2	0.074	0.916	0.069	0.938	0.789	0.849	0.959	0.564	0.823	1.504
$\rho_2 = 0.9$	SB1	0.028	0.935	0.032	0.933	0.912	0.898	0.155	0.074	5.875	12.065
	SB2	0.023	0.863	0.027	0.891	0.750	0.829	0.924	0.575	0.811	1.442
	BB1	0.041	0.924	0.041	0.923	0.868	0.861	0.120	0.066	7.210	12.999
	BB2	0.029	0.854	0.027	0.891	0.722	0.821	0.834	0.563	0.866	1.458
	PB1	0.107	0.962	0.092	0.955	0.766	0.820	0.140	0.068	5.479	11.978
	PB2	0.046	0.881	0.045	0.908	0.740	0.829	0.793	0.530	0.933	1.564
	BCaB1	0.104	0.961	0.087	0.951	0.759	0.824	0.132	0.066	5.744	12.424
	BCaB2	0.047	0.878	0.049	0.907	0.732	0.825	0.771	0.518	**0.950**	**1.593**
	CAN1	0.035	0.920	0.038	0.928	0.872	0.867	0.595	0.342	1.467	2.535
	CAN2	0.019	0.836	0.021	0.894	0.743	0.853	0.103	0.068	7.230	12.556
	tCAN1	0.013	0.953	0.026	0.938	0.955	0.911	1.034	0.406	0.924	2.247
	tCAN2	0.011	0.856	0.018	0.899	0.773	0.863	0.130	0.072	5.963	11.931
	Boott1	0.014	0.894	0.023	0.908	0.855	0.857	0.583	0.334	1.467	2.563
	Boott2	0.016	0.805	0.015	0.877	0.695	0.846	0.085	0.063	8.203	13.375
$\rho_1 = 0.5$	VST1	0.073	0.944	0.057	0.935	0.834	0.834	0.538	0.325	**1.549**	**2.565**
&	VST2	0.070	0.915	0.043	0.930	0.798	0.881	0.108	0.071	7.397	12.484
$\rho_2 = 0.1$	SB1	0.020	0.941	0.032	0.934	0.932	0.896	0.825	0.377	1.130	2.379
	SB2	0.021	0.835	0.021	0.893	0.736	0.847	0.100	0.068	7.348	12.543
	BB1	0.038	0.922	0.038	0.930	0.873	0.870	0.608	0.343	1.437	2.534
	BB2	0.029	0.823	0.023	0.888	0.720	0.839	0.089	0.065	8.123	12.919
	PB1	0.115	0.964	0.087	0.951	0.780	0.812	0.708	0.341	1.102	2.384
	PB2	0.045	0.863	0.032	0.910	0.745	0.861	0.085	0.063	8.777	13.723
	BCaB1	0.105	0.959	0.077	0.949	0.783	0.820	0.654	0.337	1.198	2.432
	BCaB2	0.046	0.864	0.032	0.909	0.739	0.850	0.083	0.062	**8.877**	**13.763**
	CAN1	0.050	0.924	0.056	0.937	0.843	0.870	0.562	0.328	1.501	2.650
	CAN2	0.021	0.841	0.024	0.893	0.760	0.856	0.946	0.594	0.803	1.441
	tCAN1	0.018	0.948	0.040	0.950	0.936	0.926	0.964	0.393	0.971	2.359
	tCAN2	0.009	0.859	0.020	0.895	0.785	0.868	1.243	0.632	0.632	1.373
	Boott1	0.018	0.879	0.026	0.913	0.820	0.865	0.542	0.333	1.514	2.595
	Boott2	0.015	0.809	0.015	0.879	0.731	0.845	0.792	0.567	0.923	1.490
	VST1	0.094	0.932	0.068	0.947	0.794	0.852	0.487	0.320	**1.630**	2.664
$\rho_1 = 05$	VST2	0.087	0.910	0.065	0.931	0.787	0.853	0.930	0.573	0.846	1.490
&	SB1	0.031	0.940	0.045	0.945	0.914	0.905	0.772	0.367	1.184	2.463
$\rho_2 = 09$	SB2	0.022	0.837	0.023	0.891	0.749	0.855	0.924	0.594	0.810	1.440
	BB1	0.050	0.919	0.057	0.934	0.852	0.868	0.574	0.324	1.484	**2.679**
	BB2	0.034	0.826	0.026	0.891	0.705	0.848	0.804	0.572	0.877	1.483
	PB1	0.129	0.967	0.095	0.960	0.759	0.834	0.706	0.351	1.075	2.378
	PB2	0.052	0.861	0.045	0.908	0.746	0.859	0.769	0.536	0.970	1.602
	BCaB1	0.119	0.958	0.090	0.959	0.746	0.836	0.631	0.343	1.183	2.440
	BCaB2	0.052	0.866	0.047	0.909	0.746	0.850	0.761	0.530	**0.980**	**1.605**
	CAN1	0.033	0.906	0.052	0.934	0.841	0.846	1.046	0.597	0.804	1.416
	CAN2	0.022	0.864	0.019	0.898	0.768	0.837	0.102	0.068	7.495	12.351
	tCAN1	0.008	0.944	0.035	0.945	0.947	0.912	1.913	0.715	0.495	1.276
	tCAN2	0.011	0.878	0.016	0.905	0.784	0.852	0.130	0.073	6.016	11.733
	Boott1	0.007	0.878	0.028	0.919	0.835	0.851	1.029	0.612	0.812	1.391
	Boott2	0.019	0.842	0.012	0.882	0.747	0.852	0.087	0.064	8.568	13.347
$\rho_1 = 09$	VST1	0.071	0.925	0.073	0.946	0.795	0.824	0.936	0.571	**0.849**	**1.444**
&	VST2	0.079	0.925	0.058	0.938	0.797	0.862	0.102	0.065	7.821	13.206
$\rho_2 = 01$	SB1	0.018	0.932	0.041	0.941	0.916	0.877	1.454	0.668	0.630	1.313
	SB2	0.025	0.861	0.023	0.895	0.761	0.830	0.099	0.066	7.691	12.666
	BB1	0.031	0.910	0.052	0.935	0.849	0.847	1.095	0.599	0.775	1.413
	BB2	0.031	0.849	0.025	0.893	0.742	0.823	0.089	0.063	8.349	13.002
	PB1	0.115	0.953	0.096	0.967	0.778	0.816	1.163	0.662	0.669	1.233
	PB2	0.043	0.882	0.042	0.917	0.760	0.848	0.087	0.061	8.729	13.991
	BCaB1	0.108	0.949	0.088	0.954	0.777	0.809	1.099	0.610	0.707	1.326
	BCaB2	0.044	0.878	0.045	0.912	0.756	0.832	0.084	0.059	**8.948**	**14.141**
	CAN1	0.047	0.911	0.030	0.936	0.837	0.890	1.018	0.643	0.822	1.384
	CAN2	0.019	0.860	0.028	0.919	0.765	0.865	0.536	0.342	1.426	2.526
	tCAN1	0.023	0.942	0.016	0.949	0.928	0.932	1.699	0.791	0.546	1.179
	tCAN2	0.014	0.878	0.022	0.923	0.800	0.881	0.650	0.369	1.232	2.391
	Boott1	0.021	0.886	0.015	0.912	0.820	0.892	1.014	0.624	0.808	1.430
	Boott2	0.018	0.830	0.022	0.905	0.755	0.861	0.441	0.323	1.713	2.666
	VST1	0.085	0.927	0.050	0.949	0.793	0.883	0.915	0.614	**0.867**	**1.438**
$\rho_1 = 0.9$	VST2	0.083	0.923	0.069	0.946	0.805	0.864	0.523	0.325	1.539	2.656
&	SB1	0.035	0.932	0.024	0.943	0.902	0.915	1.385	0.713	0.651	1.284
$\rho_2 = 0.5$	SB2	0.022	0.858	0.030	0.918	0.761	0.860	0.519	0.337	1.467	2.549
	BB1	0.046	0.915	0.028	0.935	0.849	0.892	1.062	0.647	0.799	1.379
	BB2	0.027	0.848	0.031	0.913	0.730	0.844	0.469	0.326	1.556	2.588
	PB1	0.131	0.953	0.082	0.966	0.742	0.868	1.178	0.664	0.630	1.308
	PB2	0.044	0.880	0.049	0.931	0.769	0.865	0.446	0.316	1.725	2.734
	BCaB1	0.116	0.954	0.077	0.958	0.748	0.864	1.143	0.626	0.654	1.381
	BCaB2	0.047	0.879	0.052	0.932	0.753	0.859	0.432	0.314	**1.742**	**2.738**

