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On Solutions of the Nonlinear Oscillators by Modified Homotopy Perturbation Method

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Abstract: The nonlinear oscillator problem has extensive applications. In this paper, we examined the modified homotopy perturbation method to obtain the solution of the problem and to reduce the computational efforts. For the accuracy purpose the Laplace transform and Pade approximation have also been used. We construct the deformation equations by two components of the homotopy series, then by means of the Laplace transform the obtain solution is converted to series form and Pade approximation is exploited to increase the accuracy of the achieved result. The computational efforts are significantly reduced. And, the calculated analytic solutions are in excellent agreement with the numerical solutions.

Keywords: oscillator; Laplace transformation; Pade approximants; differential equations.

1 Introduction

The most common methods for constructing approximate and analytical solutions to nonlinear oscillatory equations are the perturbation method. In the past few decades, new perturbation methods and non-perturbative methods are proposed. A review of the recently developed analytical methods is given in review article and the comprehensive book by He [1-2]. There also exists considerable number of works dealing with the problem of approximate analytical solution of nonlinear oscillators by using different methodologies. For example, the energy method [3-4], homotopy perturbation method [5-10], the variational iteration method [11-12], the Lindstedt-Poincare methods [13-14], the variational methods [15-16], the parameter-expanding method [17,18], the Adomian Pade approximation [19] and the differential Transformation method [20]. Our main concern in this paper to find the approximate analytical solutions for the nonlinear equation

$$\ddot{u}+\alpha\dot{u}=\gamma f(u,\dot{u}), \tag{1}$$

With initial conditions

$$u(0)=a,\ \dot{u}=b. \tag{2}$$

where the over dot denotes differentiation with respect to time and and are arbitrary parameters.

In the present study, we used a modified version of homotopy perturbation method which is based on two components of homotopy series. The two components HPM [21-22] provides an efficient analytical solution without any transformation, Adomian polynomials, complicated Lagrange Multiplier with repeated integration process and independent of the solution of functional differential equation for finding each component of the solution. In order to improve the accuracy of the solution, we first apply the Laplace transformation then convert the transformed series into a meromorphic function by forming the Pade approximants, and finally adopt an inverse Laplace transform to obtain an analytic solution. Finally, numerical comparison has been made between the proposed approach and the Runge-Kutta method.

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2 Analysis of the method

Let us consider the nonlinear differential equation

$$\mathscr{A}(u) = f(z),\ z\varepsilon\Omega, \tag{3}$$

where $\mathscr{A}$ is operator, f is a known function and u is a sought function. Assume that operator $\mathscr{A}$ can be written as:

$$\mathscr{A}(u) = \mathscr{L}(u) + \mathscr{N}(u), \tag{4}$$

where $\mathscr{L}$ is the linear operator and $\mathscr{N}$ is the nonlinear operator. Hence, equation (4) can be rewritten as follows:

$$\mathscr{L}(u) + \mathscr{N}(u) = f(z),\ z\varepsilon\Omega. \tag{5}$$

We define an operator $\mathscr{H}$ as:

$$\mathscr{H}(v;p) = (1-p)(\mathscr{L}(v) - \mathscr{L}(v_0)) + p(\mathscr{A}(v) - f), \tag{6}$$

where $p\varepsilon[0,1]$ is an embedding or homotopy parameter,$v(z:p):\Omega\times[0,1]\to\mathfrak{R}$ and u_0 is an initial approximation of solution of the problem in equation (6) can be written as:

$$\mathscr{H}(v;p) = \mathscr{L}(v) - \mathscr{L}(u_0) + p(\mathscr{N}(v) - f(z)) = 0, \tag{7}$$

Clearly, the operator equations $\mathscr{H}(v,0) = 0$ and $\mathscr{H}(v,1) = 0$ are equivalent to the equations $\mathscr{L}(v) - \mathscr{L}(u_0) = 0$ and $\mathscr{A}(v) - f(z) = 0$ respectively. Thus, a monotonous change of parameter p from zero to one corresponds to a continuous change of the trivial problem $\mathscr{L}(v) - \mathscr{L}(u_0) = 0$ to the original problem. Operator $\mathscr{H}(v,p)$ is called a homotopy map. Next, we assume that the solution of equation $\mathscr{H}(v,p)$ can be written as a power series in embedding parameter p, as follows:

$$v = v_0 + pv_1, \tag{8}$$

Now let us write the equation (7) in the following form

$$L(v) = u_0(z) + p(f - N(v) - u_0(z)), \tag{9}$$

By applying the inverse operator, $\mathscr{L}^{-1}$ to both sides of the equation (9), we have

$$v = \mathscr{L}^{-1}u_0(z) + p(\mathscr{L}^{-1}f - \mathscr{L}^{-1}N(v) - \mathscr{L}^{-1}u_0(z)), \tag{10}$$

