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Reduced Bias Estimation of the Reinsurance Premium of Loss Distribution

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Abstract: In this paper we propose a new asymptotically normal estimator of the reinsurance premium for the losses distribution. Our estimator is based on the reduced bias of the extreme quantile and the index of an heavy-tailed distribution. Moreover, we illustrate the behaviour of the proposed estimator and give a comparison between this estimator and the classical semi parametric estimator proposed by Necir et al. (2007) in terms of the bias and the root mean squared error (rmse).

Keywords: Reduced Bias; Extreme Values; Heavy Tails; Proportional Hazard Principle Premium; Risk Theory.

1 Introduction

In insurance, the worst scenarios are those caused by extreme events such as natural disasters, financial crashes, industrial catastrophes. These extreme events increase the bill of insurance and reinsurance companies. A typical requirement for actuaries is the determination of the optimal or adequate premium for such risks, according to an appropriate principal pricing. In insurance literature, many premium calculation principles are proposed such as: mean, value at risk, variance, etc. In our study, we consider the Wang premium calculation principle (1996) based on a proportional transformation of the hazard function. The proportional hazard (denoted PH) premium of an insured risk X with continuous distribution function F, depends on the hazard function S=1-F and a parameter $r\leq 1$ called risk aversion index. In some actuarial problems, as in the reinsurance treaty, one is interested in the estimation of a premium for a given retention level R>0, we note by $\Pi_{r,R}$ to a reinsurance premium of the high layer $[R,\infty)$. This kind of problem can be found whenever the insured represent a dangerous level of risk for the insurance company, and decides to give a part of this loss to another reinsurance company, because it may not have sufficient capital to cover the entire risk.

The PH premium is defined as function of r and S by:

$$\Pi_r = \int_0^{\infty} \left(S(x)\right)^{1/r} dx$$

For R>0 being the reinsurance retention level of a risk X, the corresponding PH premium of loss with a high layer is defined as:

$$\Pi_{r,R} = \int_R^{\infty} \left(S(x)\right)^{1/r} dx \qquad (1)$$

Now, consider $X_1, X_2, \ldots . X_n$ are iid random variables with common distribution function F of an insured risk X. We assume that *F* has regular variation function near infinity with index $-1/\gamma$, that is:

$$\frac{S(tx)}{S(t)} = x^{-1/\gamma} for any x > 0 \; and \; \frac{1}{2} < \gamma < 1. \qquad (2)$$

(see, e.g., de Haan and Ferreria, (2006), page 19). Such cdf's constitute a major subclass of the family of heavy-tailed distributions. It includes distributions such as Pareto, Burr, Student, α-stable (0<α<2), and log-gamma, which are known to be appropriate models of fitting large insuranceclaims, largefluctuations of prices, log-returns, etc. (see, e.g. Reiss and Thomas, (2007); Beirlant et al. (2001); Rolski et al. (1999)). Let

$$Q(s) = inf\{x \in R: F(x) \geq s\}, 0 < s < 1,$$

denote the quantile function associated to the df F and $U(t) = Q(1 - 1/t)$ is the tail quantile of the df F. Note that the condition (2) is equivalent to the condition

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{\gamma} \ for \ any \ x > 0. \quad (3)$$

Further, we assume that F is second order regularly varying at infinity, that is, there exist a function A with constant sign near infinity, such that $A(s) \to 0 \ as \ s \downarrow 0$, and the following refinement of (3) satisfying

$$\lim_{t \to \infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^{\gamma} \right) = x^{\gamma} \frac{x^{\rho} - 1}{x} \ for \ all \ x > 0. \quad (4)$$

The constant γ is called the first order parameter and $\rho \leq 0$ is the second order parameter of the df F.

In this paper, we are interesting with the construction of a bias-reduced asymptotically normal estimator of the reinsurance premium $\Pi_{r,R}$ given by formula (1) for a heavy tailed distribution.

Let $X_{n:1} \leq X_{n:2} \leq \cdots \leq X_{n:n}$ the order statistics of $X_1, X_2, \ldots . X_n$. The estimation of high quantiles in the case of heavy-tailed distributions has got a great deal of interest, see for instance Weissman (1978), Dekkers and de Haan (1989), Matthys and Beirlant (2003) and Gomes et al. (2005).

