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New Exact Traveling Wave Solutions for a Class of Nonlinear PDEs of Fractional Order

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Abstract: In this article, the (G'/G)-expansion method has been implemented to find the travelling wave solutions of nonlinear evolution equations of fractional order. For this, the fractional complex transformation method has been used to convert fractional order partial differential equation to ordinary differential equation. Then, (G'/G)-expansion method has been implemented to celebrate the series of travelling wave solutions to fractional order partial differential equations.

Keywords: travelling wave solutions, fractional complex transformation, (G'/G)-expansion method, fractional calculus, nonlinear PDEs.

1 Introduction

Investigating the new exact travelling wave solutions to nonlinear evolution equations have been the area under discussion in different branches of mathematical and physical sciences such as in physics, biology, chemistry, *etc*. The analytical solutions of such equations are of fundamental importance since a lot of mathematical and physical models are described by the nonlinear evolution equations. Among the possible solutions nonlinear evolution equations, certain special form solutions may depend only on a single combination of variables such as traveling wave variables.

The variety of techniques exists to construct the travelling wave and find the numerical solutions to nonlinear problems. Some of its cited here, the adomian decomposition method [1] and generalized differential transform method [2] have been used to find the numerical solutions for the space- and time-fractional coupled Burgers equations. The (G'/G)-expansion method was introduced, by Wang *et al.* [3], and this method was further extended [4] to find the solutions of fractional order differential equations. The Jacobi elliptic function expansion method [5], the tanh-function method for finding solitary wave solutions [6], the homotopy perturbation method [7], the extended fractional subequation method [8] can also be applied to handle the nonlinear evolution equations *etc.*

In this article, a new approach has been used to find the series of travelling wave solutions to nonlinear evolution equations of fractional order using the fractional complex transformation [9] and the (G'/G)-expansion method [3]. For this, we first use the fractional complex transformation, in the sense of Jumaries modified Riemann-Liouville derivative, to convert into ordinary differential equations. Then obtained ODE can be converted into (G'/G) polynomial form. Using the homogenous balance and second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$ with the aid of computation, the new travelling wave solutions of (G'/G) polynomial form can be constructed. As application the nonlinear PDE [10] with time-space fractional derivatives foam has been considered of the form:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} + a\frac{\partial^{2\beta}u}{\partial x^{2\beta}} + bu + cu^3 = 0, \;\; t > 0, \; 0 < \alpha, \; \beta \leq 1. \quad (1)$$

The following equations can also be obtained for different values of a, b and c.*i.e*,

–If we take $a = c = -1$ and $b = 1$, then equation (1) leads to Phi-Four equation:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} - \frac{\partial^{2\beta}u}{\partial x^{2\beta}} + u - u^3 = 0, \;\; t > 0, \; 0 < \alpha, \; \beta \leq 1. \quad (2)$$

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–If we take $a=-1, b=m^2$ and $c=n$, then equation (1) leads to Klein-Gordon equation:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\beta}u}{\partial x^{2\beta}}+m^2u+nu^3=0,\ \ t>0,\ 0<\alpha,\ \beta\leq 1. \tag{3}$$

–If we take $a=-1, b=-m^2$ and $c=n^2$, then equation (1) leads to Landau-Ginburg-Higgs equation:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\beta}u}{\partial x^{2\beta}}-m^2u+n^2u^3=0,\ \ t>0,\ 0<\alpha,\ \beta\leq 1. \tag{4}$$

–If we take $a=0$, then equation (1) leads to Duffing equation:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+bu+cu^3=0,\ \ t>0,\ 0<\alpha,\ \beta\leq 1. \tag{5}$$

–If we take $a=-1, b=1$ and $c=\frac{-1}{6}$ then equation (1) leads to Sine-Gordon equation:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}-\frac{\partial^{2\beta}u}{\partial x^{2\beta}}+u-\frac{1}{6}u^3=0,\ \ t>0,\ 0<\alpha,\ \beta\leq 1. \tag{6}$$

The rest of the article is organized as follows, in section 2 the basic definitions and properties for the fractional calculus are being considered regarding to modified Riemann-Liouville derivative. In section 3, the extended (G'/G)-expansion method has been proposed to find the new travelling wave solutions for NPDEs of fractional order with the help of fractional complex transformation. As an application, the new travelling wave solutions of nonlinear equation (1) have been discussed in section 4. In the last section 5, the conclusion has been drawn.

