Journal of Statistics Applications & Probability
--- *An International Journal*

Lower Generalized Order Statistics of Generalized Exponential Distribution

R.U. Khan, Anamika Kulshrestha and Devendra Kumar

Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202 002, India
*Email Address:*aruke@rediffmail.com

Received Jan. 31, 2012; Revised April 18, 2012, Accepted April, 22, 2012

Abstract: In this paper we consider three parameter generalized exponential distribution. Exact expressions and some recurrence relations for single and product moments of lower generalized order statistics are derived. Further the results are deduced for moments of order statistics and lower records and characterization of this distribution has been considered on using the conditional moment of the lower generalized order statistics.

Keywords: Characterization; generalized exponential distribution; lower generalized order statistics; order statistics; record values; recurrence relations; single and product moment.

1 Introduction

Kamps [7] introduced the concept of generalized order statistics (gos). It is known that ordinary order statistics, sequential order statistics, Stigler's order statistics and upper record values are special cases of gos. In this article we will consider the lower generalized order statistics $(l\,gos)$ defined as follows:

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0 \quad \text{for all} \quad 1 \leq r \leq n.$$

Then $X^*(1,n,m,k),\ldots,X^*(n,n,m,k)$ are called $l\,gos$ from an absolutely continuous distribution function (df) $F(x)$ with the probability distribution function (pdf) $f(x)$ if their joint pdf has the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

The marginal pdf of $r-$th $l\,gos$, $X^*(r,n,m,k)$, is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{1.2}$$

and the joint pdf of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$, is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y), \quad x > y, \tag{1.3}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\,,\quad h_m(x)=\begin{cases}-\dfrac{1}{m+1}x^{m+1}, & m\neq -1\\ -\ln x, & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1)\,,\ x\in[0,1)\,.$$

We shall also take $X^*(0,n,m,k)=0$. If $m=0$, $k=1$, then $X^*(r,n,m,k)$ reduces to the $(n-r+1)-$th order statistic, $X_{n-r+1:n}$ from the sample $X_1,X_2,\ldots,X_n$ and when $m=-1$, then $X^*(r,n,m,k)$ reduces to the $r-$th lower k record value (Pawlas and Szynal [11]). The work of Burkschat *et al.* [3] may also refer for $l\,gos$.

Recurrence relations for single and product moments of $l\,gos$ from the inverse Weibull distribution are derived by Pawlas and Syznal [11]. Ahsanullah [1] and Mbah and Ahsanullah [10] characterized the uniform and power function distributions based on distributed properties of $l\,gos$ respectively. Khan *et al.*[8] and Khan and Kumar [9] have established and recurrence relations for moment of $l\,gos$ for exponentiated Weibull and Pareto distributions.

In the present study, we have established explicit expressions and some recurrence relations for single and product moments of $l\,gos$ from generalized exponential distribution. Results for order statistics and lower record values are deduced as special cases and characterization of this distribution has been considered on using the conditional moment of the lower generalized order statistics.

A random variable X is said to have generalized exponential distribution (Gupta and Kundu [4]) if its pdf is of the form

$$f(x)=\frac{\alpha}{\lambda}[1-e^{-(x-\theta)/\lambda}]^{\alpha-1}e^{-(x-\theta)/\lambda},\qquad x>\theta,\ \alpha,\ \lambda>0 \tag{1.4}$$

and the corresponding df is

$$F(x)=[1-e^{-(x-\theta)/\lambda}]^{\alpha},\qquad x>\theta,\ \alpha,\ \lambda>0\,. \tag{1.5}$$

Here α is a shape parameter, λ is a scale parameter and θ is a location parameter. Gupta and Kundu [4, 5, 6] pointed out that the above given generalized exponential distribution will be useful as a good alternative to the gamma or the Weibull model in analyzing many lifetime data. Gupta and Kundu [6] have mentioned some drawbacks for the gamma and Weibull distributions.

2 Single Moments

Note that for generalized exponential distribution defined in (1.5)

$$\frac{\alpha}{\lambda}F(x)=[e^{(x-\theta)/\lambda}-1]f(x)\,. \tag{2.1}$$

The relation in (2.1) will be exploited in this paper to derive recurrence relations for the moments of lower generalized order statistics from the generalized exponential distribution.