Table-4 : Simulation results: 90% calibrated confidence intervals of $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$.

Intensity Parameters	Estimation Approches	$n=10$		$n=29$		Coverage Percentages		Average Lengths		Relative Coverage	
		$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$n=10$	$n=29$	$n=10$	$n=29$	$n=10$	$n=29$
	CAN1	0.034	0.881	0.029	0.907	0.844	0.885	0.118	0.071	7.153	12.388
	CAN2	0.019	0.852	0.037	0.902	0.773	0.852	0.554	0.331	1.395	2.578
	tCAN1	0.014	0.922	0.023	0.925	0.926	0.929	0.200	0.084	4.627	11.103
	tCAN2	0.010	0.876	0.032	0.909	0.804	0.869	0.724	0.354	1.110	2.456
	Boott1	0.013	0.864	0.018	0.880	0.787	0.845	0.114	0.067	6.924	**12.690**
	Boott2	0.014	0.830	0.030	0.878	0.735	0.825	0.469	0.306	1.568	**2.695**
	VST1	0.083	0.899	0.052	0.923	0.761	0.856	0.104	0.068	**7.341**	12.531
$\rho_1=0.1$	VST2	0.079	0.917	0.072	0.944	0.779	0.850	0.543	0.331	1.435	2.566
&	SB1	0.024	0.902	0.029	0.917	0.891	0.914	0.157	0.077	5.668	11.881
$\rho_2=0.5$	SB2	0.018	0.858	0.037	0.903	0.772	0.854	0.566	0.331	1.365	2.579
	BB1	0.032	0.885	0.032	0.906	0.849	0.881	0.123	0.070	6.891	12.554
	BB2	0.030	0.845	0.038	0.897	0.746	0.836	0.484	0.319	1.541	2.618
	PB1	0.114	0.933	0.078	0.946	0.754	0.849	0.123	0.071	6.135	12.031
	PB2	0.049	0.883	0.051	0.918	0.763	0.846	0.465	0.315	1.639	2.685
	BCaB1	0.108	0.925	0.076	0.944	0.742	0.848	0.116	0.069	6.386	12.217
	BCaB2	0.049	0.874	0.055	0.924	0.743	0.845	0.449	0.316	**1.657**	2.674
	CAN1	0.044	0.905	0.027	0.928	0.836	0.888	0.116	0.074	7.212	11.937
	CAN2	0.024	0.854	0.034	0.898	0.761	0.833	0.970	0.590	0.785	1.412
	tCAN1	0.022	0.940	0.017	0.940	0.930	0.925	0.191	0.089	4.864	10.415
	tCAN2	0.015	0.876	0.028	0.899	0.799	0.845	1.222	0.630	0.654	1.342
	Boott1	0.019	0.874	0.011	0.914	0.814	0.883	0.112	0.074	7.296	11.885
	Boott2	0.015	0.818	0.023	0.880	0.692	0.815	0.822	0.561	0.842	1.452
	VST1	0.084	0.922	0.059	0.941	0.800	0.857	0.105	0.068	**7.651**	**12.606**
$\rho_1=0.1$	VST2	0.085	0.923	0.080	0.928	0.751	0.818	0.959	0.556	0.783	1.472
&	SB1	0.028	0.926	0.021	0.936	0.903	0.914	0.158	0.082	5.708	11.094
$\rho_2=0.9$	SB2	0.025	0.851	0.033	0.896	0.765	0.834	0.963	0.591	0.795	1.411
	BB1	0.039	0.911	0.028	0.929	0.854	0.890	0.123	0.074	6.934	11.999
	BB2	0.036	0.834	0.039	0.892	0.726	0.817	0.832	0.560	0.873	1.460
	PB1	0.103	0.952	0.079	0.960	0.793	0.847	0.135	0.074	5.854	11.376
	PB2	0.056	0.885	0.064	0.910	0.739	0.813	0.829	0.532	0.891	1.529
	BCaB1	0.092	0.949	0.073	0.955	0.793	0.850	0.130	0.072	6.111	11.782
	BCaB2	0.053	0.876	0.062	0.910	0.735	0.804	0.803	0.531	**0.916**	1.514
	CAN1	0.038	0.899	0.030	0.910	0.829	0.880	0.582	0.351	1.425	2.505
	CAN2	0.027	0.891	0.029	0.903	0.782	0.849	0.112	0.068	6.997	12.562
	tCAN1	0.008	0.931	0.021	0.924	0.913	0.918	1.073	0.415	0.851	2.211
	tCAN2	0.011	0.912	0.022	0.912	0.829	0.878	0.150	0.074	5.509	11.839
	Boott1	0.009	0.872	0.012	0.895	0.816	0.888	0.574	0.350	1.420	2.536
	Boott2	0.014	0.861	0.020	0.887	0.774	0.858	0.099	0.064	7.802	13.379
	VST1	0.084	0.918	0.064	0.930	0.785	0.863	0.516	0.326	**1.521**	**2.648**
$\rho_1=0.5$	VST2	0.097	0.944	0.066	0.942	0.788	0.875	0.104	0.066	7.559	13.165
&	SB1	0.022	0.918	0.023	0.921	0.889	0.911	0.797	0.392	1.116	2.321
$\rho_2=0.1$	SB2	0.028	0.890	0.027	0.903	0.784	0.852	0.112	0.068	7.026	12.456
	BB1	0.039	0.901	0.030	0.913	0.836	0.884	0.601	0.353	1.391	2.503
	BB2	0.039	0.877	0.032	0.897	0.748	0.834	0.097	0.064	7.687	12.935
	PB1	0.127	0.942	0.079	0.950	0.779	0.859	0.620	0.352	1.256	2.437
