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Projection Iterative Methods for Multivalued General Variational Inequalities

Eman Al-Shemas
Mathematics Department, College of Basic Education Main Campus, Shamiya, Kuwait
Email Address: e_al_shemas@hotmail.com

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In this paper, we introduce a new class of variational inequalities involving two operators. Using the projection technique, we establish the equivalence between the multivalued general variational inequalities and the fixed point problems. This equivalent formulation is used to suggest and analyze some iterative algorithms for solving the multivalued general variational inequalities. We also discuss the convergence analysis of these iterative methods. Several special cases are also discussed.

Keywords: Variational inequalities, projection equations, iterative methods, convergence criteria.

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1 Introduction

Variational inequalities, which were introduced by Stampacchia [24] in early sixties, have played an important role in the development of various fields of pure and applied sciences. Variational inequalities have been generalized and extended in several directions using novel and new techniques, see [1-26] and the references therein. An important extension of the variational inequalities is known as the generalized variational inequality, which was introduced and studied by Fang and Peterson [16] and Chan and Pang [15] in 1982 independently. It is well known that the variational inequalities are related to the simple fact that the minimum of a differentiable convex function on a convex set in a normed linear space can be characterized by the variational inequalities. Very recently, Noor [9] has shown that the minimum of a class of differentiable nonconvex functions on a nonconvex set can be characterized by a class of variational inequalities, which is known as general variational inequalities. We would like to emphasize that these generalizations are quite different from each other in properties and applications point of view. It is natural to unify these different classes of variational inequalities.

Motivated and inspired by the research going on in this field, we introduce and study a new class of variational inequalities, which is called the multivalued general variational inequalities. Using the projection techniques, we establish the equivalence between the multivalued general variational inequalities and the fixed point problems. We would like to point out that the projection method and its variant forms represent an important tool on the study of the existence results and developing numerical methods for solving variational inequalities and related optimization problems. We also would like to point out that the iterative methods serve to solve a variety of problems which are either of the feasibility or the optimization type. This class of algorithms has witnessed great progress in recent years. Apart from theoretical interest, the main advantage of these iterative methods is computational. These methods have the ability to handle large-size problems of dimensions beyond which other methods cease to be efficient, see [1-26] and the references therein. We use alternative equivalent formulation to suggest and analyze a class of projection iterative methods for solving the multivalued general variational inequalities. We also study the convergence criteria of the proposed iterative methods under suitable conditions. Results obtained in this paper include the previously obtained results of Chan and Pang [15], Fang and Peterson [16] and Noor [9] as special cases. Results obtained in this paper represent refinements and improvements of the previously known results in this area.

2 Preliminaries

Let K be a nonempty closed and convex set in a real Hilbert space H, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$ respectively. For given nonlinear operators $g : H \longrightarrow H$ and $T : H \rightarrow 2^H$, consider the problem of finding $u \in H : \eta \in Tu$ such that

$$\langle \rho\eta + u - g(u), g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.1}$$

where $\rho > 0$ is a constant. Inequality of type (2.1) is called the multivalued general variational inequality involving two operators.

For $g \equiv I$, the identity operator, the multivalued general variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle \eta, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is called the multivalued variational inequality, introduced and studied by Chan and Pang [15] and Fang and Peterson [16] in 1982. For the applications, generalizations and other aspects of these multivalued variational inequalities, see [1–8] and the references therein.

If T is single valued then the multivalued general variational inequality (2.1) is equivalent to finding $u \in H : g(u) \in H$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.3}$$

which is called the general variational inequality introduced and studied by Noor [9] in connection with nonconvex functions (see also [10–12] for more details).

If $g \equiv I$, the identity operator, then problem (2.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [24] in 1964. For the recent trends and developments in variational inequalities, see [1-26] and the references therein.

We also need the following well known concepts and results.

Lemma 2.1 ([14]). *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the the projection operator of H onto the closed convex set K.

It is well known that the projection operator P_K is a nonexpansive operator.