We shall first establish the following Lemma which may be helpful in proving the main result.

Lemma 2.1 For the distribution as given in (1.5) and any non-negative finite integers a and b,

$$I_j(a,b)=\frac{\alpha}{(m+1)^b}\sum_{p=0}^{\infty}\sum_{w=0}^{j}\sum_{u=0}^{b}(-1)^u\binom{b}{u}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{[\alpha\{a+u(m+1)+1\}+w+p]},\quad m\neq -1 \tag{2.2}$$

$$=b!\alpha^{b+1}\sum_{p=0}^{\infty}\sum_{w=0}^{j}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{[\alpha(a+1)+w+p]^{b+1}},\qquad m=-1, \tag{2.3}$$

where

$$I_j(a,b)=\int_\theta^\infty x^j[F(x)]^a f(x)g_m^b(F(x))dx. \tag{2.4}$$

Proof On expanding $g_m^b(F(x))=\left[\frac{1}{m+1}\{1-(F(x))^{m+1}\}\right]^b$ binomially in (2.4), we get when $m\neq -1$

$$I_j(a,b)=A\int_\theta^\infty x^j[F(x)]^{a+u(m+1)}f(x)dx, \tag{2.5}$$

where

$$A=\frac{1}{(m+1)^b}\sum_{u=0}^{b}(-1)^u\binom{b}{u}.$$

Making the substitution $t=[F(x)]^{1/\alpha}$ in (2.5), we find that

$$I_j(a,b)=A\alpha\int_0^1[-\lambda\ln(1-t)+\theta]^j t^{\alpha[a+u(m+1)+1]-1}dt$$

$$=A\alpha\sum_{w=0}^{j}\binom{j}{w}\lambda^w\theta^{j-w}\int_0^1[-\ln(1-t)]^w t^{\alpha[a+u(m+1)+1]-1}dt \tag{2.6}$$

On using the logarithmic expansion

$$[-\ln(1-t)]^w=\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^w=\sum_{p=0}^{\infty}a_p(w)t^{w+p},\quad |t|<1, \tag{2.7}$$

where $a_p(w)$ is the coefficient of t^{w+p} in the expansion of $\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^w$ [see Balakrishnan and Cohen [2], p - 44], (2.6) can be expressed as

$$I_j(a,b)=A\alpha\sum_{p=0}^{\infty}\sum_{w=0}^{j}\binom{j}{w}\lambda^w\theta^{j-w}a_p(w)\int_0^1 t^{\alpha[a+u(m+1)+1]+w+p-1}dt$$

and hence the result given in (2.2).

When $m=-1$, we have

$$I_j(a,b)=\frac{0}{0}\quad\text{as}\quad\sum_{u=0}^{b}(-1)^u\binom{b}{u}=0.$$

Since (2.2) is of the form $\frac{0}{0}$ at $m=-1$, therefore, we have

$$I_j(a,b)=A^*\sum_{u=0}^{b}(-1)^u\binom{b}{u}\frac{[\alpha\{a+u(m+1)+1\}+w+p]^{-1}}{(m+1)^b}, \tag{2.8}$$

where

$$A^{*}=\alpha\sum_{p=0}^{\infty}\sum_{w=0}^{j}\binom{j}{w}\lambda^{w}\theta^{j-w}a_{p}(w).$$

Differentiating numerator and denominator of (2.8) b times with respect to m, we get

$$I_{j}(a,b)=A^{*}\alpha^{b}\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^{b}}{[\alpha\{a+u(m+1)+1\}+w+p]^{b+1}},\ b>0.$$

On applying L' Hospital rule, we have

$$\lim_{m\to -1}I_{j}(a,b)=A^{*}\alpha^{b}\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^{b}}{[\alpha(a+1)+w+p]^{b+1}}. \tag{2.9}$$

But for all integers $n\geq 0$ and for all real numbers x, we have Ruiz [12]

$$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(x-i)^{n}=n!. \tag{2.10}$$

Therefore,

$$\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}u^{b}=b!. \tag{2.11}$$

Now on substituting (2.11) in (2.9) and simplifying the resulting expression, we have the result given in (2.3).