Suppose that the initial approximation of equation (5) has the form

$$u_0(z) = \sum_{n=0}^{\infty} a_n P_n(z), \tag{11}$$

where a_n, $n = 0,1,2,...$, are unknown coefficients and $P_n(z)$, $n = 0,1,2,...$, are specific functions on the problem. By substituting equations (8) and (11) into the equation (10), we get

$$v_0 + pv_1 = \mathscr{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right)$$
$$+p\left(\mathscr{L}^{-1}f - \mathscr{L}^{-1}\left(\sum_{n=0}^{l} v_n p^n\right) - \mathscr{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right)\right), \tag{12}$$

Equating the coefficients of like powers of p, we get following set of equations

$$p^0 : v_0 = \mathscr{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right) \tag{13}$$

$$p^1 : v_1 = \mathscr{L}^{-1}(f) + \mathscr{L}^{-1}\left(\sum_{n=0}^{\infty} v_n p^n - \mathscr{L}^{-1}\mathscr{N}(v_0)\right) \tag{14}$$

Now, if we solve these equations in such a way that . Therefore, the approximate solution may be obtained as

$$u(z) = v_0(z) = \mathscr{L}^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(z)\right) \tag{15}$$

3 Implementation of the method

To obtain the solution of equation (1) by MHPM, we construct the following homotopy:

$$(1-p)(\ddot{U} - u_0(t)) + p(\ddot{U} + \alpha\dot{U} - \gamma f(U,\dot{U})) = 0 \tag{16}$$

Applying the inverse operator, $L^{-1}(\bullet) = \int_0^t \int_0^s (\bullet) d\xi ds$ to the both sides of the above equation (15), we obtain

$$U(t) = U(0) + t\dot{U}(0) + \int_0^t \int_0^s u_0(\xi) d\xi ds$$
$$-p \int_0^t \int_0^s (u_0(\xi) + \alpha \dot{U}(\xi) - \gamma f(U(\xi), \dot{U}(\xi))) d\xi ds \quad (17)$$

The solution of equations (1) to have the following form

$$U(t) = U_0(t) + pU_1(t) \quad (18)$$

Substituting equation (17) in equation (16) and equating the coefficients of like powers of p, we get following set of equations

$$U_0(t) = U_0 + t\dot{U}(0) + \int_0^t \int_0^s u_0(\xi) d\xi ds \quad (19)$$

$$U_1(t) = \int_0^t \int_0^s (-u_0(\xi) - \alpha \dot{U}_0 - \gamma f(U_0, \dot{U}_0)) d\xi ds \quad (20)$$

Assuming $u_0(t) = \sum_{n=0}^N a_n P_n$, $P_n = t^n$, and solving the above equation for $U_0(t)$ leads to the result

$$U_0(t) = a + bt + \frac{a_0}{2}t^2 + \frac{a_1}{6}t^3 + \frac{a_2}{12}t^4 + \ldots \quad (21)$$

On using equation (20) in equation (19) we arrived at $U_1(t)$. With vanishing $U_1(t)$, we have the coefficients a_i, $i = 0, 1, \ldots 8$. Therefore, we obtain the solution of equation (11) as

$$u(t) = a + bt + \frac{a_0}{2}t^2 + \frac{a_1}{6}t^3 + \frac{a_2}{12}t^4 + \ldots \quad (22)$$

The solution of equation (1) does not exhibit behavior for a large region. In order to improve the accuracy of the two component solution, we implement the modification as follows: Applying the Laplace transform to the series solution (21), yields

$$L[u(t)] = \frac{a}{s} + \frac{b}{s^2} + \frac{a_0}{s^3} + \frac{a_1}{s^4} + \frac{2a_2}{s^5} + \ldots \quad (23)$$

For simplicity, let $s = \frac{1}{t}$; then

$$L[u(t)]_{[m,n]} = \frac{at + A_1t^2 + A_2t^3 + \ldots}{1 + B_1t + B_2t^2 + B_3t^3 + \ldots} \quad (24)$$

Recalling $t = \frac{1}{s}$, we obtain $[m,n]$ Pade approximation in terms of s. By using the inverse Laplace transform to the $[m,n]$ Pade approximant, we obtain the desired approximate solution of the nonlinear oscillator equation.