Definition 2.1. A mapping $g : H \to H$ is called
(i). δ-Lipschitz, if for all $u_1, u_2 \in H$, there exists a constant $\delta > 0$, such that

$$||g(u_1) - g(u_2)|| \leq \delta ||u_1 - u_2||.$$

(ii). σ-strongly monotone, if for all $u_1, u_2 \in K$, there exists a constant $\sigma > 0$, such that

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \sigma ||u_1 - u_2||^2.$$

Definition 2.2. A multivalued operator $T : H \to 2^H$ is called
(i). M-Lipschitz, if for all $u_1, u_2 \in H$, there exists a constant $\beta > 0$ such that

$$\eta \in Tu : M(Tu_1 - Tu_2) \leq \beta ||u_1 - u_2||.$$

(ii). α-strongly monotone, if for all $u_1, u_2 \in K$, there exists a constant $\alpha > 0$, such that

$$\langle \eta_1 - \eta_2, u_1 - u_2 \rangle \geq \alpha ||u_1 - u_2||^2, \eta_1 \in Tu_1, \eta_2 \in Tu_2$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

3 Main Results

In this section, we suggest and analyze some new approximation schemes for solving the multivalued general variational inequality (2.1) using the projection operator technique. One can prove that the multivalued general variational inequality (2.1) is equivalent to the fixed-point problem by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in H : \eta \in Tu$ such that $g(u) \in H$ is a solution of the multivalued general variational inequality (2.1) if and only if $u \in H : \eta \in Tu$ satisfies the relation*

$$u = P_K[g(u) - \rho\eta], \tag{3.1}$$

where $\rho > 0$ is a constant and P_k is the projection operator.

Proof. Let $u \in H : g(u) \in K$ be a solution of (2.1). Then

$$\langle u - (g(u) - \rho\eta), g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is equivalent to, using Lemma 2.1,

$$u = P_K[g(u) - \rho\eta],$$

the required result. □

It is clear from Lemma 3.1 that the multivalued general variational inclusion (2.1) and the fixed point problems (3.1) are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems. Using the fixed-point formulation (3.1), we suggest and analyze the following iterative method for solving the multivalued general variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in H : \eta_0 \in Tu_0$, compute the approximate solution u_{n+1} by the iterative schemes:

$$u_{n+1} = (1 - a_n)u_n + a_n P_K[g(u_n) - \rho\eta_n], \ldots \tag{3.2}$$

$$\eta_n \in Tu_n : \ ||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1}), \tag{3.3}$$

where $\rho > 0$ is a constant and $a_n \in [0, 1]$ for all $n \geq 0$.

Algorithm 3.1 is known as Mann iteration.

Note that, if $g = I$, the identity operator, then Algorithm 3.1 reduces to the following iterative method for solving the multivalued variational inequalities (2.2), which appears to be a new one.

Algorithm 3.2. For a given $u_0 \in H : \eta_0 \in Tu_0$, compute the approximate solution u_{n+1} by the iterative schemes

$$u_{n+1} = (1 - a_n)u_n + a_n P_K[u_n - \rho\eta_n], \ldots$$
$$\eta_n \in Tu_n : ||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1}),$$

where $\rho > 0$ is a constant and $a_n \in [0, 1]$ for all $n \geq 0$.

If T is single valued operator then Algorithm 3.1 reduces to the following algorithm 3.3 for solving the general variational inequalities (2.3), which is due to Noor [9] .

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = (1 - a_n)u_n + a_n P_K[g(u_n) - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

where $\rho > 0$ is a constant and $a_n \in [0, 1]$ for all $n \geq 0$.

If $g = I$ and T is single valued, then Algorithm 3.1 reduces to the following Algorithm 3.4 for solving the problem (2.4).

Algorithm 3.4. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative schemes:

$$u_{n+1} = (1 - a_n)u_n + a_n P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots,$$

where $\rho > 0$ is a constant and $a_n \in [0, 1]$ for all $n \geq 0$.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result. In a similar way, one can study the convergence criteria of other Algorithms.

Theorem 3.1. *Let the multivalued operator $T : H \to 2^H$ be strongly monotone with constant $\alpha > 0$ and M-Lipschitz continuous with constant $\beta > 0$ and $g : H \to H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. If*

$$\left|\rho - \frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\alpha^2 - \mu^2(2k - k^2)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2 - k)}, \quad k < 1, \tag{3.4}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \tag{3.5}$$

and $a_n \in [0, 1]$, $\sum_{n=0}^{\infty} a_n = \infty$, then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution of the multivalued general variational equations (2.1).