Theorem 2.1 Forgeneralized exponential distribution as given in (1.4) and $1\leq r\leq n$, $k=1,2,\ldots,$ $m\neq -1$,

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}I_{j}(\gamma_{r}-1,r-1) \tag{2.12}$$

$$=\frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{w=0}^{j}\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\binom{j}{w}\frac{\lambda^{w}\theta^{j-w}a_{p}(w)}{(\alpha\gamma_{r-u}+w+p)}, \tag{2.13}$$

where $I_{j}(\gamma_{r}-1,r-1)$ is as defined in (2.4).

Proof From (1.2), we have

$$E[X^{*j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!}\int_{\theta}^{\infty}x^{j}[F(x)]^{\gamma_{r}-1}f(x)g_{m}^{r-1}(F(x))dx$$

and hence the result given in (2.12).

Making use of (2.2) in (2.12), we establish the result given in (2.13).

Identity 2.1 For $\gamma_{r}\geq 1$, $k\geq 1$, $1\leq r\leq n$ and $m\neq -1$,

$$\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{1}{\gamma_{r-u}}=\frac{(r-1)!(m+1)^{r-1}}{\prod_{t=1}^{r}\gamma_{t}}. \tag{2.14}$$

Proof At $j=0$ in (2.13), we have

$$1=\frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{a_{p}(0)}{(\alpha\gamma_{r-u}+p)}.$$

Note that, if $w=0$, then the summation $\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^w$ given in (2.7) is equal to unity, so $a_p(0)=1$, $p=0$ and $a_p(0)=0$, $p>0$,

and hence the result given in (2.14).

Special cases

i) Putting $m=0$, $k=1$ in (2.13), the exact expression for the single moments of order statistics of the generalized exponential distribution can be obtained as

$$E[X_{n-r+1:n}^{j}]=\alpha C_{r:n}\sum_{p=0}^{\infty}\sum_{w=0}^{j}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{[\alpha(n-r+1+u)+w+p]}.$$

That is

$$E[X_{r:n}^{j}]=\alpha C_{r:n}\sum_{p=0}^{\infty}\sum_{w=0}^{j}\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{[\alpha(r+u)+w+p]},$$

where

$$C_{r:n}=\frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $m=-1$ in (2.13), we deduce the exact expression for the single moment of lower k record values for the generalized exponential distribution in view of (2.12) and (2.3) in the form

$$E[X^{*j}(r,n,-1,k)]=E[(Z_r^{(k)})^j]=(\alpha k)^r\sum_{p=0}^{\infty}\sum_{w=0}^{j}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{(\alpha k+w+p)^r}$$

and hence for lower records

$$E[(Z_r^{(1)})^j]=E[X_{L(r)}^{j}]=\alpha^r\sum_{p=0}^{\infty}\sum_{w=0}^{j}\binom{j}{w}\frac{\lambda^w\theta^{j-w}a_p(w)}{(\alpha+w+p)^r}.$$

A recurrence relation for single moment of $l\,gos$ from df (1.5) is obtained in the following theorem.

Theorem 2.2 For the distribution as given in (1.5) and for $2\le r\le n$, $n\ge 2$ and $k=1,2,\dots,$

$$E[X^{*j}(r,n,m,k)]-E[X^{*j}(r-1,n,m,k)]$$
$$=\frac{j\lambda}{\alpha\gamma_r}\{E[X^{*j-1}(r,n,m,k)]-E[\phi(X^*(r,n,m,k))]\}, \tag{2.15}$$

where

$$\phi(x)=x^{j-1}e^{(x-\theta)/\lambda}.$$

Proof Khan *et al.* [8] have shown that for $1\le r\le n$, $n\ge 2$ and $k=1,2,\dots,$

$$E[X^{*j}(r,n,m,k)]-E[X^{*j}(r-1,n,m,k)]$$
$$=-\frac{j\,C_{r-1}}{\gamma_r(r-1)!}\int_{\alpha}^{\beta}x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx. \tag{2.16}$$

Upon substituting for $F(x)$ from (2.1) in (2.16) and simplifying the resulting expression, we derive the relation given in theorem 2.2.