For small values of s, we want to estimate $\chi_{\{1-s\}}$, such as $F(\chi_{\{1-s\}}) = 1 - s$, we shall work in Hall's class (Hall 1982), where it exists for $\gamma > 0, \beta \neq 0$and $\rho \leq 0$, such that

$$U(t) = Ct^{\gamma}\left(1 + A(t)/\rho + o(t^{\rho})\right) \quad (5)$$

This class contains most of the heavy-tailed models important in applications, like the Fréchet, the Generalized Pareto, and the Student's-t. We are going to base inference on the largest k order statistics (o.s.), and as usual in semi-parametric estimation of parameters of extreme events, we shall assume that k is an intermediate sequence of integers in [1,n[, i.e.,

$$k \to \infty and \, k/n \to 0 \ asn \to \infty,$$

then, from (5), there is

$$\chi_{\{1-s\}} = U(1/s) \sim Cs^{-\gamma}, ass \to \infty.$$

An obvious estimator of $\chi_{\{1-s\}}$is $\hat{C}s^{-\hat{\gamma}}$, with $\hat{C}$ and $\hat{\gamma}$ any consistent estimators of C and γ, respectively. Consequently, an obvious estimator of C, proposed by Hall (1982), is

$$\hat{C} = X_{n-k:n}(k/n)^{\hat{\gamma}},$$

and

$$\hat{Q}_n(s) = X_{n:n-k}(k/n)^{\hat{\gamma}} s^{-\hat{\gamma}}, 0 < s < \frac{k}{n},$$

is the obvious quantile estimator at the level s (Weissman 1978). Then, an estimator of the hazard function is given by:

$$\hat{S}(x) = (k/n)(X_{n:n-k})^{1/\hat{\gamma}} x^{-1/\hat{\gamma}}, \qquad asx \to \infty. \quad (8)$$

By replacing (8) in (1) and at an optimal retention level $R = R_{opt} \coloneqq F^{-1}(1 - k/n)$, we obtain a semi parametric asymptotic normal estimator for $\Pi_{r,R}$ for a fixed risk aversion index $r \geq 1$with the condition $k \to \infty, k/n \to 0 \ and \ \sqrt{k}A(n/k) \to 0 \ as \ n \to \infty$, given by :

$$\hat{\Pi}_{r,\hat{R}_{opt}} = (k/n)^{1/r} \frac{r\hat{\gamma}_n}{1 - r\hat{\gamma}_n} X_{n:n-k}, \quad (9)$$

where $\hat{R}_{opt} = X_{n:n-k}$ and $\hat{\gamma}_n$ is the classical Hill's estimator (Hill, 1975) of the tail index γ, defined by:

$$\hat{\gamma}_n = \frac{1}{k} \sum_{i=1}^{k} log(X_{n:n-i+1}) - log(X_{n:n-k+1}). \quad (10)$$

The asymptotic Normality of $\hat{\Pi}_{r,\hat{R}_{opt}}$ it is studied by Necir et al. (2007).

Hill's estimator $\hat{\gamma}_n$ plays a pivotal role in statistical inference on distribution tails. This estimator has been thoroughly studied, improved and even generalized to any real parameter γ. Weak consistency of

$\hat{\gamma}_n$ was established by Mason, (1982) assuming only that the underlying cdf F satisfies condition (3). The asymptotic normality of $\hat{\gamma}_n$ has been established (see de Haan and Peng, (1998)) under the condition (5) where the function A is defined as $A(t) = \gamma\beta t^{\rho}$, for an adequate k as:

$$\hat{\gamma}_n = \gamma + \frac{\gamma}{\sqrt{k}} Z_k + \frac{A(n/k)}{(1-\rho)}\left(1 + o_p(1)\right).$$

with Z_k is asymptotically standard normal r.v's.

Both estimators proposed in (9) and (10) are built under the strong assumptions $\sqrt{k}A(n/k) \to 0$ as $n \to \infty$ and present significant bias for moderate k with low stability areas around respectively $\Pi_{r,R}$ and γ.

Peng (1998) initiated the concept of bias reduction by constructing a new estimator of γ based on the Hall's second order model, and followed by many works as Beirlant et al. (1999), Feuerverger and Hall (1999), Gomes et al. (2000).

In this paper, we use the bias-reduced estimator of the high quantile, proposed by Gomes et. al. to propose a new estimator for premium $\Pi_{r,R}$ and we establish its asymptotic normality.

The paper is organized as follows. In Section 2, we introduce a reduced bias theory and construct the new semi parametric estimator for $\Pi_{r,R}$ and present our main result. Section 3 is devoted to simulated results and to compare these results with the estimator of Necir et al. (2007). The proof of the main result is postponed until section 4.