4 Analytical Solutions

In order to verify the procedure of the method, we consider the following particular cases and comparison will be made with Runge-Kutta method as well as ref. [19-20].

4.1 Application I

Consider the vander-pol equation [20] by taking $\alpha = 1$, $f(u, \dot{u}) = \gamma(1 - u^2)\dot{u}$

$$\ddot{u} + \dot{u} = \gamma(1 - u^2)\dot{u} \quad (25)$$

With initial conditions

$$u(0) = 0, \; \dot{u}(0) = 2 \quad (26)$$

The approximate analytical solution of equation (25) with conditions (26) can be obtained by applying the procedure mentioned in previous section. Assuming $u_0(t) = \sum_{n=0}^8 a_n P_n$, $P_n = t^n$, and solving the above equation for $U_0(t)$ leads to the result

$$U_0(t) = 2t + \frac{a_0}{2}t^2 + \frac{a_1}{6}t^3 + \frac{a_2}{12}t^4 + \ldots \quad (27)$$

On using equation (27) in (19), we get $U_1(t)$. With vanishing $U_1(t)$, we have the coefficients

$$a_0 = 2\gamma, a_1 = 2(\gamma^2 - 1), a_2 = \gamma(\gamma^2 - 10),$$
$$a_3 = \frac{\gamma}{3}(\gamma^4 - 59\gamma^2 + 1), \ldots, i = 0, 1, \ldots 8. \quad (28)$$

Therefore, we obtain the solutions of equation (25) as

$$u(t) = 2t + \gamma t^2 - \frac{1}{3}t^3 - \frac{5\gamma}{6}t^4 + \frac{1}{60}t^5 + \ldots \quad (29)$$

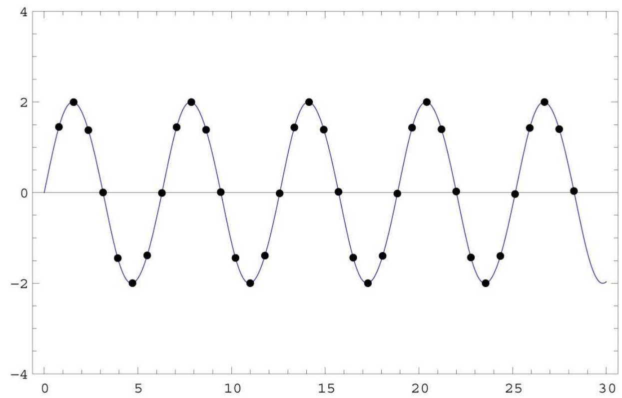

Fig. 1: Plots of the comparisions of RK4 and present solution of van der pol equation t vs. u

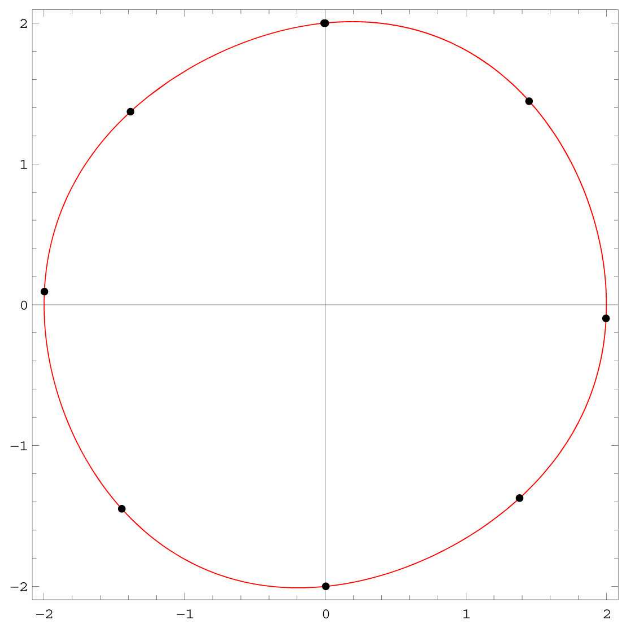

Fig. 2: Plots of the comparisions of RK4 and present solution of van der pol equation phase plane diagram u vs. $\dot{u}$