Remark 2.1 Putting $m=0$, $k=1$ in (2.15), we obtain a recurrence relation for single moment of order statistics of the generalized exponential distribution in the form

$$E[X_{n-r+1:n}^{j}]=E[X_{n-r+2:n}^{j}]+\frac{j\lambda}{\alpha(n-r+1)}\{E[X_{n-r+1:n}^{j-1}]-E[\phi(X_{n-r+1:n})]\}.$$

Replacing $(n-r+1)$ by $(r-1)$, we have

$$E[X_{r:n}^{j}]=E[X_{r-1:n}^{j}]-\frac{j\lambda}{\alpha(r-1)}\{E[X_{r-1:n}^{j-1}]-E[\phi(X_{r-1:n})]\}.$$

Remark 2.2 Setting $m=-1$ and $k\geq 1$ in theorem 2.2, we get a recurrence relation for single moment of lower k record values from generalized exponential distribution in the form

$$E[(Z_r^{(k)})^j]=E[(Z_{r-1}^{(k)})^j]+\frac{j\lambda}{\alpha k}\{E[(Z_r^{(k)})^{j-1}]-E[\phi(Z_r^{(k)})]\}.$$

3 Product moments

Before coming to the main results we shall prove the following Lemmas.

Lemma 3.1 For generalized exponential distribution as given in (1.4) and non-negative integers a, b, c with $m\neq -1$,

$$I_{i,j}(a,0,c)=\alpha^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\binom{j}{w_1}\binom{i}{w_2}\frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}a_p(w_1)}{[\alpha(c+1)+w+p]}$$

$$\times\frac{a_q(w_2)}{[\alpha(a+c+2)+w_1+w_2+p+q]},\tag{3.1}$$

where

$$I_{i,j}(a,b,c)=\int_\theta^\infty\int_\theta^x x^i y^j[F(x)]^a f(x)[h_m(F(y))-h_m(F(x))]^b[F(y)]^c f(y)dydx.\tag{3.2}$$

Proof From (3.2), we have

$$I_{i,j}(a,0,c)=\int_\theta^\infty x^i[F(x)]^a f(x)I(x)dx,\tag{3.3}$$

where

$$I(x)=\int_\theta^x y^j[F(y)]^c f(y)dy.\tag{3.4}$$

By setting $z=[F(y)]^{1/\alpha}$ in (3.4), we get

$$I(x)=\alpha\sum_{p=0}^{\infty}\sum_{w_1=0}^{j}\binom{j}{w_1}\frac{\lambda^{w_1}\theta^{j-w_1}a_p(w_1)[F(x)]^{c+1+(w_1+p)/\alpha}}{[\alpha(c+1)+w_1+p]}.$$

On substituting the above expression of $I(x)$ in (3.3), we find that

$$I_{i,j}(a,0,c)=\alpha\sum_{p=0}^{\infty}\sum_{w_1=0}^{j}\binom{j}{w_1}\frac{\lambda^{w_1}\theta^{j-w_1}\alpha_p(w_1)}{[\alpha(c+1)+w_1+p]}\int_\theta^\infty x^i[F(x)]^{a+c+1+(w_1+p)/\alpha}f(x)dx.\tag{3.5}$$

Again by setting $t=[F(x)]^{1/\alpha}$ in (3.5) and simplifying the resulting expression, we establish the result given in (3.1).

Lemma 3.2 For the distribution as given in (1.5) and any non-negative integers a, b and c,

$$I_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} I_{i,j}(a+(b-v)(m+1), 0, \; c+v(m+1)). \tag{3.6}$$

$$= \frac{\alpha^2}{(m+1)^b} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\sum_{v=0}^{b} (-1)^v \binom{b}{v}\binom{j}{w_1}\binom{i}{w_2}$$

$$\times \frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)} a_p(w_1) a_q(w_2)}{[\alpha\{c+1+v(m+1)\}+w+p][\alpha\{a+c+2+b(m+1)\}+w_1+w_2+p+q]}, \quad m \neq -1 \tag{3.7}$$

$$= b!\,\alpha^{b+2} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i} \binom{j}{w_1}\binom{i}{w_2} \frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}}{[\alpha(c+1)+w_1+p]^{b+1}}$$

$$\times \frac{a_p(w_1) a_q(w_2)}{[\alpha(a+c+2)+w_1+w_2+p+q]}, \qquad m=-1, \tag{3.8}$$

where $I_{i,j}(a,b,c)$ is as given in (3.2).