	PB2	0.071	0.918	0.044	0.921	0.789	0.870	0.098	0.064	**8.037**	**13.650**
	BCaB1	0.119	0.942	0.072	0.946	0.779	0.860	0.603	0.346	1.292	2.487
	BCaB2	0.068	0.916	0.050	0.922	0.780	0.855	0.097	0.063	8.007	13.650
	CAN1	0.036	0.895	0.035	0.922	0.828	0.874	0.591	0.354	1.401	2.467
	CAN2	0.021	0.864	0.023	0.904	0.788	0.850	1.010	0.635	0.780	1.338
	tCAN1	0.012	0.934	0.024	0.936	0.908	0.920	1.029	0.422	0.883	2.180
	tCAN2	0.014	0.883	0.021	0.908	0.832	0.863	1.260	0.673	0.660	1.282
	Boott1	0.012	0.867	0.018	0.902	0.781	0.862	0.567	0.347	1.377	2.481
	Boott2	0.015	0.834	0.021	0.880	0.753	0.826	0.849	0.571	0.887	1.446
	VST1	0.075	0.921	0.059	0.942	0.766	0.857	0.542	0.342	**1.412**	**2.509**
$\rho_1=0.5$	VST2	0.079	0.923	0.059	0.937	0.800	0.847	0.994	0.615	0.805	1.376
&	SB1	0.020	0.921	0.028	0.932	0.877	0.907	0.817	0.395	1.074	2.298
$\rho_2=0.9$	SB2	0.024	0.868	0.023	0.903	0.794	0.849	1.002	0.635	0.792	1.337
	BB1	0.035	0.897	0.037	0.922	0.831	0.871	0.611	0.351	1.360	2.480
	BB2	0.034	0.850	0.023	0.902	0.754	0.843	0.869	0.620	0.868	1.359
	PB1	0.134	0.949	0.080	0.956	0.726	0.840	0.648	0.365	1.120	2.304
	PB2	0.056	0.890	0.047	0.918	0.791	0.837	0.846	0.571	0.935	1.467
	BCaB1	0.119	0.945	0.077	0.952	0.721	0.837	0.620	0.354	1.163	2.367
	BCaB2	0.057	0.886	0.048	0.918	0.778	0.830	0.823	0.565	**0.946**	**1.468**
	CAN1	0.039	0.896	0.037	0.908	0.837	0.863	1.056	0.607	0.793	1.423
	CAN2	0.011	0.860	0.022	0.917	0.770	0.876	0.120	0.073	6.403	12.003
	tCAN1	0.013	0.927	0.025	0.931	0.932	0.910	1.835	0.735	0.508	1.238
	tCAN2	0.006	0.881	0.016	0.926	0.812	0.894	0.158	0.080	5.149	11.109
	Boott1	0.011	0.872	0.020	0.889	0.820	0.853	1.042	0.591	0.787	**1.443**
	Boott2	0.008	0.827	0.015	0.901	0.750	0.888	0.097	0.069	7.710	12.913
	VST1	0.073	0.910	0.055	0.934	0.788	0.858	0.971	0.604	**0.811**	1.420
$\rho_1=0.9$	VST2	0.079	0.926	0.066	0.951	0.792	0.891	0.111	0.069	7.161	12.986
&	SB1	0.021	0.913	0.029	0.924	0.903	0.898	1.464	0.684	0.617	1.312
$\rho_2=0.1$	SB2	0.010	0.859	0.023	0.920	0.773	0.880	0.122	0.073	6.340	12.042
	BB1	0.034	0.897	0.036	0.909	0.846	0.868	1.115	0.610	0.759	1.424
	BB2	0.018	0.845	0.025	0.914	0.744	0.869	0.104	0.070	7.130	12.437
	PB1	0.115	0.939	0.070	0.956	0.773	0.861	1.134	0.656	0.681	1.312
	PB2	0.048	0.888	0.046	0.933	0.771	0.874	0.095	0.067	8.131	**13.120**
	BCaB1	0.107	0.937	0.067	0.951	0.780	0.854	1.102	0.636	0.708	1.342
	BCaB2	0.049	0.889	0.039	0.934	0.764	0.883	0.094	0.068	**8.158**	13.023
	CAN1	0.026	0.911	0.037	0.921	0.843	0.875	1.139	0.633	0.740	1.383
	CAN2	0.024	0.873	0.021	0.897	0.798	0.846	0.572	0.353	1.396	2.397
	tCAN1	0.010	0.937	0.024	0.934	0.922	0.921	1.907	0.757	0.483	1.217
	tCAN2	0.014	0.888	0.016	0.901	0.824	0.858	0.722	0.383	1.142	2.239
	Boott1	0.009	0.885	0.017	0.895	0.830	0.843	1.083	0.618	0.766	1.364
	Boott2	0.016	0.846	0.016	0.884	0.743	0.843	0.491	0.328	1.514	2.568
	VST1	0.074	0.924	0.063	0.935	0.782	0.840	0.979	0.599	**0.799**	**1.402**
$\rho_1=0.9$	VST2	0.078	0.936	0.050	0.942	0.780	0.870	0.581	0.359	1.342	2.425
&	SB1	0.016	0.924	0.028	0.927	0.897	0.901	1.513	0.703	0.593	1.281
$\rho_2=0.5$	SB2	0.026	0.877	0.024	0.896	0.800	0.842	0.571	0.347	1.401	2.428
	BB1	0.025	0.908	0.035	0.921	0.848	0.880	1.174	0.637	0.722	1.380
	BB2	0.029	0.864	0.026	0.893	0.773	0.837	0.516	0.334	1.497	2.510
	PB1	0.121	0.953	0.082	0.956	0.777	0.834	1.204	0.656	0.645	1.272
	PB2	0.052	0.891	0.037	0.917	0.764	0.851	0.486	0.325	1.573	2.619
	BCaB1	0.111	0.945	0.083	0.949	0.771	0.827	1.117	0.624	0.690	1.326
	BCaB2	0.048	0.894	0.035	0.914	0.770	0.844	0.488	0.321	**1.579**	**2.626**