2 Background on Fractional Calculus

In this section, the proposed method has been applied in the sense of the Jumaries modified Riemann-Liouville derivative [11] of order α. For this, some basic definitions and properties of the fractional calculus theory are being considered (for details see [12]). Thus, the fractional derivatives can be defined following [11] as:

$$\frac{\partial^{\alpha}f(x)}{\partial x^{\alpha}}=\begin{cases}1/\Gamma(-\alpha)\frac{d}{dx}\int_0^x(x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, & \text{for } \alpha<0;\\ 1/\Gamma(1-\alpha)\frac{d}{dx}\int_0^x(x-\xi)^{-1}(f(\xi)-f(0))d\xi, & \text{for } 0<\alpha<1;\\ (f^n(x))^{\alpha-n}, & \text{for } n\leq\alpha<n+1,\ n\geq 1.\end{cases}$$

Moreover, some properties for the modified Riemann-Liouville derivative have also been given as follows

$$\frac{\partial^{\alpha}x^{\gamma}}{\partial x^{\alpha}}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}, \tag{7}$$

$$\frac{\partial^{\alpha}(u(x)v(x))}{\partial x^{\alpha}}=v(x)\frac{\partial^{\alpha}u(x)}{\partial x^{\alpha}}+u(x)\frac{\partial^{\alpha}v(x)}{\partial x^{\alpha}}, \tag{8}$$

$$\frac{\partial^{\alpha}f[u(x)]}{\partial x^{\alpha}}=f'_u[u(x)]\frac{\partial^{\alpha}u(x)}{\partial x^{\alpha}}, \tag{9}$$

$$\frac{\partial^{\alpha}f[u(x)]}{\partial x^{\alpha}}=\frac{\partial^{\alpha}f[u(x)]}{\partial x^{\alpha}}(u'(x))^{\alpha}. \tag{10}$$

As far as concerned about the above results (8)-(10), the function $u(x)$ is said to be non-differentiable in equations (8) and (9) and it is differentiable in (10). The function $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (9) and non-differentiable in (10). Thus, the formulas (8)(10) should be used carefully.

In view of [13], [1] the fractional complex transformation can be defined as follows:

$$u(t,x,y)=u(\xi), \tag{11}$$

$$where\ \xi=\frac{Lt^{\alpha}}{\Gamma(\alpha+1)}+\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{My^{\gamma}}{\Gamma(\gamma+1)}.$$

It helps us to convert the partial differential equation of fractional order into an ordinary differential equation in very simple and easy manner (where K, L and M are non-zero arbitrary constants). In the following section, the (G'/G)-expansion method has been described to find the travelling wave solutions.

3 Description of the (G'/G)-expansion method

The (G'/G)-expansion method [4], [3] can be performed using the following steps. For this, we consider the following NPDE (nonlinear partial differential equation) of fractional order

$$P\left(u,\ \frac{\partial^{\alpha}u}{\partial t^{\alpha}},\ \frac{\partial^{\beta}u}{\partial x^{\beta}},\ \frac{\partial^{\gamma}u}{\partial y^{\gamma}},\ldots\right)=0, \tag{12}$$

where u is an unknown function and P is a polynomial of u and its partial fractional derivatives along with the involvement of higher order derivatives and nonlinear terms.
To find the exact solutions, the following steps can be performed.

Step 1: First, we convert the NPDE of fractional order into nonlinear ordinary differential equations using the fractional complex transformation (11) introduced by Li *et al.* [9]. Hence, the travelling wave variable, defined in equation (11), permits us to reduce equation (12) to an ODE of $u=u(\xi)$ in the following form

$$P(u,u',u'',u''',\ldots)=0. \tag{13}$$

If the possibility occurs, the above equation can be integrated term by term once or more times.

Step 2: Suppose that the solution of equation (13) can be expressed as a polynomial of (G'/G) in the following form

$$u(\xi)=\sum_{i=-m}^{m}\alpha_i\left(\frac{G'}{G}\right)^i,\quad \alpha_m\neq 0, \tag{14}$$

where α_i's are constants and $G(\xi)$ satisfies the following second order linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{15}$$

with λ and μ as constants.

