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The Fractional-order SIR and SIRS Epidemic Models with Variable Population Size

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Abstract: In this work we deal with the fractional-order SIR and SIRS epidemic models with constant recruitment rate, mass action incidence and variable population size. The stability of equilibrium points is studied. Numerical solutions of these models are given. Numerical simulations have been used to verify the theoretical analysis.

Keywords: Fractional-order, SIR and SIRS epidemic models, variable population size, stability, numerical solutions.

1 Introduction

The epidemic models incorporates constant recruitment, disease-induced death and mass action incidence rate.

Some infectious disease confers permanent immunity and other diseases confers temporal acquired immunity. This type of diseases can be modelled by SIR and SIRS models, respectively. The total population N is divided into three compartments with $N = S+I+R$, where S is the number of individuals in the susceptible class, I is the number of individuals who are infectious and R is the number of individuals recovered [18].

The use of fractional-orders differential and integral operators in mathematical models has become increasingly widespread in recent years [17]. Several forms of fractional differential equations have been proposed in standard models.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, economic, viscoelasticity, biology, physics and engineering. Recently, a large amount of literature has been developed concerning the application of fractional differential equations in nonlinear dynamics [17].

In this paper we study the fractional-order SIR and SIRS epidemic models. The stability of equilibrium points is studied. Numerical solutions of these models are given.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena.

We like to argue that fractional order equations are more suitable than integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important. In sec. 2 the equilibrium points and their asymptotic stability of differential equations of fractional order are studied. In sec. 3 the models are presented and discussed. In sec. 4 numerical solutions of the models are given.

Now we give the definition of fractional-order integration and fractional-order differentiation:

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t), t > 0$ is defined by

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1,n)$ of $f(t), t > 0$ is defined by

$$D_*^{\alpha} f(t) = I^{n-\alpha} D^n f(t), \;\; D_* = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [6,7,8,10,14,16,17]).

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Let $\beta,\gamma \in R^+$ and $\alpha \in (0,1)$. Then
(i) $I_a^\beta : L^1 \to L^1$, and if $f(y) \in L^1$, then $I_a^\gamma I_a^\beta f(y) = I_a^{\gamma+\beta} f(y)$.
(ii) $\lim_{\beta\to n} I_a^\beta f(y) = I_a^n f(y)$ uniformly on [a,b], $n = 1, 2, 3, \cdots$,
where $I_a^1 f(y) = \int_a^y f(s)\, ds$.
(iii) $\lim_{\beta\to 0} I_a^\beta (y) = f(y)$ weakly.
(iv) If $f(y)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha\to 1} D_*^\alpha f(y) = \frac{df(y)}{dy}$.
(v) If $f(y) = k \neq 0$, k is a constant, then $D_*^\alpha k = 0$.
The following lemma can be easily proved (see [10]).
Lemma 1 Let $\beta \in (0,1)$ if $f \in C[0,T]$, then $I^\beta f(t)|_{t=0} = 0$.

2 Equilibrium points and their asymptotic stability

Let $\alpha \in (0,1]$ and consider the system ([1,2,3,11,12,13])

$$\begin{aligned} D_*^\alpha y_1(t) &= f_1(y_1,y_2,y_3), \\ D_*^\alpha y_2(t) &= f_2(y_1,y_2,y_3), \\ D_*^\alpha y_3(t) &= f_3(y_1,y_2,y_3), \end{aligned} \tag{3}$$

with the initial values

$$y_1(0) = y_{o1} \text{ and } y_2(0) = y_{o2} \text{ and } y_3(0) = y_{o3}. \tag{4}$$

To evaluate the equilibrium points, let

$$D_*^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}, y_3^{eq}) = 0, \quad i = 1,2,3$$

from which we can get the equilibrium points y_1^{eq}, y_2^{eq}, y_3^{eq}.
To evaluate the asymptotic stability, let

$$y_i(t) = y_i^{eq} + \varepsilon_i(t),$$

So the the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable if the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{bmatrix}$$

evaluated at the equilibrium point satisfiesis $(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2, |\arg(\lambda_3)| > \alpha\pi/2)$ ([2,3,13,15]).The stability region of the fractional-order system with order α is illustrated in Fig. 1 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 1, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.