2 Reduced bias estimator and reinsurance premium

2.1 Reduced-bias estimation of γ

The reduced bias estimator of the tail index γ is proposed by Caeiro et al. (2005) and given by :

$$\overline{H}(k) = \hat{\gamma}_n\left(1 - \frac{\beta(k)(n/k)^{\rho(k)}}{1-\rho}\right). \tag{11}$$

where $\hat{\gamma}_n$ is the Hill estimator of γ given by (10). Caeiro et al. (2005) state that if the secondorder condition in (4) holds, and for $k \to \infty$, $k/n \to 0$ and $\sqrt{k}A(n/k) \to \boldsymbol{\lambda}$ as $n \to \infty$, finite and non necessarily null, then

$$\sqrt{k}\left(\overline{H}_{\hat{\rho},\hat{\beta}}(k) - \gamma\right) =^d \gamma Z_k + O_p\left(\sqrt{k}A(n/k)\right). \tag{12}$$

where Z_k is an asymptotic standard normal r.v's.

2.2 Estimators of the shape second order parameter ρ

The expression of $\overline{H}(k)$ requires the knowledge (or the estimation) of the second order parameters ρ and β, we can state here the works of Fraga Alves (2003) for the class of the shape second order parameter estimator. We define the statistics functions of the j-moment of the log-excesses

$$M_n^{(j)}(k) = \frac{1}{k}\sum_{i=1}^{k}\left(log(X_{n:n-i+1}) - log(X_{n:n-k+1})\right)^j, j = 1,2, \ldots.$$

and a tuning parameter τ in R as :

$$T_n^{\tau}(k) = \frac{\left(M_n^{(1)}(k)\right)^{\tau} - \left(M_n^{(1)}(k)/2\right)^{\tau/2}}{\left(M_n^{(1)}(k)/2\right)^{\tau/2} - \left(M_n^{(1)}(k)/6\right)^{\tau/3}}$$

for all τ in R. These statistics converge towards the value $\frac{3(1-\rho)}{(3-\rho)}$.

The expression of the shape second order parameter estimator is:

$$\hat{\rho}_n^{(\tau)}(k) = -\left|\frac{3(T_n^{\tau}(k)-1)}{T_n^{\tau}(k)-3}\right|. \quad (13)$$

2.3Estimators of the scale second order parameter β

Gomes and Martins (2002) proposed a functional estimator of the second order scale parameter depending with the expression of ρ and k taking the form:

$$\hat{\beta}_{\hat{\rho}}(k) = \left(\frac{k}{n}\right)^{\hat{\rho}} \frac{v_n^{(1-\hat{\rho})}(k)N_n^{(1)}(k) - N_n^{(1-\hat{\rho})}(k)}{v_n^{(1-\hat{\rho})}(k)N_n^{(1-\hat{\rho})}(k) - N_n^{(1-2\hat{\rho})}(k)}, \quad (14)$$

Where

$$v_n^{(\alpha)}(k) = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{\alpha-1}.$$

And

$$N_n^{(\alpha)}(k) = \frac{1}{k}\sum_{i=1}^{k} i\left(\,log(X_{n:n-i+1}) - log(X_{n:n-i})\right)\left(\frac{i}{k}\right)^{\alpha-1}.$$

Fraga Alves (2003) and Gomes (2007) have respectively shown the consistency of the estimators in (13) and (14) at an appropriate value of the intermediate integer sequence $k_1 = [n^{1-\varepsilon}]$ for $\varepsilon > 0$.

3New estimator of reinsurance premium

Let us define the quantile function for heavy tailed distribution satisfaying the first order condition in (3) as: $Q(1-s) = Cs^{-\gamma}$, with the hazard function given by : $S(x) = C^{1/\gamma}x^{-1/\gamma}$. In the literature, several works were dedicated to the estimation of quantile function Q, as Weissman (1978), Caeiro (2006) and Caeiro et al. (2008), based on the estimation of the parameter C and the reduction of its bias. In this section, we propose an appropriate estimator of the parameter C, based on the bias reduction technics :

$$\bar{C}_{\bar{H}}(k) = \frac{X_{n:n-[k/2]} - X_{n:n-k}}{2^{\bar{H}}-1}.\left(1 - \beta_{1/2}(\bar{H},\hat{\rho},\hat{\beta})\right)\left(\frac{k}{n}\right)^{\bar{H}},$$

Where

$$\beta_{\theta}(H,\rho,\beta) = \frac{\theta^{-(\gamma+\rho)}-1}{\theta^{-\gamma}-1}\frac{\gamma\beta(n/k)^{\rho}}{\rho}, \quad (15)$$

With $\hat{A}(t) = \bar{H}\hat{\beta}t^{\hat{\rho}}$, and $0 < \theta < 1$.