Table-5 : Simulation results: 90% calibrated confidence intervals of $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$.

Intensity Parameters	Estimation Approches	$n=10$		$n=29$		Coverage Percentages		Average Lengths		Relative Coverage	
		$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$\beta(\alpha_i)$	$\beta(1-\alpha_i)$	$n=10$	$n=29$	$n=10$	$n=29$	$n=10$	$n=29$
$\rho_1=0.1$ & $\rho_2=0.5$	CAN1	0.029	0.888	0.030	0.910	0.830	0.857	0.076	0.047	10.891	18.421
	CAN2	0.028	0.902	0.057	0.935	0.837	0.874	0.389	0.223	2.151	3.925
	tCAN1	0.020	0.912	0.023	0.912	0.875	0.866	0.094	0.050	9.295	17.310
	tCAN2	0.018	0.917	0.045	0.942	0.884	0.895	0.490	0.244	1.805	3.672
	Boott1	0.021	0.866	0.023	0.894	0.793	0.842	0.068	0.044	11.664	19.064
	Boott2	0.019	0.879	0.040	0.922	0.796	0.874	0.350	0.222	2.272	3.943
	VST1	0.069	0.923	0.051	0.928	0.811	0.851	0.071	0.045	11.424	18.881
	VST2	0.064	0.920	0.077	0.949	0.790	0.866	0.351	0.215	2.250	4.029
	SB1	0.037	0.887	0.030	0.908	0.818	0.851	0.073	0.046	11.210	18.426
	SB2	0.029	0.905	0.056	0.936	0.839	0.876	0.395	0.224	2.121	3.902
	BB1	0.041	0.873	0.034	0.906	0.778	0.836	0.066	0.044	11.804	18.897
	BB2	0.040	0.893	0.056	0.931	0.795	0.869	0.342	0.216	**2.322**	**4.031**
	PB1	0.068	0.920	0.054	0.921	0.809	0.835	0.068	0.042	11.851	**19.670**
	PB2	0.065	0.928	0.077	0.948	0.802	0.860	0.354	0.221	2.263	3.894
	BCaB1	0.066	0.919	0.051	0.920	0.815	0.832	0.068	0.043	**11.979**	19.533
	BCaB2	0.070	0.925	0.076	0.947	0.786	0.853	0.347	0.219	2.267	**3.888**
$\rho_1=0.1$ & $\rho_2=0.9$	CAN1	0.033	0.892	0.033	0.917	0.835	0.870	0.076	0.046	11.002	19.017
	CAN2	0.054	0.908	0.042	0.925	0.830	0.868	0.648	0.410	1.281	2.119
	tCAN1	0.024	0.909	0.027	0.921	0.876	0.891	0.092	0.049	9.565	18.166
	tCAN2	0.037	0.926	0.037	0.933	0.901	0.889	0.808	0.441	1.115	2.017
	Boott1	0.028	0.856	0.025	0.902	0.768	0.864	0.065	0.044	11.780	19.715
	Boott2	0.041	0.885	0.032	0.905	0.803	0.842	0.601	0.392	1.336	2.150
	VST1	0.074	0.910	0.058	0.932	0.788	0.857	0.069	0.044	11.428	19.683
	VST2	0.097	0.931	0.066	0.945	0.790	0.852	0.586	0.395	1.347	2.155
	SB1	0.036	0.893	0.036	0.912	0.834	0.864	0.075	0.045	11.177	19.346
	SB2	0.057	0.910	0.039	0.919	0.835	0.869	0.655	0.410	1.275	2.120
	BB1	0.042	0.878	0.037	0.910	0.788	0.862	0.067	0.043	11.817	19.945
	BB2	0.070	0.899	0.045	0.917	0.777	0.844	0.571	0.388	**1.361**	**2.176**
	PB1	0.071	0.912	0.059	0.930	0.792	0.852	0.067	0.042	11.860	**20.091**
	PB2	0.100	0.937	0.070	0.950	0.807	0.845	0.631	0.405	1.279	2.084
	BCaB1	0.069	0.910	0.056	0.930	0.799	0.851	0.066	0.043	**12.019**	19.968
	BCaB2	0.097	0.934	0.070	0.951	0.801	0.846	0.622	0.406	1.289	2.086
$\rho_1=0.5$ & $\rho_2=0.1$	CAN1	0.029	0.900	0.031	0.921	0.837	0.891	0.386	0.237	2.171	3.758
	CAN2	0.033	0.899	0.041	0.937	0.826	0.900	0.075	0.047	10.975	19.200
	tCAN1	0.016	0.917	0.025	0.929	0.890	0.910	0.488	0.257	1.824	3.544
	tCAN2	0.022	0.920	0.037	0.946	0.879	0.919	0.095	0.051	9.257	18.162
	Boott1	0.021	0.879	0.024	0.901	0.820	0.861	0.346	0.224	2.371	3.850
	Boott2	0.027	0.870	0.034	0.919	0.785	0.878	0.066	0.045	11.817	19.686
	VST1	0.081	0.926	0.050	0.940	0.806	0.872	0.339	0.233	2.377	3.750
	VST2	0.079	0.916	0.061	0.954	0.782	0.886	0.066	0.046	11.848	19.466
	SB1	0.032	0.899	0.032	0.921	0.829	0.889	0.378	0.235	2.196	3.775
	SB2	0.037	0.897	0.040	0.935	0.819	0.899	0.075	0.047	10.945	19.137
	BB1	0.045	0.891	0.034	0.912	0.784	0.878	0.332	0.224	2.362	**3.917**
	BB2	0.049	0.884	0.047	0.932	0.778	0.883	0.065	0.044	**12.020**	**19.926**
	PB1	0.067	0.924	0.054	0.939	0.813	0.872	0.343	0.225	2.368	3.882