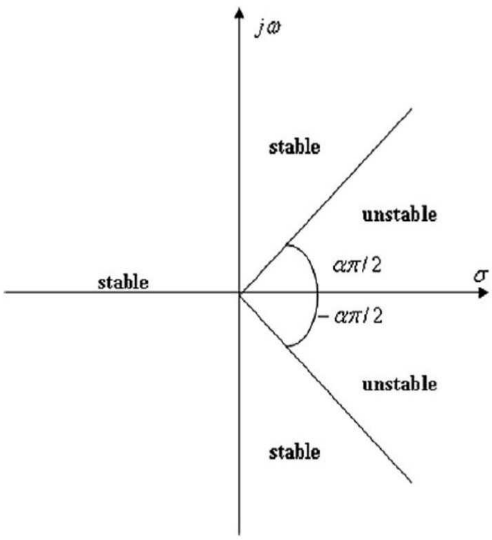

Fig. 1. Stability region of the fractional-order system.

The eigenvalues equation of the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is given by the following polynomial:

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{5}$$

and its discriminant $D(P)$ is given as:

$$\begin{aligned} D(P) = \; & 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 \\ & -4(a_2)^3 - 27(a_3)^2, \end{aligned} \tag{6}$$

using the results of Ref. [2], we have the following fractional Routh-Hurwitz conditions:

(i) If $D(P) > 0$, then the necessary and sufficient condition for the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$, to be locally asymptotically stable, is $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.

(ii) If $D(P) < 0$, $a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then all roots of Eq. (5) satisfy the condition $|\arg(\lambda)| < \alpha\pi/2$.

(iii) If $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 - a_3 = 0$, then $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable for all $\alpha \in (0,1)$.

(iv) The necessary condition for the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$, to be locally asymptotically stable, is $a_3 > 0$.

3 Fractional-order SIR and SIRS epidemic models.

Let $S(t)$ be the number of individuals in the susceptible class at time t, $I(t)$ be the number of individuals who are infectious at time t and $R(t)$ be the number of individuals recovered at time t.

The fractional-order SIRS epidemic model is given by

$$\begin{aligned} D_*^{\alpha_1}S(t) &= \Lambda - \beta SI - \mu S + \gamma R, \\ D_*^{\alpha_1}I(t) &= \beta SI - (\kappa + \mu + \alpha)I, \\ D_*^{\alpha_1}R(t) &= \kappa I - (\mu + \gamma)R, \end{aligned} \tag{7}$$

where $0 < \alpha_1 \leq 1$ and the parameters Λ, μ, β, κ and α are positive constants, and γ is non-negative constant. Here we assume that κ is the rate at which infectives recover. If the individuals recovered acquired permanent immunit $\gamma = 0$ then one gets SIR model and if $\gamma \neq 0$ the individuals acquired temporal immunity, then one gets SIRS model.

Which, together with $N = S + I + R$, implies

$$D_*^{\alpha_1}N = \Lambda - \mu N - \alpha I.$$

Thus the total population size N may vary in time.

To evaluate the equilibrium points, let

$$\begin{aligned} D_*^{\alpha_1}S &= 0, \\ D_*^{\alpha_1}I &= 0, \\ D_*^{\alpha_1}R &= 0, \end{aligned}$$

then $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\Lambda}{\mu}, 0, 0)$, (S_*, I_*, R_*), are the equilibrium points where,

$$\begin{aligned} S_* &= \frac{1}{\beta}(\kappa + \mu + \alpha), \\ I_* &= \frac{(\mu + \gamma)[\Lambda\beta - \mu(\kappa + \mu + \alpha)]}{\beta[\kappa\mu + (\mu + \gamma)(\mu + \alpha)]}, \\ R_* &= \frac{\kappa[\Lambda\beta - \mu(\kappa + \mu + \alpha)]}{\beta[\kappa\mu + (\mu + \gamma)(\mu + \alpha)]}. \end{aligned}$$