Proof. Let $u \in H, \eta \in Tu$ be a solution of (2.1). Then, using Lemma 3.1, we have

$$u = (1 - a_n)u + a_n P_K\{g(u) - \rho\eta\}, \tag{3.6}$$

where $a_n \in [0, 1]$. To prove the result, we need first to evaluate $||u_{n+1} - u||$ for all $n \geq 0$. From (3.2) and (3.6), and the nonexpansivity of P_K, we have

$$\begin{aligned} ||u_{n+1} - u|| &= ||(1 - a_n)u_n + a_n P_K\{g(u_n) - \rho\eta_n\} - (1 - a_n)u - a_n P_K\{g(u) - \rho\eta\}|| \\ &\leq (1 - a_n)||u_n - u|| + a_n||g(u_n) - g(u) - \rho(\eta_n - \eta)|| \\ &\leq (1 - a_n)\|u_n - u\| + a_n\|u_n - u - (g(u_n) - g(u))\| \\ &\quad + a_n\|u_n - u - \rho(\eta_n - \eta)\|. \end{aligned} \tag{3.7}$$

From the strongly monotonicity and M-Lipschitz continuity of the operator T, we have

$$\begin{aligned} ||u_n - u - \rho(\eta_n - \eta)||^2 &= ||u_n - u||^2 - 2\rho\langle \eta_n - \eta, u_n - u\rangle + \rho^2||\eta_n - \eta||^2 \\ &\leq ||u_n - u||^2 - 2\rho\alpha\|u_n - u\|^2 + \rho^2(M(Tu_1 - Tu_2))^2 \\ &\leq ||u_n - u||^2 - 2\rho\alpha\|u_n - u\|^2 + \rho^2\beta^2||u_n - u||^2] \\ &= \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_n - u\|^2. \end{aligned} \tag{3.8}$$

In a similar way, using the strongly monotonicity and Lipschitz continuity of the operator g with constants $\sigma > 0$ and $\delta > 0$ respectively, we have

$$\begin{aligned} \|u_n - u - (g(u_n) - g(u))\| &\leq [1 - 2\sigma + \delta^2]\|u_n - u\|^2 \\ &= k^2\|u_n - u\|^2, \end{aligned} \tag{3.9}$$

where k is defined by (3.5).

Combining (3.8), (3.9) and (3.7), we have

$$\|u_{n+1} - u\| \leq (1 - a_n)\|u_n - u\| + a_n\theta\|u_n - u\|, \tag{3.10}$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + k$.

From (3.4), it follows that $\theta < 1$. Thus

$$\begin{aligned} ||u_{n+1} - u|| &\leq [1 - a_n(1 - \theta)]\|u_n - u\| \\ &\leq \prod_{i=0}^{n}[1 - (1 - \theta)a_i]\|u_0 - u\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n = \infty$ and $1 - \theta > 0$, we have $\lim_{n\to\infty} \prod_{i=0}^{n}[1 - (1 - \theta)a_i] = 0$ and then $\lim_{n\to\infty} ||u_{n+1} - u|| = 0$. Consequently the sequence $\{u_n\}$ converges strongly to u.

Now we prove that $\eta_n \to \eta \in Tu$. From (3.3), we have

$$||\eta_n - \eta_{n-1}|| \leq M(Tu_n - Tu_{n-1}) \leq \beta\|u_n - u_{n-1}\|,$$

which implies that $\{\eta_n\}$ is a Cauchy sequence in H, so there exists $\eta \in H$ such that $\eta_n \to \eta$. Further,

$$\begin{aligned} d(\eta, Tu) &= Inf\{\|\eta - t\| : t \in Tu\} \leq ||\eta - \eta_n|| + d(\eta_n, Tu) \\ &\leq ||\eta - \eta_n|| + M(Tu_n - Tu_{n-1}) \\ &\leq ||\eta - \eta_n|| + \beta\|u_n - u\| \to 0. \end{aligned}$$

Since Tu is closed, we have $\eta \in Tu$, which completes the proof. □

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Eman H. N. Al-Shemas received her B.S., M.S., and Ph.D. from the Girls College in Riyadh, Saudi Arabia, in 1981, 1983, and 1991, respectively. She visited Colorado University at Boulder, Colorado USA in August 1999 and in June 2001; Colorado University Denver, Colorado, USA in 2001; Catholic University in Leuven, Belgium 2002; University of Cambridge, Cambridge, UK 2006. She received the award of "Arab bureau of Education for the Gulf States" for the program of examining the basic competences of teachers in 2004. She was Deputy head of Mathematics Department, College of Basic Education, Kuwait, from 2003 to 2007. She published some papers in various international refereed journals. She now works as an Associate Professor and has been serving as the Head of Mathematics Department, College of Basic Education, Kuwait since 2007. She has published six books in Arabic Language.