Proof Expanding $[h_m(F(y))-h_m(F(x))]^b$ binomially in (3.2) after noting that $h_m(F(y))-h_m(F(x)) = g_m(F(y))-g_m(F(x))$, we get when $m \neq -1$

$$I_{i,j}(a,b,c) = \frac{1}{(m+1)^b} \sum_{v=0}^{b} (-1)^v \binom{b}{v} \int_{\theta}^{\infty}\int_{\theta}^{x} x^i y^j [F(x)]^{a+(b-v)(m+1)} f(x) [F(y)]^{c+v(m+1)} f(y)\,dy\,dx$$

and hence the result given in (3.6).

Making use of Lemma 3.1 in (3.6), we establish the result given in (3.7).

When $m=-1$, we have

$$I_{i,j}(a,b,c) = \frac{0}{0} \text{ as } \sum_{v=0}^{b} (-1)^v \binom{b}{v} = 0.$$

On applying L' Hospital rule and then using (2.11), (3.8) can be proved on the lines of (2.3).

Theorem 3.1 For generalized exponential distribution as given in (1.5) and for $1 \le r < s \le n$, $k = 1,2,\ldots$, $m \neq -1$,

$$E[X^{*i}(r,n,m,k)\, X^{*j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u}$$

$$\times I_{i,j}(m+(m+1)u, s-r-1, \gamma_s - 1) \tag{3.9}$$

$$= \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times \binom{j}{w_1}\binom{i}{w_2} \frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)} a_p(w_1) a_q(w_2)}{[\alpha\gamma_{s-v}+w_1+p][\alpha\gamma_{r-u}+w_1+w_2+p+q]}. \tag{3.10}$$

Proof From (1.3), we have

$$E[X^{*i}(r,n,m,k)\,X^{*j}(s,n,m,k)]=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{\theta}^{\infty}\int_{\theta}^{x}x^{i}y^{j}[F(x)]^{m}f(x)$$

$$\times g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx. \tag{3.11}$$

On expanding $g_m^{r-1}(F(x))$ binomially in (3.11) and simplifying the resulting expression, we have the result given in (3.9).

Making use of (3.7) in (3.9), we establish the relation given in (3.10).

Identity 3.1 For γ_r, $\gamma_s\geq 1$, $k\geq 1$, $1\leq r<s\leq n$ and $m\neq -1$,

$$\sum_{v=0}^{s-r-1}(-1)^{v}\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}}=\frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s}\gamma_t}. \tag{3.12}$$

Proof At $i=j=0$ in (3.10), we have

$$1=\frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}$$

$$\times\binom{s-r-1}{v}\frac{a_p(0)a_q(0)}{(\alpha\gamma_{s-v}+p)(\alpha\gamma_{r-u}+p+q)},$$

where

$$a_p(0)=1,\, a_q(0)=1,\; p,\; q=0$$

and

$$a_p(0)=0,\, a_q(0)=0,\; p,\; q>0.$$

Therefore,

$$\sum_{v=0}^{s-r-1}(-1)^{v}\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}}=\frac{(r-1)!(s-r-1)!(m+1)^{s-2}}{C_{s-1}\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{1}{\gamma_{r-u}}}.$$

Now on using (2.14), we get the result given in (3.12).

At $r=0$, (3.12) reduces to (2.14).