For $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\Lambda}{\mu}, 0, 0)$ we find that

$$A = \begin{bmatrix} -\mu & 0 & \gamma \\ 0 & \frac{\beta\Lambda}{\mu} - (\kappa + \mu + \alpha) & 0 \\ 0 & \kappa & -(\mu + \gamma) \end{bmatrix},$$

and its eigenvalues are

$$\lambda_1 = -\mu < 0,$$

$$\lambda_2 = -(\mu + \gamma) < 0,$$

$$\lambda_3 = \frac{\beta\Lambda}{\mu} - (\kappa + \mu + \alpha) < 0 \quad \text{if} \quad \frac{\beta\Lambda}{\mu} < (\kappa + \mu + \alpha).$$

Hence the equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\Lambda}{\mu}, 0, 0)$ is local asymptotically stable if

$$\frac{\beta\Lambda}{\mu} < (\kappa + \mu + \alpha). \tag{8}$$

For $(S_{eq}, I_{eq}, R_{eq}) = (S_*, I_*, R_*)$ we find that

$$A = \begin{bmatrix} -\beta I_* - \mu & -(\kappa + \mu + \alpha) & \gamma \\ \beta I_* & 0 & 0 \\ 0 & \kappa & -(\mu + \gamma) \end{bmatrix}.$$

A sufficient condition for the local asymptotic stability of the equilibrium point

$(S_{eq}, I_{eq}, R_{eq}) = (S_*, I_*, R_*)$ is

$$\begin{aligned} &|\arg(\lambda_1)| > \alpha_1\pi/2, |\arg(\lambda_2)| > \alpha_1\pi/2, \\ &|\arg(\lambda_3)| > \alpha_1\pi/2. \end{aligned} \tag{9}$$

The characteristic polynomial of the equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (S_*, I_*, R_*)$ is given by:

$$\begin{aligned} &\lambda^3 + [\beta I_* + 2\mu + \gamma]\lambda^2 + \\ &[(\beta I_* + \mu)(\mu + \gamma) + \beta I_*(\kappa + \mu + \alpha)]\lambda + \\ &\beta I_*(\kappa + \mu + \alpha)(\mu + \gamma) - \beta I_*\kappa\gamma = 0. \end{aligned} \tag{10}$$

4 Numerical methods and results

An Adams-type predictor-corrector method has been introduced and investigated further in ([1,2,3,4,5,9]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (7) by an equivalent fractional integral equations

$$\begin{aligned} S(t) &= S(0) + I^{\alpha_1}[\Lambda - \beta SI - \mu S + \gamma R], \\ I(t) &= I(0) + I^{\alpha_1}[\beta SI - (\kappa + \mu + \alpha)I], \\ R(t) &= R(0) + I^{\alpha_1}[\kappa I - (\mu + \gamma)R], \end{aligned} \tag{11}$$

and then apply the **PECE** (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions are displayed in Figs. 2-12 for $S(0) = 20.0$, $I(0) = 1.0$, $R(0) = 1.0$ and different $0 < \alpha_1 \leq 1$.

In Figs. 2-8 we take $\Lambda = 0.1$, $\beta = 0.1$, $\mu = 0.2$, $\gamma = 0.3$, $\alpha = 0.1$, $\kappa = 0.1$ and found that the equilibrium point $(\frac{\Lambda}{\mu}, 0, 0) = (0.5, 0, 0)$ is local asymptotically stable where the condition (8) $(\frac{\beta\Lambda}{\mu} = 0.05 < (\kappa + \mu + \alpha) = 0.4)$ is satisfied.

In figs. 8 we found that in the fractional order case the peak of the infection is reduced. But the disease take a longer time to be eradicated.