We can now conclude the reduced bias semi-parametric estimator of the high quantile, as follow:

$$\bar{Q}_{\bar{H}}(s,k) = \frac{X_{n:n-[k/2]} - X_{n:n-k}}{2^{\bar{H}}-1}.\left(1 - \beta_{1/2}(\bar{H},\hat{\rho},\hat{\beta})\right)\left(\frac{k}{n}\right)^{\bar{H}} s^{-\bar{H}},$$

Then, the relative hazard function S(x) is equivalent to :

$$\bar{S}_{\bar{H}}(s,k) = \left(\frac{k}{n}\right)\left(\frac{X_{n:n-[k/2]} - X_{n:n-k}}{2^{\bar{H}}-1}.\left(1 - \beta_{1/2}(\bar{H},\hat{\rho},\hat{\beta})\right)\right)^{\bar{H}} s^{-1/\bar{H}}. \quad (16)$$

In order to estimate the PH-premium of loss, we must give the semi parametric estimation of the optimal reinsurance retentionlevel R_{opt} , which is obtained by definition as

$$\bar{R}_{opt} = \frac{X_{n:n-[k/2]} - X_{n:n-k}}{2^{\bar{H}}-1}.\left(1 - \beta_{1/2}(\bar{H},\hat{\rho},\hat{\beta})\right), \quad (17)$$

Finally, by replacing thehazard function (16) in the expression (1), we obtain the new estimator of the PH-premium of loss distribution, as follow:

$$\overline{\Pi}_{r,\bar{R}_{opt}} = \left(\frac{k}{n}\right)^{1/r}\left(\frac{r\overline{H}}{1-r\overline{H}}\right)\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\overline{H}}-1}\right)\left(1-\mathcal{B}_{1/2}(\overline{H},\hat{\rho},\hat{\beta})\right),$$

The asymptotic normality of our estimator is given in the following theorem.

Theorem (1): We assume that the distribution function F satisfies the second order condition (4), with the integer k, such that: $k\to\infty,\ k/n\to 0\ and\ \sqrt{k}A(n/k)\to\boldsymbol{\lambda}\ as\ n\to\infty$, where $\boldsymbol{\lambda}$ finite and non necessarily null. Then, for $1>r\geq 1/\boldsymbol{\lambda}$, we have as $n\to\infty$:

$$\sqrt{k}\frac{(k/n)^{-1/r}}{X_{n:n-[k/2]}-X_{n:n-k}}\left(\overline{\Pi}_{r,\bar{R}_{opt}}-\Pi_{r,R}\right)\to^{d} N\left(0,\sigma^2(\gamma,r,\rho)\right),$$

where

$$\sigma^2(\gamma,r,\rho)=\left(\frac{\gamma}{2^\gamma-1}\right)\left(\frac{r\gamma}{1-r\gamma}\right)\left(1-\frac{\gamma 2^\gamma(2^\rho-1)}{(2^\gamma-1)(2^{\rho+\gamma}-1)}+\frac{r\gamma}{1-r\gamma}-\frac{\gamma 2^\gamma \ln 2}{(2^\gamma-1)}\right).$$

4. Finite sample behavior - Simulation study

We use the **R** statistical software (Ihaka and Gentleman, 1996) to apply the above result to the most usual distribution of Hall class, namely, the Generalized Pareto's model:

$$F(x)=1-(1+\gamma x)^{-1/\gamma}, 0.5<\gamma<1, C=1, \rho=-\gamma\ and\ \beta=1, for\ x>0,$$

and the Fréchet model :

$$F(x)=exp\left(-x^{-1/\gamma}\right), 0.5<\gamma<1, C=1, \rho=-1\ and\ \beta=1/2, for\ x>0.$$

In the first part, by using the results of theorem (1), we fix $\zeta\in]0,1[$ and $q_{\zeta/2}\}$ is the $(1-\zeta/2)$-quantile of the standard normal distribution $N(0,1)$. The $(1-\zeta)$-confidence bounds of $\Pi_{r,R_{opt}}$ is given by:

$$\Pi_{r,R}=\overline{\Pi}_{r,\bar{R}_{opt}}\pm\frac{\sigma(\gamma,r,\rho)}{\sqrt{k}}q_{\zeta/2}\left(X_{n:n-[k/2]}-X_{n:n-k}\right)(k/n)^{1/r}.$$

The simulation results are presented in the following Tables, and the values of the optimal fraction integer k_{opt} is such that$k_{opt}=argmin_k RMSE\left(\Pi_{r,R_{opt}};\overline{\Pi}^{i}_{r,\bar{R}_{opt}}\right)$ for 2000 sample replications.

