Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070647

Hájek-Rényi Type Inequalities and Strong Law of Large Numbers for NOD Sequences

Ma Songlin *

Department of Mathematics, Chao Hu University, Chaohu, 238000, HeFei, P. R. China

Received: 15 Apr. 2013, Revised: 18 Aug. 2013, Accepted: 19 Aug. 2013
Published online: 1 Nov. 2013

Abstract: In the paper, we get the precise results of Hájek-Rényi type inequalities for the partial sums of negatively orthant dependent sequences, which improve the results of Theorem 3.1 and Corollary 3.2 in Kim (2006). In addition, the Marcinkiewicz type strong law of large numbers is obtained. At last, the strong stability for weighted sums of negatively orthant dependent sequences is discussed.

Keywords: Hájek-Rényi inequality; negatively orthant dependent sequences; strong law of large numbers.

1 Introduction

We use the following notations.Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space. Denote $S_n \doteq \sum_{i=1}^n X_i$ and $T_n \doteq \sum_{i=1}^n (X_i - EX_i)$ for each $n \geq 1$ and $I(A)$ be the indicator function of the set A.

Hájek and Rényi (1955) proved the following important inequality. If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with mean zero, and $\{b_n, n \geq 1\}$ is a nondecreasing sequence of positive real numbers, then for any $\varepsilon > 0$ and positive integer $m < n$,

$$P\left(\max_{m \leq k \leq n} \left|\frac{\sum_{i=1}^k X_i}{b_k}\right| \geq \varepsilon\right) \leq \varepsilon^{-2}\left(\sum_{j=1}^m \frac{EX_j^2}{b_m^2} + \sum_{j=m+1}^n \frac{EX_j^2}{b_j^2}\right) \quad (1.1)$$

In the paper, we will further study the Hájek-Rényi type inequality and the strong law of large numbers for negatively orthant dependent sequences.

Definition 1.1. A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively upper orthant dependent (NUOD), if for all real numbers $x_1, x_2, \cdots, x_n$,

$$P(X_i > x_i, i = 1, 2, \cdots, n) \leq \prod_{i=1}^n P(X_i > x_i), \quad (1.2)$$

and negatively lower orthant dependent (NLOD), if for all real numbers $x_1, x_2, \cdots, x_n$,

$$P(X_i \leq x_i, i = 1, 2, \cdots, n) \leq \prod_{i=1}^n P(X_i \leq x_i). \quad (1.3)$$

A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.

An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD (NUOD Or NLOD),if every finite subcollection is NOD (NUOD Or NLOD).

Lemma 1.1 (cf. Bozorgnia et al., 1996). Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables, $f_1, f_2, \cdots$ be all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of NOD random variables.

Lemma 1.2 (cf. Kim, 2006).Let $X_1, X_2, \cdots, X_n$ be NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geq 1$. Then we have

$$E\left(\sum_{i=1}^p X_{m+i}\right)^2 \leq \sum_{i=1}^p EX_{m+i}^2 \quad (1.4)$$

for all integers $m, p \geq 1$ and $m + p \leq n$. Moreover, we have

$$E\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^k X_i\right|^2\right) \leq (\log_3 n + 2)^2 \sum_{i=1}^n EX_i^2 \quad (1.5)$$

By Lemma 1.1 and Lemma 1.2, we can get the following Corollary.

Corollary 1.1 (Khintchine-Kolmogorov theorem). Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables. If

$$\sum_{n=1}^\infty Var(X_n) \log^2 n < \infty \quad (1.6)$$

then

$$\sum_{n=1}^\infty (X_n - EX_n) \text{ converges } a.s..$$

* Corresponding author e-mail: 781719212@qq.com

2 Hájek-Rényi type inequalities for NOD sequences

In this section, we will give Hájek-Rényi type inequalities for NOD sequences, which improve the results of Kim (2006).