Special cases

i) Putting $m=0$, $k=1$ in (3.10), the exact expression for the product moments of order statistics of the generalized exponential distribution is obtained as

$$E[X_{n-r+1:n}^{i}\,X_{n-s+1:n}^{j}]=\alpha^2 C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\binom{j}{w_1}\binom{i}{w_2}\frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}a_p(w_1)a_q(w_2)}{[\alpha(n-s+1+v)+w_1+p][\alpha(n-r+1+u)+w_1+w_2+p+q]}.$$

That is

$$E[X_{r:n}^{i}\,X_{s:n}^{j}]=\alpha^{2}C_{r,s:n}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\sum_{u=0}^{n-s}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{n-s}{u}\binom{s-r-1}{v}$$

$$\times\binom{j}{w_1}\binom{i}{w_2}\frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}a_p(w_1)a_q(w_2)}{[\alpha(r+v)+w_1+p][\alpha(s+u)+w_1+w_2+p+q]},$$

where $C_{r,s:n}=\dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

ii) Putting $m=-1$ in (3.10), we deduce the explicit expression for the product moments of lower k record values for generalized exponential distribution in view of (3.9) and (3.8) in the form

$$E[(Z_r^{(k)})^i(Z_s^{(k)})^j]=(\alpha k)^s\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\binom{j}{w_1}\binom{i}{w_2}\frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}a_p(w_1)a_q(w_2)}{(\alpha k+w_1+p)^{s-r}(\alpha k+w_1+w_2+p+q)^r}$$

and hence for lower records

$$E[X_{L(r)}^{i}X_{L(s)}^{j}]=\alpha^{s}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_1=0}^{j}\sum_{w_2=0}^{i}\binom{j}{w_1}\binom{i}{w_2}\frac{\lambda^{w_1+w_2}\theta^{i+j-(w_1+w_2)}a_p(w_1)a_q(w_2)}{(\alpha+w_1+p)^{s-r}(\alpha+w_1+w_2+p+q)^r}.$$

Making use of (2.1), we can derive recurrence relations for product moments of $l\,gos$ from (1.5).

Theorem 3.2 For the distribution as given in (1.5) and for $1\le r<s\le n$, $n\ge 2$ and $k=1,2,\dots,$

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)]-E[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)]$$

$$=\frac{j\lambda}{\alpha\gamma_s}\{E[X^{*i}(r,n,m,k)X^{*j-1}(s,n,m,k)]-E[\phi(X^{*}(r,n,m,k)X^{*}(s,n,m,k))]\},\quad(3.13)$$

where

$$\phi(x,y)=x^{i}y^{j-1}e^{(y-\theta)/\lambda}.$$

Proof Khan *et al.* [8] have shown that for $1\le r<s\le n-1$, $n\ge 2$ and $k=1,2,\dots,$

$$E[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)]-E[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)]$$

$$=-\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{\alpha}^{x}x^{i}y^{j-1}[F(x)]^{m}f(x)g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}\,dy\,dx,\ x>y. \quad (3.14)$$

Now on using (2.1) in (3.14), we have the result given in (3.13).

Remark 3.1 Putting $m=0$, $k=1$ in (3.13), we obtain recurrence relations for product moments of order statistics of the generalized exponential distribution in the form

$$E[X_{n-r+1:n}^{i}X_{n-s+1:n}^{j}]-E[X_{n-r+1:n}^{i}X_{n-s+2:n}^{j}]=\frac{j\lambda}{\alpha(n-s+1)}$$

$$\times\{E[X_{n-r+1:n}^{i}X_{n-s+1:n}^{j-1}]-E[\phi(X_{n-r+1:n}X_{n-s+1:n})]\}.$$

That is

$$E[X_{r:n}^{i}X_{s:n}^{j})=E[X_{r-1:n}^{i}X_{s:n}^{j}]-\frac{j\lambda}{\alpha(r-1)}\{E[X_{r-1:n}^{i-1}X_{s:n}^{j}]-E[\phi(X_{r-1:n}X_{s:n})]\}.$$

Remark 3.2 Setting $m=-1$ and $k\geq 1$, in (3.13), we obtain the recurrence relations for product moments of lower k record values from generalized exponential distribution in the form

$$E[(X_{L(r)}^{(k)})^i(X_{L(s)}^{(k)})^j]-E[(X_{L(r)}^{(k)})^i(X_{L(s-1)}^{(k)})^j]$$

$$=\frac{j\lambda}{\alpha k}\{E[(X_{L(r)}^{(k)})^i(X_{L(s)}^{(k)})^{j-1}]-E[\phi(X_{L(r)}^{(k)})(X_{L(s)}^{(k)})]\}.$$