	PB2	0.089	0.929	0.065	0.957	0.787	0.886	0.068	0.047	11.565	18.968
	BCaB1	0.067	0.914	0.053	0.938	0.802	0.873	0.332	0.224	**2.418**	3.895
	BCaB2	0.079	0.924	0.069	0.953	0.790	0.880	0.068	0.045	11.652	19.401
$\rho_1=0.5$ & $\rho_2=0.9$	CAN1	0.037	0.885	0.028	0.912	0.803	0.878	0.360	0.232	2.231	3.778
	CAN2	0.035	0.904	0.044	0.938	0.861	0.889	0.678	0.421	1.270	2.112
	tCAN1	0.021	0.906	0.026	0.918	0.856	0.891	0.455	0.246	1.883	3.623
	tCAN2	0.026	0.923	0.039	0.944	0.902	0.915	0.837	0.452	1.078	2.025
	Boott1	0.029	0.860	0.021	0.890	0.753	0.873	0.321	0.218	2.348	4.000
	Boott2	0.027	0.878	0.035	0.924	0.799	0.892	0.606	0.408	1.318	2.186
	VST1	0.091	0.918	0.049	0.929	0.764	0.881	0.322	0.225	2.373	3.923
	VST2	0.079	0.926	0.061	0.951	0.798	0.892	0.604	0.409	1.322	2.183
	SB1	0.040	0.883	0.028	0.912	0.798	0.881	0.352	0.232	2.264	3.796
	SB2	0.035	0.906	0.044	0.941	0.862	0.898	0.690	0.426	1.250	2.109
	BB1	0.050	0.872	0.031	0.906	0.763	0.863	0.313	0.221	**2.438**	3.909
	BB2	0.046	0.890	0.049	0.933	0.814	0.876	0.597	0.399	**1.365**	**2.197**
	PB1	0.079	0.913	0.041	0.926	0.772	0.878	0.323	0.220	2.392	3.992
	PB2	0.082	0.934	0.068	0.957	0.804	0.881	0.629	0.420	1.278	2.096
	BCaB1	0.072	0.907	0.043	0.923	0.769	0.875	0.320	0.216	2.405	**4.042**
	BCaB2	0.082	0.929	0.068	0.954	0.797	0.880	0.614	0.414	1.299	2.128
$\rho_1=0.9$ & $\rho_2=0.1$	CAN1	0.019	0.890	0.032	0.919	0.835	0.892	0.727	0.422	1.148	2.116
	CAN2	0.045	0.916	0.030	0.926	0.835	0.899	0.074	0.048	11.265	18.897
	tCAN1	0.013	0.906	0.026	0.923	0.877	0.905	0.896	0.452	0.979	2.001
	tCAN2	0.028	0.930	0.026	0.931	0.879	0.915	0.093	0.051	9.444	17.943
	Boott1	0.013	0.866	0.025	0.901	0.784	0.883	0.636	0.400	1.233	2.210
	Boott2	0.031	0.892	0.023	0.914	0.806	0.903	0.069	0.046	11.746	19.735
	VST1	0.070	0.915	0.055	0.937	0.789	0.889	0.628	0.406	1.257	2.187
	VST2	0.082	0.934	0.048	0.947	0.798	0.911	0.067	0.047	11.847	19.534
	SB1	0.018	0.888	0.031	0.918	0.833	0.891	0.727	0.422	1.146	2.113
	SB2	0.044	0.921	0.029	0.929	0.836	0.903	0.076	0.048	10.980	18.721
	BB1	0.032	0.881	0.036	0.914	0.798	0.871	0.624	0.401	1.280	2.175
	BB2	0.052	0.904	0.036	0.924	0.794	0.884	0.066	0.045	**11.961**	19.625
	PB1	0.058	0.911	0.052	0.939	0.801	0.895	0.611	0.405	1.312	2.211
	PB2	0.086	0.936	0.052	0.949	0.795	0.908	0.070	0.046	11.372	19.597
	BCaB1	0.058	0.906	0.057	0.932	0.793	0.875	0.600	0.391	**1.321**	**2.236**
	BCaB2	0.080	0.938	0.056	0.939	0.802	0.895	0.071	0.044	11.368	**20.203**
$\rho_1=0.9$ & $\rho_2=0.5$	CAN1	0.024	0.910	0.032	0.921	0.854	0.863	0.733	0.420	1.165	2.055
	CAN2	0.040	0.897	0.033	0.934	0.827	0.890	0.365	0.242	2.265	3.682
	tCAN1	0.016	0.927	0.030	0.925	0.901	0.882	0.906	0.442	0.994	1.998
	tCAN2	0.024	0.918	0.030	0.942	0.882	0.910	0.467	0.260	1.887	3.497
	Boott1	0.020	0.891	0.026	0.894	0.839	0.833	0.649	0.389	**1.294**	2.144
	Boott2	0.029	0.874	0.024	0.927	0.792	0.894	0.333	0.238	2.380	3.751
	VST1	0.073	0.932	0.055	0.935	0.822	0.845	0.641	0.401	1.283	2.106
	VST2	0.082	0.921	0.053	0.948	0.794	0.885	0.330	0.232	2.406	3.808
	SB1	0.024	0.913	0.034	0.917	0.850	0.856	0.733	0.412	1.160	2.078
	SB2	0.043	0.901	0.034	0.938	0.827	0.892	0.369	0.244	2.240	3.651
	BB1	0.037	0.902	0.039	0.916	0.807	0.846	0.636	0.394	1.268	2.145
	BB2	0.053	0.889	0.036	0.931	0.783	0.882	0.323	0.231	**2.426**	**3.819**
	PB1	0.070	0.930	0.049	0.932	0.821	0.853	0.635	0.396	1.293	2.155
	PB2	0.081	0.924	0.061	0.950	0.806	0.873	0.340	0.231	2.374	3.778
	BCaB1	0.063	0.928	0.051	0.931	0.823	0.848	0.637	0.392	1.291	**2.163**
	BCaB2	0.082	0.927	0.060	0.950	0.807	0.877	0.341	0.231	2.368	3.800