In Figs. 9-12 we take $\Lambda = 0.5$, $\beta = 0.5$, $\mu = 0.1$, $\gamma = 0.3$, $\alpha = 0.1$, $\kappa = 0.1$ and found that the equilibrium point $(\frac{\Lambda}{\mu}, 0, 0) = (5, 0, 0)$ is unstable where the condition (8) is not satisfied $(\frac{\beta\Lambda}{\mu} = 2.5 > (\kappa + \mu + \alpha) = 0.3)$ and the equilibrium point (S_*, I_*, R_*) is local asymptotically stable where the condition (9) is satisfied where the equilibrium point and the eigenvalues are given as:

$(S_*, I_*, R_*) = (0.6, 1.95556, 0.488889)$,

$\lambda_1 = -0.254728$,

$\lambda_{2,3} = -0.244858 \pm 0.162515i$.

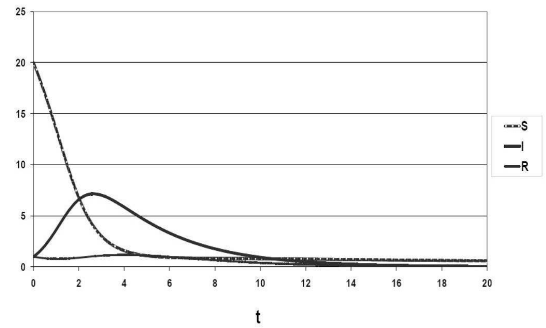

Figure 2 α_1=1.0.

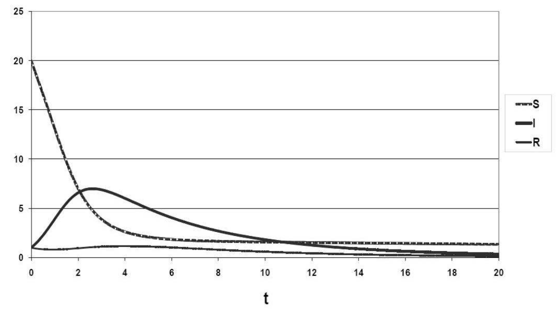

Figure 3 α_1=0.9.

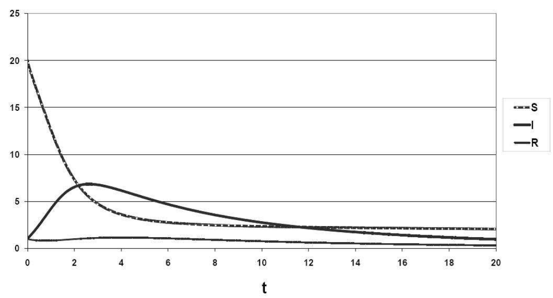

Figure 4 α_1=0.8.

The equilibrium point $(S_*, I_*, R_*) = (0.6, 1.95556, 0.488889)$ is local asymptotically stable where

$|\arg(\lambda_1)| = \pi > \alpha_1\pi/2$,

$|\arg(\lambda_{2,3})| = 2.55564 > \alpha_1\pi/2$.

In figs. 12 we found that in the fractional order case the peak of the infection is reduced. But the disease take a longer time to be eradicated.