Applying the Laplace transform to the solution (29), yields

$$L[u(t)] = 2\left(\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \frac{1}{s^{10}} - \frac{1}{s^{12}}\right)$$

$$+\gamma\left(\frac{2}{s^3} - \frac{20}{s^5} + \frac{182}{s^7} - \frac{1640}{s^9} + \frac{14762}{s^{11}} - \frac{132860}{s^{13}}\right) + \ldots \quad (30)$$

For simplicity, let $s = \frac{1}{t}$; then

$$L[u(t)] = 2(t^2 - t^4 + t^6 - t^8 + t^{10} - t^{12})$$

$$+\gamma(2t^3 - 20t^5 + 182t^7 - 1640t^9 14762t^{11} - 132860t^{13}) + \ldots \quad (31)$$

The $[2/2]$ and $[4/4]$ Pade approximation for the term containing the linear power of γ is

$$u_{[2/2]} = \frac{2t^2}{1 + t^2 - \gamma t + \gamma^2 t^2} \quad (32)$$

$$u_{[4/4]} = \frac{2t^2}{1 + t^2} + \frac{2\gamma t^2}{1 + 10t^2 + 9t^4} \quad (33)$$

Recalling $t = \frac{1}{s}$ and applying the inverse Laplace transform to the $[2/2]$ and $[4/4]$ Pade approximant, we obtained the approximate solutions

$$u_{[2/2]} = \frac{-2e^{t\left(\frac{\gamma}{2} - \frac{\sqrt{-4-3\gamma^2}}{2}\right)} + 2e^{t\left(\frac{\gamma}{2} + \frac{\sqrt{-4-3\gamma^2}}{2}\right)}}{\sqrt{-4-3\gamma^2}} \quad (34)$$

$$u_{[4/4]} = \frac{\gamma}{4}\cos[t] - \frac{\gamma}{4}\cos[3t] + 2\sin[t] \quad (35)$$

The graph of the displacement is depicted in Figure 1, and phase plane diagram is sketched in Figure 2 and compared to the fourth order Runge-Kutta method.

4.2 Application II

Consider the Unplugged van der pol equation [20] by taking $\alpha = 1,\ f(u, \dot{u}) = -\gamma u^2 \dot{u}$

$$\ddot{u} + \dot{u} = \gamma u^2 \dot{u} \quad (36)$$

With initial conditions

$$u(0) = 1,\ \dot{u}(0) = 0 \quad (37)$$

The approximate analytical solution of equation (36) with conditions (37) can be obtained by applying the procedure mentioned in previous section. Assuming

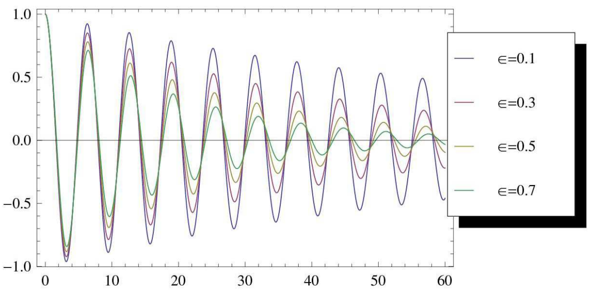

Fig. 3: Plots of the solution of Duffing-van der pol equation by present method

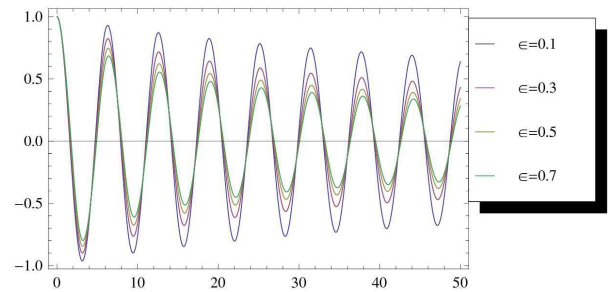

Fig. 4: Plots of the solution of Duffing-van der pol equation by RK4 method

$u_0(t) = \sum_{n=0}^{8} a_n P_n, P_n = t^n$, and solving the above equation for leads to the result

$$U_0(t) = 1 + \frac{a_0}{2}t^2 + \frac{a_1}{6}t^3 + \frac{a_2}{12}t^4 + \dots \tag{38}$$