Table 1. *IC(95%)* forΠ Fréchet Pattern's with $\gamma=0.65, r=1.1$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	234	1.7386	1.6995	1.5579	1.8410	0.2830	0.939
2000	476	1.7466	1.7094	1.6096	1.8092	0.1996	0.942
5000	713	1.5230	1.5069	1.4350	1.5788	0.1437	0.947

Table 2. *IC(95%)* forΠ Fréchet Pattern's with $\gamma=0.65, r=1.2$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	253	2.7777	2.7259	2.5076	2.9443	0.4366	0.923
2000	437	2.7007	2.4989	2.4987	2.8231	0.3243	0.944
5000	821	2.5582	2.5310	2.4184	2.6435	0.2250	0.949

Table 3. *IC(95%)* forΠ Fréchet Pattern's with $\gamma=0.75, r=1.1$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	239	3.7723	3.7056	3.3532	4.0579	0.7047	0.920
2000	368	3.6140	3.5710	3.2974	3.8447	0.5472	0.946
5000	720	3.4737	3.4455	3.2569	3.6342	0.3772	0.953

Table 4. *IC(95%)* forΠ Fréchet Pattern's with $\gamma = 0.75, r = 1.2$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	*227*	*7.9739*	*7.8563*	*7.0898*	*8.6228*	*1.5330*	*0.951*
2000	*461*	*7.9845*	*7.8732*	*7.3342*	*8.4123*	*1.0708*	*0.954*
5000	*892*	*7.8107*	*7.8350*	*7.3543*	*8.1157*	*0.7614*	*0.956*

Table 5. *IC(95%)* forΠGPD Pattern's with $\gamma = 0.65, r = 1.1$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	*138*	*2.3104*	*2.1322*	*1.9010*	*2.3634*	*0.4624*	*0.858*
2000	*255*	*2.2589*	*2.1025*	*1.9341*	*2.2709*	*0.3368*	*0.893*
5000	*520*	*2.1471*	*2.0353*	*1.9215*	*2.1490*	*0.2274*	*0.902*

Table 6. *IC(95%)* forΠGPD Pattern's with $\gamma = 0.65, r = 1.2$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	*129*	*3.7471*	*3.4793*	*3.0891*	*3.8696*	*0.7805*	*0.879*
2000	*249*	*3.7228*	*3.4846*	*3.2032*	*3.7659*	*0.5636*	*0.886*
5000	*403*	*3.4005*	*3.0910*	*3.0910*	*3.5099*	*0.4189*	*0.904*

Table 7. *IC(95%)* forΠGPD Pattern's with $\gamma = 0.75, r = 1.1$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	*146*	*4.6281*	*4.3638*	*3.8329*	*3.8947*	*1.0618*	*0.892*
2000	*204*	*4.3715*	*4.3255*	*3.7996*	*4.6715*	*0.8718*	*0.909*
5000	*541*	*4.4127*	*4.3899*	*3.9808*	*4.5179*	*0.5371*	*0.927*

Table 8. *IC(95%)* forΠ Fréchet Pattern's with $\gamma = 0.75, r = 1.2$.

n	k_{opt}	Π	$\overline{\Pi}$	*lb*	*ub*	*length*	*Cov pb*
1000	*203*	*10.5025*	*9.8591*	*8.5704*	*10.5478*	*1.9773*	*0.875*
2000	*347*	*10.3702*	*9.9169*	*8.8580*	*10.3758*	*1.5178*	*0.888*
5000	*511*	*9.9228*	*9.9579*	*8.9831*	*10.2327*	*1.2495*	*0.906*

In the second part, we compare the finite sample behavior of the proposed reduced bias reinsurance premium estimator $\overline{\Pi}_{r,\bar{R}_{opt}}$, with the estimator of Necie et al. (2007), denoted by $\widehat{\Pi_r}$, at the same level of retention R_{opt} , and risk aversion r. The results are based on 2000 samples from the Generalized Pareto model, with $0.5 < \gamma < 1$.

In Table (9) and Table (10) we present, respectively for r=1 and r=1.2, for sample sizes 1000, 2000, 5000, 10000 and 20000, the optimal level k, we use the Cheng Peng (2001) optimal sample fraction k_{opt} to compute the Π, and we choose the optimal level k, such that $k_{opt} = argmin_k RMSE(:)$ to compute our estimation Π, the bias of each estimator is given, and the root mean squared error denoted RMSE. For almost all sample sizes, our estimator has the smallest RMSE, even if we must go a little further into the sample and by choosing an appropriate fraction. Results obtained also show reduced bias of our estimator.