Table-6 : Simulation results: 90% calibrated confidence intervals of $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$

Intensity Parameters	Estimation Approches	$n=10$		$n=29$		Coverage Percentages		Average Lengths		Relative Coverage	
		$\hat{\beta}(\alpha_i)$	$\hat{\beta}(1-\alpha_i)$	$\hat{\beta}(\alpha_i)$	$\hat{\beta}(1-\alpha_i)$	$n=10$	$n=29$	$n=10$	$n=29$	$n=10$	$n=29$
	CAN1	0.025	0.883	0.029	0.910	0.819	0.860	0.076	0.046	10.731	18.524
	CAN2	0.040	0.896	0.039	0.948	0.808	0.890	0.367	0.246 2	.199	3.617
	tCAN1	0.012	0.904	0.023	0.916	0.851	0.881	0.099	0.050	8.564	17.580
	tCAN2	0.025	0.913	0.032	0.952	0.861	0.908	0.463	0.266	1.861	3.416
	Boott1	0.015	0.847	0.025	0.896	0.758	0.865	0.067	0.044	11.361	19.783
	Boott2	0.030	0.867	0.030	0.933	0.770	0.873	0.330	0.238	2.335	3.675
	VST1	0.070	0.916	0.056	0.932	0.781	0.875	0.069	0.044	11.279	19.755
$\rho_1=0.1$	VST2	0.087	0.921	0.066	0.957	0.766	0.864	0.328	0.229	2.335	**3.778**
&	SB1	0.026	0.878	0.031	0.909	0.812	0.858	0.075	0.046	10.833	18.739
$\rho_2=0.5$	SB2	0.041	0.898	0.041	0.947	0.809	0.885	0.373	0.244	2.172	3.622
	BB1	0.036	0.862	0.034	0.907	0.774	0.848	0.065	0.044	11.845	19.235
	BB2	0.052	0.879	0.046	0.945	0.764	0.876	0.320	0.233	**2.385**	3.767
	PB1	0.061	0.905	0.052	0.928	0.779	0.869	0.066	0.043	11.880	20.072
	PB2	0.089	0.928	0.067	0.954	0.777	0.855	0.343	0.233	2.268	3.676
	BCaB1	0.059	0.904	0.050	0.921	0.782	0.859	0.065	0.043	**11.951**	**20.169**
	BCaB2	0.085	0.921	0.066	0.957	0.774	0.855	0.335	0.235	2.307	3.643
	CAN1	0.030	0.894	0.033	0.913	0.836	0.863	0.076	0.046	10.972	18.734
	CAN2	0.044	0.895	0.023	0.932	0.815	0.889	0.639	0.448	1.276	1.983
	tCAN1	0.017	0.917	0.029	0.920	0.870	0.882	0.097	0.049	8.975	17.953
	tCAN2	0.031	0.921	0.023	0.938	0.885	0.904	0.803	0.475	1.102	1.904
	Boott1	0.021	0.875	0.028	0.903	0.777	0.851	0.069	0.044	11.267	19.351
	Boott2	0.035	0.870	0.020	0.914	0.778	0.873	0.576	0.417	1.351	**2.092**
	VST1	0.064	0.921	0.054	0.930	0.785	0.853	0.072	0.045	10.930	19.105
$\rho_1=0.1$	VST2	0.096	0.920	0.045	0.942	0.781	0.871	0.562	0.420	1.389	2.071
&	SB1	0.033	0.893	0.032	0.915	0.828	0.864	0.075	0.046	11.102	18.654
$\rho_2=0.9$	SB2	0.046	0.895	0.027	0.934	0.817	0.888	0.643	0.444	1.271	2.001
	BB1	0.040	0.883	0.036	0.911	0.792	0.855	0.067	0.044	**11.856**	19.340
	BB2	0.064	0.881	0.028	0.924	0.767	0.869	0.549	0.419	**1.398**	2.073
	PB1	0.061	0.917	0.054	0.932	0.791	0.854	0.068	0.044	11.590	**19.522**
	PB2	0.099	0.933	0.049	0.950	0.792	0.879	0.607	0.422	1.305	2.083
	BCaB1	0.056	0.914	0.059	0.932	0.803	0.837	0.068	0.043	11.799	19.377
	BCaB2	0.099	0.926	0.050	0.949	0.781	0.875	0.587	0.419	1.330	2.089
	CAN1	0.027	0.891	0.037	0.916	0.819	0.852	0.387	0.228	2.117	3.739
	CAN2	0.042	0.924	0.038	0.938	0.852	0.871	0.077	0.048	11.113	18.232
	tCAN1	0.016	0.915	0.029	0.922	0.862	0.872	0.492	0.247	1.753	3.534
	tCAN2	0.026	0.941	0.032	0.943	0.922	0.892	0.098	0.051	9.447	17.323
	Boott1	0.019	0.856	0.028	0.902	0.772	0.862	0.336	0.219	2.297	3.930
	Boott2	0.027	0.903	0.028	0.921	0.833	0.863	0.072	0.046	11.564	18.742
	VST1	0.059	0.922	0.052	0.934	0.808	0.869	0.369	0.227	2.192	3.837
$\rho_1=0.5$	VST2	0.095	0.940	0.066	0.953	0.803	0.859	0.066	0.045	**12.178**	**19.105**
&	SB1	0.025	0.887	0.036	0.915	0.816	0.851	0.387	0.228	2.107	3.735
$\rho_2=0.1$	SB2	0.042	0.925	0.039	0.938	0.859	0.876	0.078	0.048	11.022	18.349
	BB1	0.030	0.872	0.038	0.912	0.791	0.844	0.346	0.219	2.289	3.850
	BB2	0.061	0.920	0.044	0.933	0.803	0.853	0.067	0.045	11.921	18.953
	PB1	0.054	0.914	0.050	0.928	0.802	0.863	0.345	0.218	2.323	**3.955**