On using equation (38) in (19), we get $U_1(t)$. With vanishing $U_1(t)$, we have the coefficients

$$a_0 = -1, a_1 = \gamma, a_2 = \frac{1}{2}(1-\gamma^2),$$
$$a_3 = \frac{\gamma}{6}(\gamma^2 - 8), \dots, i = 0, 1, \dots 8. \tag{39}$$

Therefore, we obtain the solutions of equation (36) as

$$u(t) = 1 - \frac{t^2}{2} + \frac{\gamma t^3}{6} + \frac{1}{24}t^4(1-\gamma^2) + \frac{1}{120}t^5\gamma(\gamma^2-8) + \dots \tag{40}$$

Applying the Laplace transform to the solution (40), yields

$$L[u(t)] = \frac{1}{s} - \frac{1}{s^3} + \frac{\gamma}{s^4} - \frac{(1-\gamma^2)}{s^5}$$
$$+ \frac{\gamma(\gamma^2-8)}{s^6} - \frac{\gamma^4 - 29\gamma^2 + 1}{s^7} + \dots \tag{41}$$

For simplicity, let $s = \frac{1}{t}$; then

$$L[u(t)] = t - t^3 + \gamma t^4 + (1-\gamma^2)t^5 + \gamma(\gamma^2-8)t^6$$
$$-\gamma(\gamma^4 - 29\gamma^2 + 1)t^7 + \dots \tag{42}$$

The $[4/4]$ Pade approximation and recalling $t = \frac{1}{s}$ and applying the inverse Laplace transform we obtained the approximate solutions. Figures 3 and 4 represents the displacement of unplugged van der pol oscillator and all of its solutions are expected to oscillate with decreasing amplitude to zero. Ref. [19-20] derived the solutions by modified decomposition and differential transform method. Figures 5 and 6 represents the phase diagram of the oscillator equation which is good agreement with numerical solution.

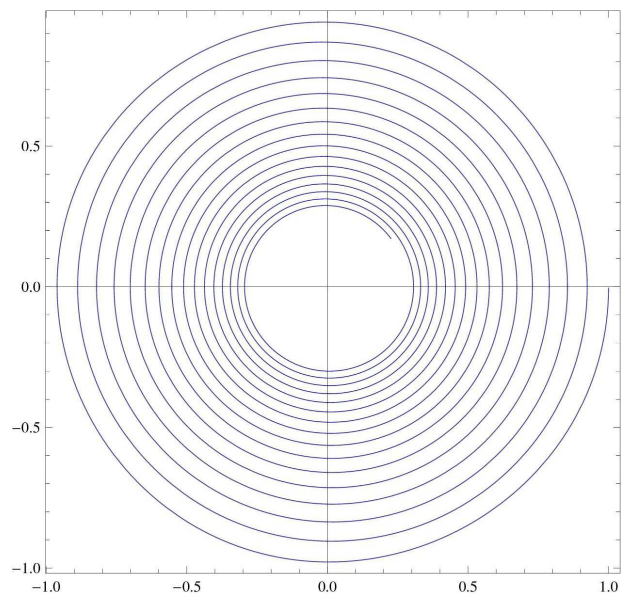

Fig. 5: Comparison for u versus $\dot{u}$ trajectory of the Duffing-van der pol equation by present method.

4.3 Application III

Consider the Duffing equation [20] by taking $\alpha = 1,\ f(u,\dot{u}) = -\gamma u^3$

$$\ddot{u} + \dot{u} = -\gamma u^3 \tag{43}$$

With initial conditions

$$u(0) = 0,\ \dot{u}(0) = 1 \tag{44}$$

Assuming
$u_0(t) = \sum_{n=0}^{8} a_n P_n, P_n = t^n$, and solving the above equation for leads to the result

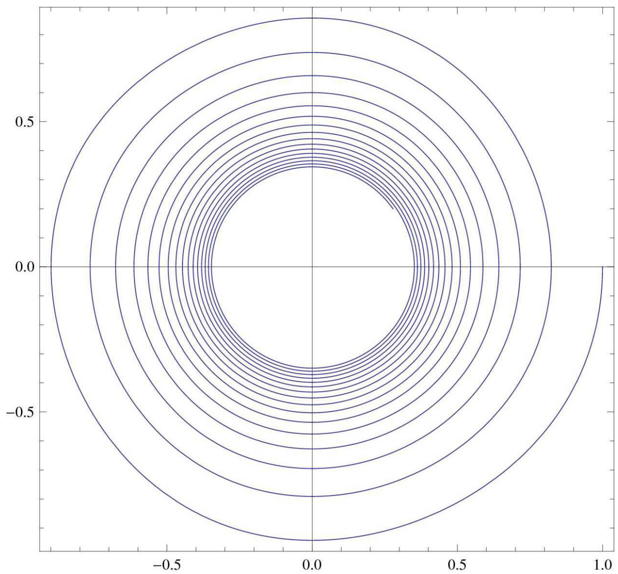

Fig. 6: Comparison for u versus $\dot{u}$ trajectory of the Duffing-van der pol equation by RK4 method