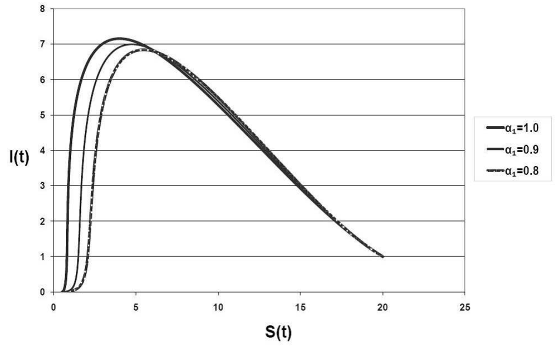

Figure 5

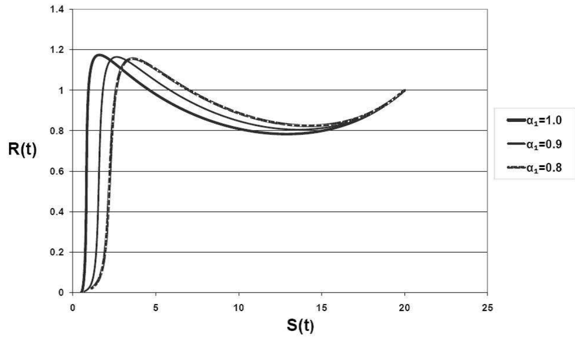

Figure 6

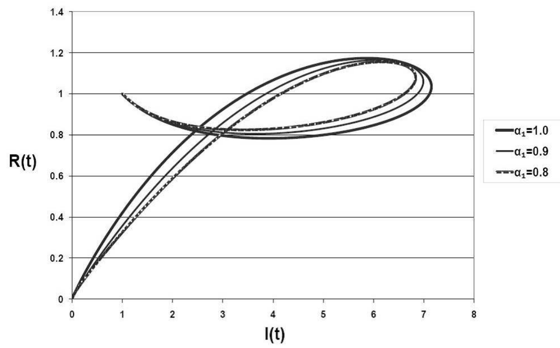

Figure 7

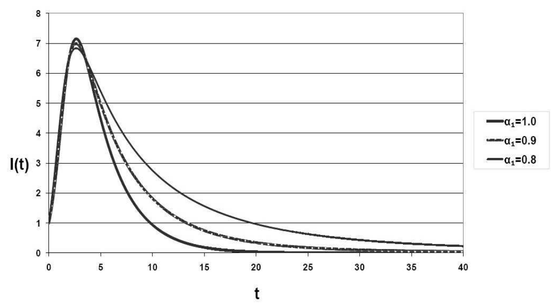

Figure 8

5 Conclusions

In this paper we study the fractional-order SIR and SIRS epidemic models with variable population size. The

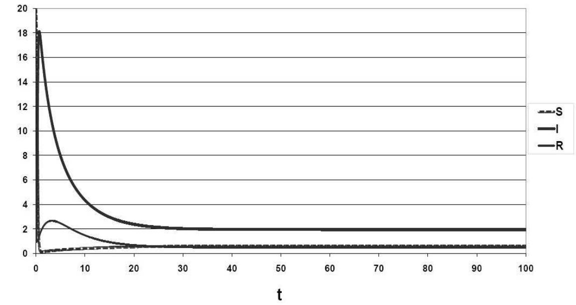

Figure 9 α_1=1.0.

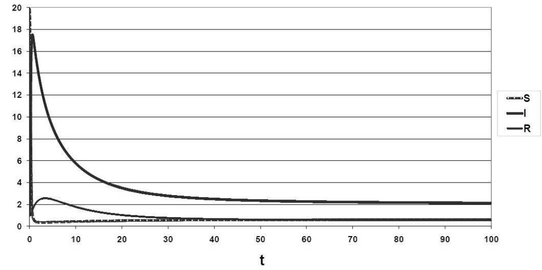

Figure 10 α_1=0.9.

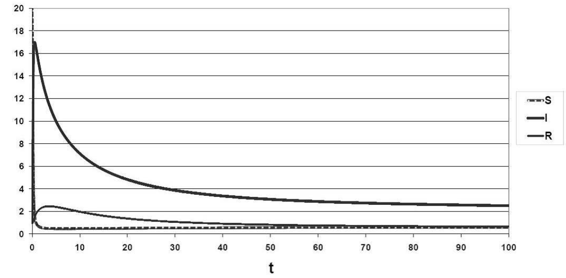

Figure 11 α_1=0.8.

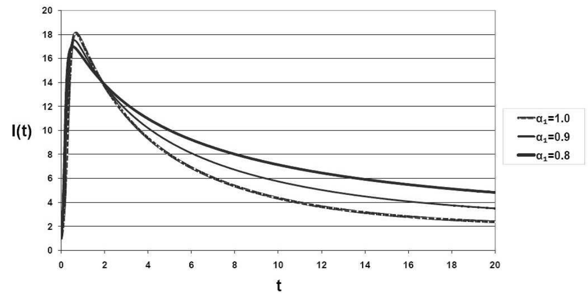

Figure 12

stability of equilibrium points is studied. Numerical solutions of these models are given.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena.

We like to argue that the fractional order models are at least as good as integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important.

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