Step 3: The homogeneous balance can be used, to determine the positive integer m, between the highest order derivatives and the nonlinear terms appearing in (13). After the substitution of equation (14) into equation (15) and using equation (15), we collect all the terms with the same order of (G'/G) together. Equate each coefficient of the obtained polynomial to zero, yields the set of algebraic equations for K,L,M,λ,μ and $\alpha_i (i=0,\pm1,\pm2,...,\pm m)$.

Step 4: After solving the system of algebraic equations, and using the equation (15), the variety of travelling wave solutions can be obtained using the generalized solutions of equation (15).

$$\left(\frac{G'}{G}\right)=\begin{cases} -\lambda/2+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}{C_1\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}\right), \\ \lambda^2-4\mu>0; \\ -\lambda/2+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}{C_1\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}\right), \\ \lambda^2-4\mu<0; \\ -\lambda/2+\frac{C_2}{C_1+C_2\xi}, \quad \lambda^2-4\mu=0, \end{cases} \tag{16}$$

where C_1 and C_2 are arbitrary constants.

4 Applications

In this section, the improved (G'/G)-expansion method has been used to construct the new travelling wave solutions for nonlinear space-time fractional equation (1). For this, the following fractional complex transformation

$$u(x,t)=u(\xi),\ \ \xi=\frac{Kx^{\beta}}{\Gamma(\beta+1)}+\frac{Lt^{\alpha}}{\Gamma(\alpha+1)} \tag{17}$$

where K and L are constants, permits to reduce the equation (1) into the following ODE

$$(aK^2+L^2)u''+bu+cu^3=0 \tag{18}$$

Now by calculating the homogeneous balance (*i.e*, $m=1$), between the highest order derivatives and nonlinear term presented in the above equations (18), we have the following form

$$u(\xi)=\alpha_{-1}\left(\frac{G'}{G}\right)^{-1}+\alpha_0\left(\frac{G'}{G}\right)^{0}+\alpha_1\left(\frac{G'}{G}\right)^{1}, \tag{19}$$

where $\alpha_i,(i=0,1)$ are arbitrary constants. To determine the constants substitute the equation (19) into the equations (18), and by collecting all the terms with the same power of (G'/G) together. After equating each coefficient equal to zero, this yields a set of following algebraic equations:

$$\begin{gathered} 2\alpha_1(L^2+aK^2)+c\alpha_1^3=0, \\ 3\lambda\alpha_1(L^2+aK^2)+3c\alpha_0\alpha_1^2=0, \\ \alpha_1(\lambda^2+2\mu)(L^2+aK^2)+b\alpha_1+3c(\alpha_{-1}\alpha_1^2+\alpha_1\alpha_0^2)=0, \\ \lambda(\alpha_{-1}+\mu\alpha_1)(L^2+aK^2)+b\alpha_0+c(\alpha_0^3+3\alpha_0\alpha_1\alpha_{-1})=0, \\ \alpha_{-1}(\lambda^2+2\mu)(L^2+aK^2)+b\alpha_{-1}+3c(\alpha_{-1}\alpha_0^2+\alpha_1\alpha_{-1}^2)=0, \\ \lambda\mu\alpha_{-1}(L^2+aK^2)+3c\alpha_0\alpha_{-1}^2=0, \\ 2\mu^2\alpha_{-1}(L^2+aK^2)+c\alpha_{-1}^3=0. \end{gathered}$$

After solving these algebraic equations with the help of software *Maple*, yields the following families for the values of $K,L,\lambda,\mu,a_(i),(i=-1,0,1)$.

Case 1:

$$a=a,\ b=-8\mu(L^2+aK^2),\ c=\frac{-2(L^2+aK^2)}{\alpha_1^2},\ K=K,$$
$$\lambda=0,\ \alpha_{-1}=-\mu\alpha_1,\ L=L,\ \mu=\mu,\ \alpha_0=0.$$

Where a,α_1,K,L and μ are arbitrary constants with $\alpha_1\neq0$.