The results in Table (11) and Table (12) are identical to those in Table (9) and Table (10), respectively with r=1.1 and γ=0.85 for Table (11) and r=1, γ=0.95 for Table (12).

*Table 9:*Results from 2000 simulated Generalized Pareto samples with $\gamma = 0.75, r = 1$.

n	*1000*	*2000*	*5000*	*10000*	*20000*
$k_{opt}(\widehat{\Pi}_{\hat{\gamma}})/n$ $k_{opt}(\overline{\Pi}_{\bar{H}})/n$	*0.0250* *0.3500*	*0.0265* *0.3135*	*0.0180* *0.1664*	*0.0238* *0.1544*	*0.02771* *0.13755*
$\text{Bias}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{Bias}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*0.0593* ***0.0211***	*0.0558* ***0.0130***	*0.0492* ***0.0070***	*0.0532* ***0.0040***	*0.03070* ***0.00100***
$\text{RMSE}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{RMSE}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*0.2427* ***0.3363***	*0.2366* ***0.2077***	*0.1355* ***0.1338***	*0.1100* ***0.0976***	*0.07750* ***0.06810***

*Table 10:*Results from 2000 simulated Generalized Pareto samples with $\gamma = 0.75, r = 1.2$.

n	*1000*	*2000*	*5000*	*10000*	*20000*
$k_{opt}(\widehat{\Pi}_{\hat{\gamma}})/n$ $k_{opt}(\overline{\Pi}_{\bar{H}})/n$	*0.0210* *0.3520*	*0.0225* *0.2285*	*0.0150* *0.1506*	*0.0154* *0.0856*	*0.00900* *0.14090*
$\text{Bias}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{Bias}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*0.0560* ***0.0450***	*0.0570* ***0.0190***	*0.0487* ***0.0150***	*0.0438* ***0.0068***	*0.03230* ***0.00190***
$\text{RMSE}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{RMSE}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*1.4509* ***1.1142***	*1.1053* ***0.9202***	*0.8004* ***0.7367***	*0.6248* ***0.4733***	*0.5048* ***0.3809***

*Table 11:*Results from 2000 simulated Generalized Pareto samples with $\gamma = 0.85, r = 1.1$.

n	*1000*	*2000*	*5000*	*10000*	*20000*
$k_{opt}(\widehat{\Pi}_{\hat{\gamma}})/n$ $k_{opt}(\overline{\Pi}_{\bar{H}})/n$	*0.0280* *0.2360*	*0.0260* *0.2405*	*0.0185* *0.1448*	*0.0132* *0.1583*	*0.0124* *0.1158*
$\text{Bias}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{Bias}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*0.0474* ***0.0400***	*0.0498* ***0.0390***	*0.0353* ***0.0270***	*0.0261* ***0.0180***	*0.02560* ***0.00110***
$\text{RMSE}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{RMSE}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*2.3328* ***2.0534***	*1.7214* ***1.3765***	*1.2642* ***1.1361***	*1.0293* ***0.7963***	*0.75830* ***0.6395***

*Table 12:*Results from 2000 simulated Generalized Pareto samples with $\gamma = 0.95, r = 1$.

n	*1000*	*2000*	*5000*	*10000*	*20000*
$k_{opt}(\widehat{\Pi}_{\hat{\gamma}})/n$ $k_{opt}(\overline{\Pi}_{\bar{H}})/n$	*0.0270* *0.2520*	*0.0210* *0.2550*	*0.0164* *0.1570*	*0.0147* *0.1382*	*0.01050* *0.12180*
$\text{Bias}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{Bias}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*0.0301* ***0.0130***	*0.0302* ***0.0179***	*0.0252* ***0.0120***	*0.0180* ***0.0037***	*0.01350* ***0.00130***
$\text{RMSE}(\widehat{\Pi}_{\hat{\gamma}}, \Pi_r)$ $\text{RMSE}(\overline{\Pi}_{\bar{H}}, \Pi_r)$	*3.0817* ***2.5939***	*2.2694* ***1.7718***	*1.6604* ***1.3619***	*1.2747* ***1.0704***	*1.0678* ***0.7887***