$$U_0(t) = 1 + \frac{a_0}{2}t^2 + \frac{a_1}{6}t^3 + \frac{a_2}{12}t^4 + \ldots \tag{45}$$

On using equation (45) in (19), we get $U_1(t)$. With vanishing $U_1(t)$, we have the coefficients

$$a_0 = 0, a_1 = -1, a_2 = 0, a_3 = \frac{1}{6}(1-6\gamma), a_4 = 0,$$
$$a_5 = \frac{1}{120}(66\gamma - 1), , ..., i = 0,1,...8. \tag{46}$$

Therefore, we obtain the solutions of equation (43) as

$$u(t) = t - \frac{t^3}{6} + \frac{(1-6\gamma)t^5}{120} + \frac{(1-\gamma^2)t^7}{5040} + \ldots \tag{47}$$

Applying the Laplace transform to the solution (47), yields

$$L[u(t)] = \frac{1}{s^2} - \frac{1}{s^4} + \frac{(1-6\gamma)}{s^6} + \frac{(66\gamma-1)}{s^8}$$
$$+ \frac{756\gamma^2 - 612\gamma + 1}{s^{10}} - \frac{33156\gamma^2 - 5532\gamma + 1}{s^6} + \ldots \tag{48}$$

For simplicity, let $s = \frac{1}{t}$; then

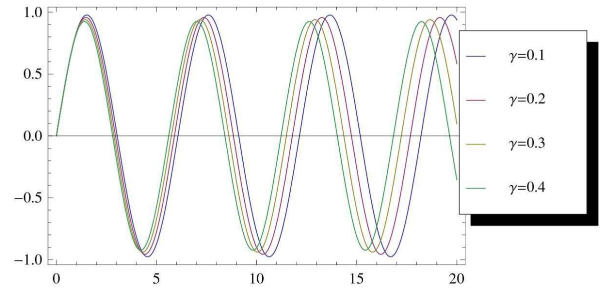

Fig. 7: Plots of the solution of Duffing equation by present method

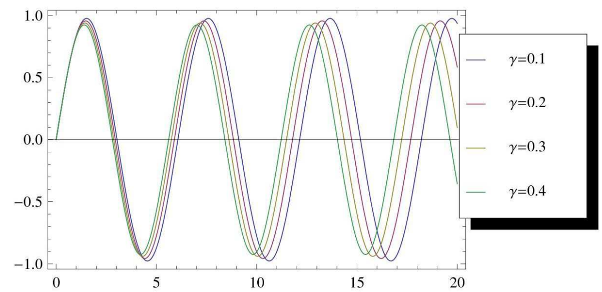

Fig. 8: Plots of the solution of Duffing equation by RK4 method

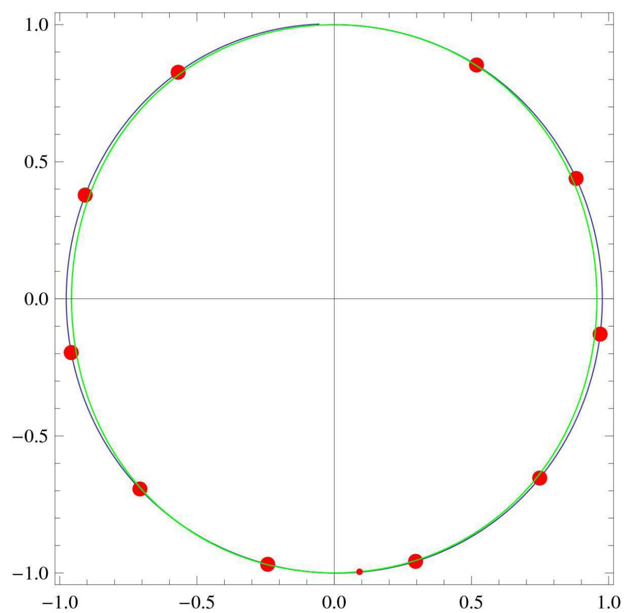

Fig. 9: Comparison for u versus $\dot{u}$ trajectory of the Duffing equation for $\gamma = 0.1$.