Case 2:

$$a=a,\ b=-\lambda^2(L^2+aK^2),\ c=0,\ K=K,\ \lambda=\lambda,$$
$$\alpha_{-1}=\lambda\alpha_0,\ L=L,\ \alpha_1=0,\ =0,\ \alpha_0=\alpha_0.$$

Where a,α_0,λ,K and L are arbitrary constants.

Case 3:

$$a=a,\ b=4\mu(L^2+aK^2),\ c=\frac{-2(L^2+aK^2)}{\alpha_1^2},\ K=K,$$
$$\lambda=0,\ \alpha_(-1)=-\mu\alpha_1,\ L=L,\ \mu=\mu,\ \alpha_0=0.$$

Where a,α_1,K,L and μ are arbitrary constants with $\alpha_1\neq0$.

Case 4:

$$a=a,\ b=\frac{2}{\alpha_1^2}(\alpha_0^2L^2+\alpha_0^2aK^2-\mu L^2\alpha_1^2-\mu aK^2\alpha_1^2),$$
$$c=\frac{-2(L^2+aK^2)}{\alpha_1^2},\ K=K,\ \lambda=\frac{2\alpha_0}{\alpha_1},\ \alpha_{-1}=0, L=L,\ \mu=\mu,$$
$$\alpha_1=\alpha_1,\ \alpha_0=\alpha_0.$$

Where a,α_0,α_1,K,L and μ are arbitrary constants with $\alpha_1\neq0$.

Substituting the above results in equation (19) and combining with the solution of equations (16), the new series of exact travelling wave solutions to the equation (1) can be constructed.

From Case 1, the following travelling wave solutions can be obtained.

When $\lambda^2-4\mu>0$, we have

$$u_1(\xi)=-\mu\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}{C_1\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}\right)\right]^{-1}+\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}{C_1\cosh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})+C_2\sinh(\frac{\xi}{2}\sqrt{\lambda^2-4\mu})}\right)\right],$$

where

$$\xi=\frac{Kx^\beta}{\Gamma(\beta+1)}+\frac{Lt^\alpha}{\Gamma(\alpha+1)}.$$

When $\lambda^2-4\mu=0$, we have the solution of the form

$$u_2(\xi)=-\mu\alpha_1\left[-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\xi}\right]^{-1}+\alpha_1\left[-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\xi}\right],$$

where

$$\xi=\frac{Kx^\beta}{\Gamma(\beta+1)}+\frac{Lt^\alpha}{\Gamma(\alpha+1)}.$$

When $\lambda^2-4\mu<0$, we have

$$u_3(\xi)=-\mu\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}{C_1\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}\right)\right]^{-1}+\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}{C_1\cos(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})+C_2\sin(\frac{\xi}{2}\sqrt{4\mu-\lambda^2})}\right)\right],$$

where

$$\xi=\frac{Kx^\beta}{\Gamma(\beta+1)}+\frac{Lt^\alpha}{\Gamma(\alpha+1)}.$$

Especially, if we take $C_2=0$, in first solution, then the following solution can be obtained.

$$u_4(\xi)=-\mu\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]^{-1}+\alpha_1\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right].$$

And if we take $C_2=0$, in third solution, then the following solution can be constructed.

$$u_5(\xi)=-\mu\alpha_1\left[-\frac{\lambda}{2}-\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]^{-1}+\alpha_1\left[-\frac{\lambda}{2}-\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right].$$

Similarly, as the solutions have been constructed for case 1, we can construct the new travelling wave solutions for cases 2, 3 and 4.

5 Conclusion

The extended (G'/G)-expansion method has been applied to solve the fractional partial differential equation. As applications, a series of new travelling wave solutions for the space-time fractional order nonlinear partial differential equations (Phi-Four equation, Klein-Gordon equation, Landau-Ginburg-Higgs equation, Duffing equation, Sine-Gordon equation) have been successfully found. It may be observed that, the nonlinear fractional complex transformation ensures that a certain fractional order differential equation can be turned into ordinary differential equation of integer order. Afterwards, the obtained ODE can be expressed by a polynomial in (G'/G), from where its solution can be obtained using the second order equation $(G''/G)(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$. Since, the homogeneous balancing principle has been used, so we can claim that this method can be applied to other fractional order partial differential equations where the homogeneous balancing principle is satisfied.

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