5Proofs

Let

$$\overline{\Pi}_{r,\bar{R}_{opt}} - \Pi_{r,R} = A_n + B_n,$$

where

$$A_n = \left(\frac{k}{n}\right)^{1/r}\left(\frac{r\overline{H}}{1-r\overline{H}}\right)\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\overline{H}}-1}\right)\left(1-\beta_{1/2}(\overline{H},\hat{\rho},\hat{\beta})\right)$$
$$-\left(\frac{k}{n}\right)^{1/r}\left(\frac{r\gamma}{1-r\gamma}\right)\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\gamma}-1}\right)\left(1-\beta_{1/2}(\gamma,\rho,\beta)\right),$$

and

$$B_n = \left(\frac{k}{n}\right)^{1/r}\left(\frac{r\gamma}{1-r\gamma}\right)\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\gamma}-1}\right)\left(1-\beta_{1/2}(\gamma,\rho,\beta)\right)-\int_R^\infty S(x)^{1/r}dx.$$

First, we shall show that, $B_n \to 0$ in probability as $n \to \infty$.

We define the quantile function for heavy tailed distribution satisfying the first order condition in (3), as:

$$Q(1-s) = Cs^{-\gamma},$$

then, the hazard function is given by: $S(x) = C^{1/\gamma}x^{-1/\gamma}$. The PH premium of loss is writting as :

$$\Pi_{r,R} = \int_{R_{opt}}^\infty C^{1/r\gamma}x^{-1/r\gamma}dx = C^{1/r\gamma}\left(\frac{r\gamma}{1-r\gamma}\right)R_{opt}{}^{1-1/r\gamma},$$

As $R_{opt} = C(n/k)^\gamma$, we obtain:

$$R_{opt} = \left(\frac{k}{n}\right)^{-\gamma+1/r} C\left(\frac{r\gamma}{1-r\gamma}\right),$$

as

$$C = \left(\frac{k}{n}\right)^{\gamma}\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\gamma}-1}\right)\left(1-\beta_{1/2}(\gamma,\rho,\beta)\right).$$

Then, we conclude that:

$$B_n = o_p(1), as\ n \to \infty.$$

For A_n, we can write

$$A_n = \left(\frac{k}{n}\right)^{\frac{1}{r}}\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\overline{H}}-1}\right)\left(1-\beta_{1/2}(\overline{H},\hat{\rho},\hat{\beta})\right)\left(\frac{r\overline{H}}{1-r\overline{H}}-\frac{r\gamma}{1-r\gamma}\right)$$
$$+\left(\frac{k}{n}\right)^{\frac{1}{r}}\left(\frac{r\gamma}{1-r\gamma}\right)\left(X_{n:n-[k/2]}-X_{n:n-k}\right)\left(1-\beta_{1/2}(\gamma,\rho,\beta)\right)\left(\frac{1}{2^{\overline{H}}-1}-\frac{1}{2^{\gamma}-1}\right)$$
$$-\left(\frac{k}{n}\right)^{\frac{1}{r}}\left(\frac{r\gamma}{1-r\gamma}\right)\left(\frac{X_{n:n-[k/2]}-X_{n:n-k}}{2^{\overline{H}}-1}\right)\left(\beta_{1/2}(\overline{H},\hat{\rho},\hat{\beta})-\beta_{1/2}(\gamma,\rho,\beta)\right)$$

$= A_{n,1} + A_{n,2} + A_{n,3}$.

Now, the delta method enables us to write

$$\frac{1}{2^{\overline{H}}-1}-\frac{1}{2^{\gamma}-1} \sim^P -\frac{2^\gamma ln2}{\left(2^{\overline{H}}-1\right)^2}(\overline{H}-\gamma),$$
$$\frac{1}{1-\overline{H}}-\frac{1}{1-\gamma} \sim^P \frac{1}{(\gamma-1)^2}(\overline{H}-\gamma),$$

and

$$\frac{\beta_{1/2}(\overline{H},\hat{\rho},\hat{\beta})}{\beta_{1/2}(\gamma,\rho,\beta)} = 1+\left(1-\frac{\gamma 2^\gamma(2^\rho-1)}{(2^\gamma-1)(2^{\rho+\gamma}-1)}\right)\frac{\overline{H}-\gamma}{\gamma}+\frac{\hat{\beta}-\beta}{\beta}+(\hat{\rho}-\rho)ln(n/k).$$

i.e., $\sqrt{k}A(n/k) \to \boldsymbol{\lambda}\ as\ n \to \infty, as$

$$\overline{H}-\gamma = o_p(1), and\ \hat{\beta}-\beta = (\hat{\rho}-\rho)ln(n/k) = o_p(1)\ as\ n \to \infty.$$

And

$$\mathrm{B}_{1/2}\left(\bar{H},\hat{\rho},\hat{\beta}\right)=\mathrm{B}_{1/2}(\gamma,\rho,\beta)\left(1+o_p(1)\right).$$