	PB2	0.098	0.946	0.072	0.958	0.811	0.850	0.072	0.047	11.243	18.243
	BCaB1	0.055	0.908	0.051	0.927	0.793	0.854	0.338	0.217	**2.348**	3.940
	BCaB2	0.095	0.942	0.072	0.954	0.811	0.840	0.071	0.046	11.483	18.445
	CAN1	0.031	0.896	0.037	0.931	0.830	0.875	0.382	0.236	2.171	3.706
	CAN2	0.034	0.904	0.040	0.911	0.843	0.863	0.697	0.401	1.209	2.153
	tCAN1	0.015	0.916	0.032	0.935	0.867	0.894	0.494	0.251	1.757	3.562
	tCAN2	0.020	0.924	0.033	0.916	0.902	0.883	0.895	0.432	1.007	2.045
	Boott1	0.021	0.873	0.031	0.914	0.790	0.868	0.343	0.223	2.303	3.885
	Boott2	0.025	0.879	0.029	0.894	0.791	0.854	0.628	0.386	1.260	**2.210**
	VST1	0.060	0.934	0.054	0.947	0.818	0.879	0.373	0.231	2.191	3.797
$\rho_1-0.5$	VST2	0.076	0.924	0.056	0.925	0.794	0.859	0.621	0.392	1.278	2.192
&	SB1	0.031	0.896	0.041	0.930	0.825	0.869	0.379	0.231	**2.178**	3.757
$\rho_2=0.9$	SB2	0.034	0.906	0.038	0.907	0.846	0.859	0.710	0.402	1.192	2.138
	BB1	0.040	0.885	0.039	0.925	0.786	0.863	0.337	0.225	2.333	3.833
	BB2	0.044	0.896	0.042	0.905	0.790	0.845	0.620	0.383	1.274	2.207
	PB1	0.062	0.924	0.051	0.947	0.811	0.883	0.348	0.231	2.327	3.814
	PB2	0.079	0.932	0.057	0.930	0.799	0.854	0.643	0.389	1.242	2.194
	BCaB1	0.060	0.922	0.053	0.945	0.813	0.873	0.346	0.229	**2.350**	3.813
	BCaB2	0.072	0.925	0.057	0.928	0.801	0.849	0.633	0.387	1.266	2.193
	CAN1	0.046	0.881	0.035	0.923	0.822	0.866	0.629	0.414	1.306	2.091
	CAN2	0.036	0.911	0.034	0.931	0.846	0.909	0.077	0.048	11.037	18.925
	tCAN1	0.028	0.898	0.031	0.930	0.872	0.886	0.780	0.442	1.118	2.004
	tCAN2	0.023	0.931	0.027	0.936	0.900	0.928	0.097	0.052	9.236	17.801
	Boott1	0.031	0.861	0.026	0.906	0.788	0.875	0.581	0.396	1.356	2.210
	Boott2	0.025	0.883	0.024	0.919	0.810	0.894	0.069	0.047	11.700	19.070
	VST1	0.082	0.904	0.054	0.942	0.787	0.883	0.591	0.406	1.331	2.175
$\rho_1=0.9$	VST2	0.070	0.928	0.054	0.944	0.819	0.888	0.070	0.046	**11.716**	19.277
&	SB1	0.047	0.878	0.034	0.923	0.814	0.866	0.622	0.415	1.310	2.087
$\rho_2=0.1$	SB2	0.032	0.914	0.031	0.931	0.855	0.910	0.080	0.049	10.745	18.656
	BB1	0.059	0.869	0.040	0.910	0.774	0.845	0.551	0.386	**1.405**	2.191
	BB2	0.041	0.901	0.037	0.929	0.808	0.895	0.069	0.046	11.660	**19.414**
	PB1	0.076	0.900	0.050	0.943	0.800	0.888	0.570	0.406	1.404	2.188
	PB2	0.072	0.935	0.056	0.950	0.813	0.893	0.072	0.047	11.288	19.065
	BCaB1	0.073	0.896	0.051	0.935	0.783	0.880	0.565	0.394	1.387	**2.234**
	BCaB2	0.071	0.932	0.057	0.947	0.808	0.884	0.071	0.046	11.396	19.167
	CAN1	0.027	0.919	0.036	0.934	0.866	0.876	0.717	0.428	1.208	2.047
	CAN2	0.032	0.900	0.042	0.934	0.826	0.885	0.379	0.233	2.181	3.791
	tCAN1	0.018	0.932	0.029	0.936	0.901	0.890	0.880	0.458	1.024	1.944
	tCAN2	0.020	0.924	0.038	0.937	0.899	0.896	0.487	0.248	1.847	3.615
	Boott1	0.021	0.895	0.025	0.921	0.851	0.873	0.639	0.418	1.333	2.088
	Boott2	0.023	0.875	0.033	0.924	0.806	0.870	0.340	0.229	2.368	3.807
	VST1	0.065	0.936	0.056	0.950	0.849	0.868	0.650	0.415	1.307	2.090
$\rho_1=0.9$	VST2	0.070	0.918	0.066	0.944	0.807	0.854	0.341	0.220	2.365	3.881
&	SB1	0.029	0.917	0.035	0.934	0.853	0.876	0.704	0.428	1.211	2.045
$\rho_2=0.5$	SB2	0.035	0.903	0.042	0.933	0.835	0.888	0.382	0.234	2.186	3.798
	BB1	0.036	0.906	0.038	0.928	0.812	0.856	0.627	0.408	1.294	**2.098**
	BB2	0.045	0.888	0.049	0.932	0.779	0.873	0.331	0.221	2.350	**3.947**
	PB1	0.056	0.935	0.052	0.951	0.853	0.870	0.647	0.421	1.318	2.068
	PB2	0.071	0.931	0.068	0.946	0.817	0.850	0.353	0.224	2.314	3.796
	BCaB1	0.056	0.931	0.052	0.949	0.850	0.866	0.636	0.418	**1.336**	2.074
	BCaB2	0.068	0.924	0.070	0.943	0.818	0.840	0.345	0.220	**2.371**	3.816