Remark 3.3 At $j=0$ in (3.10), we have

$$E[X^{*i}(r,n,m,k)]=\frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w_2=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\times\binom{i}{w_2}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\lambda^{w_2}\theta^{i-w_2}a_p(0)a_q(w_2)}{(\alpha\gamma_{s-v}+p)(\alpha\gamma_{r-u}+w_2+p+q)},$$

where

$$a_p(0)=1,\ p=0 \quad\text{and}\quad a_p(0)=0,\ p>0.$$

Therefore,

$$E[X^{*i}(r,n,m,k)]=\frac{\alpha C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{q=0}^{\infty}\sum_{w_2=0}^{i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{w_2}\frac{\lambda^{w_2}\theta^{i-w_2}a_q(w_2)}{\gamma_{s-v}(\alpha\gamma_{r-u}+w_2+q)}. \tag{3.15}$$

Making use of (3.12) in (3.15) and simplifying the resulting expression, we get

$$E[X^{*i}(r,n,m,k)]=\frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{q=0}^{\infty}\sum_{w_2=0}^{i}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{\lambda^{w_2}\theta^{i-w_2}a_q(w_2)}{(\alpha\gamma_{r-u}+w_2+q)}$$

as obtained in (2.13).

Remark 3.4 At $i=0$, Theorem 3.2 reduces to Theorem 2.2.

4 Characterization

Let $X^*(r,n,m,k)$, $r=1,2,\ldots,n$ be $l\,gos$ from a continuous population with df $F(x)$ and pdf $f(x)$, then the conditional pdf of $X^*(s,n,m,k)$ given $X^*(r,n,m,k)=x$, $1\leq r<s\leq n$, in view of (1.3) and (1.2), is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y\,|\,x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$

$$\times[(h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \tag{4.1}$$

Theorem 4.1 Let X be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$E[X^*(s,n,m,k) \mid X^*(l,n,m,k)=x] = \lambda \sum_{p=1}^{\infty} \frac{(1-e^{-(x-\theta)/\lambda})^p}{p} \prod_{j=1}^{s-l} \left(\frac{\gamma_{l+j}}{\gamma_{l+j}+p/\alpha} \right) + \theta \,,$$

$l = r,\ r+1$ (4.2)

if and only if

$$F(x) = [1-e^{-(x-\theta)/\lambda}]^{\alpha} \,, x > \theta, \alpha,\ \lambda > 0 \,.$$

Proof: From (4.1), we have

$$E[X^*(s,n,m,k) \mid X^*(r,n,m,k)=x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_{\theta}^{x} y \left[1 - \left(\frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \left(\frac{F(y)}{F(x)} \right)^{\gamma_s -1} \frac{f(y)}{F(x)} dy \,. \tag{4.3}$$

By setting $u = \frac{F(y)}{F(x)} = \left(\frac{1-e^{-(y-\theta)/\lambda}}{1-e^{-(x-\theta)/\lambda}} \right)^{\alpha}$ from (1.5) in (4.3), we obtain

$$E[X^*(s,n,m,k) \mid X^*(r,n,m,k)=x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_0^1 [(-\lambda \ln\{1-(1-e^{-(x-\theta)/\lambda})u^{1/\alpha}\}) + \theta] u^{\gamma_s -1} (1-u^{m+1})^{s-r-1} du$$

$$= \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} (A_1 + A_2) \,, \tag{4.4}$$

where

$$A_1 = \lambda \sum_{p=1}^{\infty} \frac{[1-e^{-(x-\theta)/\lambda}]^p}{p} \int_0^1 u^{(p/\alpha)+\gamma_s -1} (1-u^{m+1})^{s-r-1} du \tag{4.5}$$

and

$$A_2 = \theta \int_0^1 u^{\gamma_s -1} (1-u^{m+1})^{s-r-1} du \tag{4.6}$$

Again by setting $t = u^{m+1}$ in (4.5) and (4.6), we get

$$A_1 = (m+1)^{s-r-1} \Gamma(s-r) \lambda \sum_{p=1}^{\infty} \frac{(1-e^{-(x-\theta)/\lambda})^p}{p \prod_{j=1}^{s-r} (\gamma_{r+j} + p/\alpha)}$$

$$A_2 = \frac{\theta (m+1)^{s-r-1} \Gamma(s-r)}{\prod_{j=1}^{s-r} \gamma_{r+j}}$$