Next, we have

$$\sqrt{k}\frac{(k/n)^{-1/r}}{X_{n:n-[k/2]}-X_{n:n-k}}A_{n,1}=\left(\frac{1}{2^{\gamma}-1}\right)\left(\frac{r}{(1-r\gamma)^2}\right)\left(1-\mathrm{B}_{1/2}(\gamma,\rho,\beta)\right)\sqrt{k}(\bar{H}-\gamma)+o_p(1).$$

And

$$\sqrt{k}\frac{(k/n)^{-1/r}}{X_{n:n-[k/2]}-X_{n:n-k}}A_{n,2}=-\frac{2^{\gamma}\,ln2}{(2^{\gamma}-1)^2}\left(\frac{r\gamma}{1-r\gamma}\right)\left(1-\mathrm{B}_{1/2}(\gamma,\rho,\beta)\right)\sqrt{k}(\bar{H}-\gamma)+o_p(1).$$

And

$$\sqrt{k}\frac{(k/n)^{-1/r}}{X_{n:n-[k/2]}-X_{n:n-k}}A_{n,3}=-\left(\frac{1}{2^{\gamma}-1}\right)\left(\frac{r}{1-r\gamma}\right)\left(1-\frac{\gamma 2^{\gamma}(2^{\rho}-1)}{(2^{\gamma}-1)(2^{\rho+\gamma}-1)}\right)\sqrt{k}(\bar{H}-\gamma)+o_p(1).$$

and, with same assumptions of theorem 1, from Caeiro et al. 2005, we have

$$\sqrt{k}(\bar{H}-\gamma)\to^{d} N(0,\gamma^2), as\ n\to\infty.$$

Finally, we conclude that,

$$\sqrt{k}\frac{(k/n)^{-1/r}}{X_{n:n-[k/2]}-X_{n:n-k}}\left(\bar{\Pi}_{r,\bar{R}_{opt}}-\Pi_{r,R}\right)\to^{d} N\left(0,\sigma^2(\gamma,r,\rho)\right),$$

This completes the proof of Theorem 1.

References

[1] H. Bäuhlmann, , 1980. An economic premium principle. ASTIN Bull. 11, 52-60.

[2] F Caeiro and Gomes I. Asymptotic comparaison at optimal levels of reduced bias extreme value index estimators. September 2010.

[3] F. Caeiro and Gomes I. Semi-parametric second order reduced bias high quatile estimation. Sociedad de Estadistica e Investigacion Operativa, 6, Number 1:1 20, 2008.

[4] S. Cheng and L. Peng. Confidence intervals for the tail index. Bernoulli, 7:751 760, 2001.

[5] C. Cheong. Heavy tailed value at risk analysis for malaysian stock exchange. Physica A., 387:4285 4298, 2008.

[6] de Haan and A. Ferreira. Extreme Value Theory, An Introduction. Springer Series for Operations research, 2006.

[7] I. Gomes, F. Figueiredo, and S. Mendon ca. asymptotically best linear unbiased tail estimators under second order regular variation condition. Statistics Plann. Inference, 134:409 433, 2005.

[8] I. Gomes, M. Martins, and M. Neves. Improving second order reduced-bias extreme value index estimation. Revstat, 5:177 207, 2007a.

[9] A. Necir and K. Boukhetala. Estimating the risk adjusted premium of the largest reinsurance covers. Processus Computing Statistics, 1:1577 1584, 2004.

[10] A. Necir, D. Merghani, and F. Meddi. Statistical estimate of the proportional hazard premium of loss. Scandinavian Actuarial Journal, 3:147 161, 2007.

[11] K. Pavlo, M. Zabaranskin, and S. Uryasev. Modeling and optimization of risk. Surveys in Operations Research and managent science, 16:49 66, 2001.

[12] A. Rassoul. Application of Extreme Value Theory to Estimate Some Tools in Insurance and Finance. PhD thesis, Univ. Biskra, 2010.

[13] R.D. Reiss and M. Thomas. Statistical Analysis of Extreme Values, Third Edition. Deutsche Bibliothek, 2007.

[14] S. Wang. Premium calculation by transformation the layer premium density. ASTIN Bull., 26:71 92, 1996.

[15] I.Weissman. Estimation of parameters and large quantiles based on the k largest observations. J. American Statistics Assoc., 73:812 815, 1978.

[16] J.L. Wirch and M.R. Hardy. A synthesis of risk measures for capital adequacy. Insurance: Mathematics & Economics, 25:337 347, 1999.