Note that:

(1) boldface denotes the greatest relative coverage among estimation approaches.
(2) Calibrated confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, Exact-t1, Boot-t1, VST1, SB1, BB1, PB1, BCaB1 and that of ρ_2 are denoted by CAN2 Exact-t2, Boot-t2 , VST2, SB2, BB2, PB2 and BCaB2.

According to the simulation results in Tables 3 to 6, we find that average lengths are decreasing with sample size n, but both coverage percentages and relative coverage are increasing with sample size n. From Tables 3 to 6, one can observe that the coverage percentage can approach to 90% when n increases to 29. Also some interesting results are summarized in Table 7, about queueing network models or estimation approaches give greater relative coverage.

Table-7 : Performances of the estimation approaches of intensities

Queueing Network Type	Queuing Network simulated	Queueing Network with greater relative coverage	Intensity Parameters	Estimation approach with greatest relative coverage	
				$n = 10$	$n = 29$
$M/G/1$ to $G/M/1$	$M/E_4/1$to $E_4/M/1$ and $M/E_4^{Pe}/1$ to $H_4^{Pe}/M/1$	$M/E_4/1$ to $E_4/M/1$	$\rho_1 = 0.1$ $\&\rho_2 = 0.5$	VST BCaB	VST PB
			$\rho_1 = 0.1$ $\& \rho_2 = 0.9$	VST BCaB	VST BCaB
			$\rho_1 = 0.5$ $\&\rho_2 = 0.1$	VST BCaB	VST BCaB
			$\rho_1 = 0.5$ $\&\rho_2 = 0.9$	VST BCaB	BB BCaB
			$\rho_1 = 0.9$ $\&\rho_2 = 0.1$	VST BCaB	VST BCaB
			$\rho_1 = 0.9$ $\&\rho_2 = 0.5$	VST BCaB	VST BCaB
$G/G/1$ to $G/G/1$	$E_4/H_4^{pe}/1$ to $H_4^{Pe}/E_4/1$ and $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$	$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	$\rho_1 = 0.1$ $\&\rho_2 = 0.5$	BCaB BB	PB BB
			$\rho_1 = 0.1$ $\& \rho_2 = 0.9$	BCaB BB	PB BB
			$\rho_1 = 0.5$ $\&\rho_2 = 0.1$	BCaB BB	BB BB
			$\rho_1 = 0.5$ $\&\rho_2 = 0.9$	BB BB	BCaB BB
			$\rho_1 = 0.9$ $\&\rho_2 = 0.1$	BCaB BB	BCaB BCaB
			$\rho_1 = 0.9$ $\&\rho_2 = 0.5$	Boott BB	BCaB BB

Based on Table 7, we note that:

(1) Under $M/G/1$ to $G/M/1$ queueing network models, the calibrated confidence intervals corresponding to queueing network models with inter-arrival/service time distribution of small CV (< 1) have greater relative coverage than those of large CV (> 1).
(2) Among the simulated $M/G/1$ to $G/M/1$ queueing network models, estimation approaches VST or BCaB calibrated confidence interval have the greatest relative coverage.
(3) Among the simulated $M/G/1$ to $G/M/1$ queueing network models, the calibrated confidence intervals of queueing network model $M/E_4/1$ to $E_4/M/1$ shows the greatest relative coverage.
(4) Under $G/G/1$ to $G/G/1$ queueing network models, the calibrated confidence interval corresponding to queueing network models with inter-arrival distribution/ service time distribution of large $CV(> 1)$ have greatest relative coverage than those of small CV (< 1).
(5) Among $G/G/1$ to $G/G/1$ queueing network models, the estimation approach BCaB or BB calibrated confidence intervals has the greatest relative coverage.
(6) Among $G/G/1$ to $G/G/1$ queueing network models, the calibrated confidence intervals of model $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$ shows the greatest relative coverage.

Limitations of the Study:

(1) This is a comparative numerical simulation study based on two-stage open queueing network systems as shown in Figure -1.
(2) Parameters selected in simulation study are arbitrary as shown in Table-1.

4 Conclusions

This paper provides the calibrated confidence interval estimations of intensities ρ_1 and ρ_2 for two-stage open queueing network. Eight different calibrated estimation approaches CAN, Exact-t, Boot-t, VST, SB, BB, PB and BCaB are applied to produce confidence intervals for intensities ρ_1 and ρ_2. The relative coverage is adopted to understand, compare and assess performance of the resulted confidence intervals. The simulation results imply that VST and BCaB method has the best performance on calibrated confidence interval estimations of intensities ρ_1 and ρ_2 for $M/G/1$ to $G/M/1$ queueing network type models with short run data. Under $G/G/1$ to $G/G/1$ queueing network type models with short run data, the estimation approach BCaB or BB out performs the other estimation approach in terms of the relative coverage. This approach is easily applied to practical queueing network models.

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