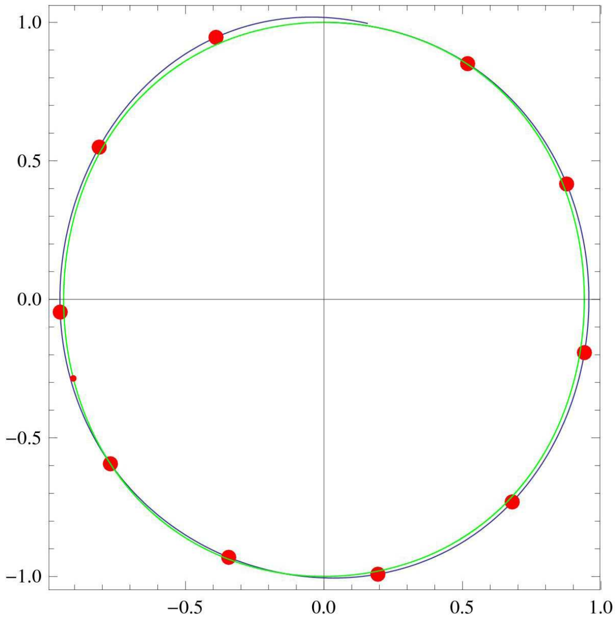

Fig. 10: Comparison for u versus $\dot{u}$ trajectory of the Duffing equation for $\gamma = 0.2$.

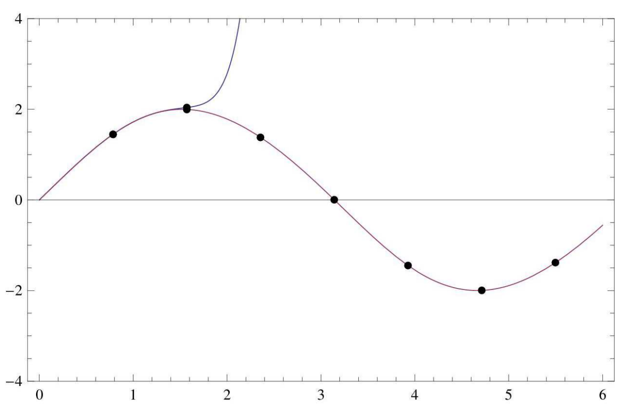

Fig. 11: Comparison of displacement plot for HPM versus RK4 for $\gamma = 0.1$

$$L[u(t)] = t^2 - t^4 + (1-6\gamma)t^6 + (66\gamma - 1)t^8 + (756\gamma^2 - 612\gamma + 1)t^{10} + \ldots \quad (49)$$

The $[4/4]$ Pade approximation and recalling $s = \frac{1}{t}$ and applying the inverse Laplace transform we obtained the approximate solutions. The Graphs of the displacement are depicted in Figures 7 and 8 and phase diagram are depicted in Figures 9 and 10 and are compared with the numerical solution.