Substituting these expressions for A_1 and A_2 in equation (4.4) and simplifying the resulting expression, we derive the relation given in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_{\theta}^{x} y[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-1}$$

$$\times[F(y)]^{\gamma_s-1} f(y)dy=[F(x)]^{\gamma_{r+1}} H_r(x)\,, \tag{4.7}$$

where

$$H_r(x)=\lambda\sum_{p=1}^{\infty}\frac{(1-e^{-(x-\theta)/\lambda})^p}{p}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\right)+\theta\,.$$

Differentiating (4.7) both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_{\theta}^{x} y[(F(x))^{m+1}-(F(y))^{m+1}]^{s-r-2}[F(y)]^{\gamma_s-1} f(y)dy$$

$$=H_r'(x)[F(x)]^{\gamma_{r+1}}+\gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x)$$

or

$$\gamma_{r+1}H_{r+1}(x)[F(x)]^{\gamma_{r+2}+m} f(x)=H_r'(x)[F(x)]^{\gamma_{r+1}}+\gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x)\,,$$

where

$$H_r'(x)=e^{-(x-\theta)/\lambda}\sum_{p=1}^{\infty}[1-e^{-(x-\theta)/\lambda}]^{p-1}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\right),$$

$$H_{r+1}(x)-H_r(x)=\sum_{p=1}^{\infty}[1-e^{-(x-\theta)/\lambda}]^p\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}\right)\frac{\lambda}{\alpha\gamma_{r+1}}\,.$$

Therefore,

$$\frac{f(x)}{F(x)}=\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]}$$

$$=\frac{\alpha\, e^{-(x-\theta)/\lambda}}{\lambda[1-e^{-(x-\theta)/\lambda}]}$$

which proves that

$$F(x)=[1-e^{-(x-\theta)/\lambda}]^{\alpha}\,,\, x>\theta\,,\alpha\,,\ \lambda>0\,.$$

Acknowledgments

The authors are grateful to the referee for giving valuable comments that led to an improvement in the presentation of the study.

References

[1] Ahsanullah, M.A characterization of the uniform distribution by dual generalized order statistics. *Comm. Statist. Theory Methods*, 33(2004), 2921-2928.

[2] Balakrishnan, N. and Cohen, A. C. *Order Statistics and Inference: Estimation Methods*(1991). Academic Press, San Diego.

[3] Burkschat, M., Cramer, E. and Kamps, U. Dual Generalized order statistics. *Metron*, LXI(2003), 13-26.

[4] Gupta, R. D. and Kundu, D. Generalized exponenetial distribution. *Aust. N. Z. J. Stat*., 41(1999), 173-188.

[5] Gupta, R. D. and Kundu, D.Exponenetiated exponential distribution.: an alternative to gamma and Weibull distributions. *Biometrical J*., 43 (2001a), 117-130.

[6] Gupta, R. D. and Kundu, D.Generalized exponential distributions: different methods of estimations. *J. Statist. Comput. Simulation*, 69 (2001b), 315-338.

[7] Kamps, U.*A Concept of Generalized Order Statistics*(1995). B. G. Teubner Stuttgart, Germany.

[8] Khan, R. U., Anwar, Z. and Athar, H.Recurrence relations for single and product moments of generalized order statistics from exponentiated Weibull distribution. *Aligarh J. Statist*., 28 (2008), 37-45..

[9] Khan, R. U. and Kumar, D.On moments of lower generalized order statistics from exponentiated Pareto distribution and its characterization. *Appl. Math. Sci.* (*Ruse*), 4 (2010), 2711-2722.

[10] Mbah, A. K. and Ahsanullah, M.Some characterization of the power function distribution based on lower generalized order statistics. *Pakistan J. Statist*., 23 (2007), 139-146.

[11] Pawlas, P. and Syznal, D.Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. *Demonstratio Math*., XXXIV (2001), 353-358.

[12] Ruiz, S. M.An algebraic identity leading to Wilson’s theorem. *Mathematical Gazette*, 80 (1996), 579-582.