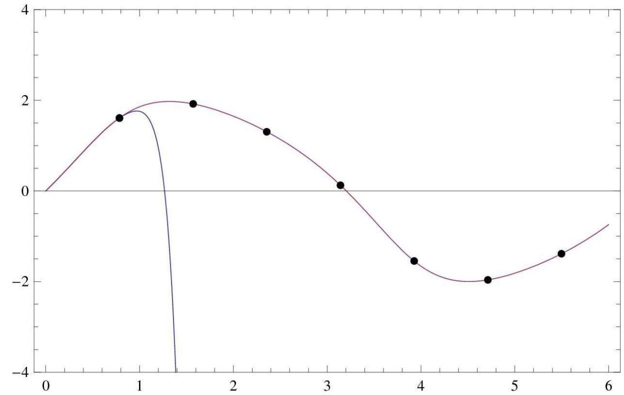

Fig. 12: Comparison of displacement plot for HPM versus RK4 $\gamma = 0.6$

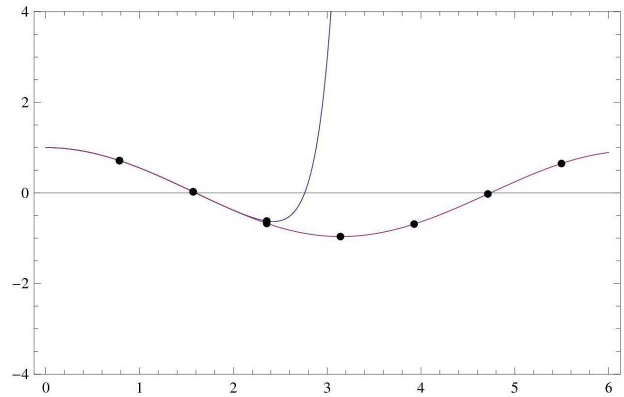

Fig. 13: Comparison of displacement plot for HPM versus RK4 $\gamma = 0.1$

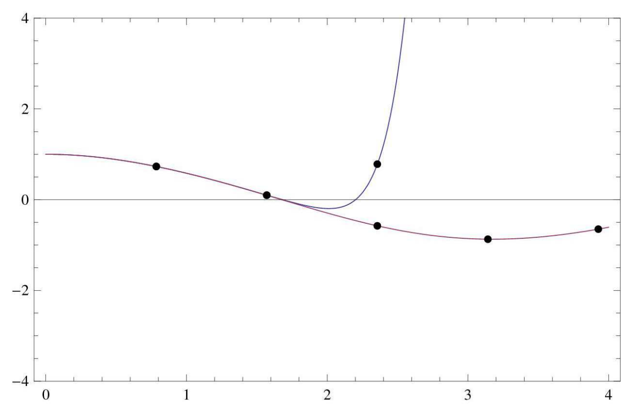

Fig. 14: Comparison of displacement plot for HPM versus RK4 $\gamma = 0.4$

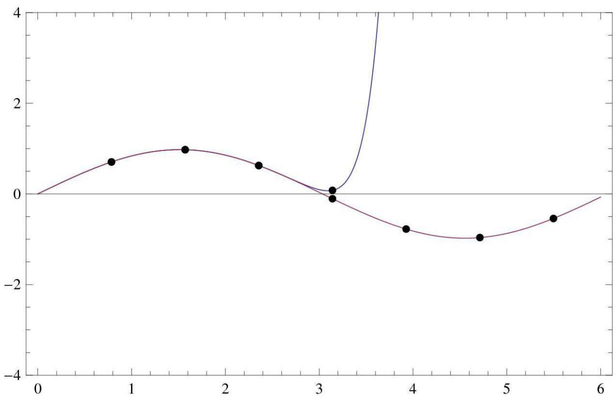

Fig. 15: Comparison of displacement plot for HPM versus RK4 $\gamma = 0.1$

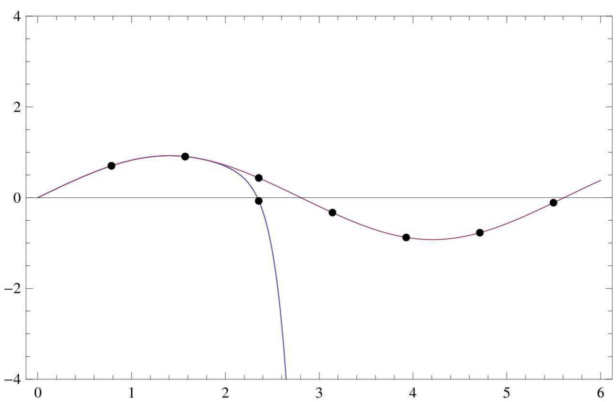

Fig. 16: Comparison of displacement plot for HPM versus RK4 $\gamma = 0.4$

5 Conclusions

In this paper, the application of modified HPM was extended to obtain analytical solution of nonlinear oscillators. The results obtained from this method have been compared with those obtained from numerical method using Runge Kutta method and ref. [19-20]. The effects of variation of the parameters on the accuracy of the modified homotopy- perturbation method have studied. The presented scheme provides concise and straightforward solution to approach reliable results, and it overcomes the difficulties that have been arisen in conventional methods. The present method is an extremely simple method, leading to high accuracy of the obtained results. A numerical comparison between classical HPM and Runge-Kutta method is depicted in figures $11-16$. The HPM is converges in a small time interval and computationally taken long time for large interval. The HPM series gives reasonable results in the small time interval.

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