Theorem 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any integer $n \geq 1$,

$$P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\} \leq \frac{4}{\varepsilon^2}(\log_3 n + 2)^2 \sum_{j=1}^{n}\frac{Var(X_j)}{b_j^2}. \quad (2.1)$$

Proof. Without loss of generality, we assume that $b_n \geq 1$ for all $n \geq 1$. Let $\alpha = \sqrt{2}$. For $i \geq 0$, define

$$A_i = \{1 \leq k \leq n : \alpha^i \leq b_k < \alpha^{i+1}\}.$$

For $A_i \neq \emptyset$, we let $v(i) = \max\{k : k \in A_i\}$ and t_n be the index of the last nonempty set A_i. Obviously, $A_iA_j = \emptyset$ if $i \neq j$ and $\sum_{i=0}^{t_n} A_i = \{1, 2, \cdots, n\}$. It is easily seen that $\alpha^i \leq b_k \leq b_{v(i)} < \alpha^{i+1}$ if $k \in A_i$ and $\{X_n - EX_n, n \geq 1\}$ is also a sequence of NOD random variables by Lemma 1.1. By Markov's inequality and (1.6) in Lemma 1.2, we have

$$P\left\{\max_{1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\}$$

$$= P\left\{\max_{0\leq i\leq t_n, A_i\neq\emptyset}\max_{k\in A_i}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\}$$

$$\leq \sum_{i=0, A_i\neq\emptyset}^{t_n} P\left\{\frac{1}{\alpha^i}\max_{1\leq k\leq v(i)}\left|\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right\}$$

$$\leq \frac{1}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}E\left\{\max_{1\leq k\leq v(i)}\left|\sum_{j=1}^{k}(X_j - EX_j)\right|^2\right\}$$

$$\leq \frac{1}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}(\log_3 v(i) + 2)^2\sum_{j=1}^{v(i)}Var(X_j)$$

$$\leq \frac{1}{\varepsilon^2}(\log_3 n + 2)^2\sum_{j=1}^{n}Var(X_j)\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}}. \quad (2.2)$$

Now we estimate $\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}}$.

Let $i_0 = \min\{i : A_i \neq \emptyset, v(i) \geq j\}$, then $b_j \leq b_{v(i_0)} < \alpha^{i_0+1}$ follows from the definition of $v(i)$. Therefore,

$$\sum_{i=0, A_i\neq\emptyset, v(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}} < \sum_{i=i_0}^{\infty}\frac{1}{\alpha^{2i}} = \frac{1}{1-\frac{1}{\alpha^2}}\frac{1}{\alpha^{2i_0}}$$

$$< \frac{\alpha^2}{1-\frac{1}{\alpha^2}}\frac{1}{b_j^2} = \frac{4}{b_j^2}. \quad (2.3)$$

Thus (2.1) follows from (2.2) and (2.3) immediately.

Theorem 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any positive integers $m < n$,

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right) \leq$$

$$\frac{4}{\varepsilon^2}\left(\sum_{j=1}^{m}\frac{Var(X_j)}{b_m^2} + 4[\log_3(n-m)+2]^2\sum_{j=m+1}^{n}\frac{Var(X_j)}{b_j^2}\right) \quad (2.4)$$

Proof. Observe that

$$\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \leq \left|\frac{1}{b_m}\sum_{j=1}^{m}(X_j - EX_j)\right|$$

$$+ \max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j - EX_j)\right|,$$

thus

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| \geq \varepsilon\right)$$

$$\leq P\left(\left|\frac{1}{b_m}\sum_{j=1}^{m}(X_j - EX_j)\right| \geq \frac{\varepsilon}{2}\right)$$

$$+ P\left(\max_{m+1\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}(X_j - EX_j)\right| \geq \frac{\varepsilon}{2}\right)$$

By Markov's inequality , Lemma 1.2 and Theorem 2.1,(2.4) can be obtained.

3 Marcinkiewicz type strong law of large numbers for NOD sequences

Theorem 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables with $\sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p})\log^2 n < \infty$ for $0 < p < 2$.

Assume that $EX_1 = 0$ if $1 \leq p < 2$,then

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_k \to 0 \ a.s., \ n \to \infty. \quad (3.1)$$

Proof. Denote $Y_n = -n^{1/p}I(X_n \leq -n^{1/p}) + X_nI(|X_n| < n^{1/p}) + n^{1/p}I(X_n \geq n^{1/p})$ then

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| \geq n^{1/p}) < \infty,$$

which implies that $P(X_n \neq Y_n, \ i.o.) = 0$ by the Borel-Cantelli lemma. Thus $\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_k \to 0 \ a.s.$ if and only if $\frac{1}{n^{1/p}}\sum_{k=1}^{n} Y_k \to 0 \ a.s.$ So we need to show that

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n}(Y_k-EY_k)\to 0\ a.s.,\ n\to\infty, \tag{3.2}$$

and

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n}EY_k\to 0,\ n\to\infty. \tag{3.3}$$

By Corollary 1.1 and Kronecker's lemma, to prove (3.2), it suffices to show that

$$\sum_{n=1}^{\infty}Var\left(\frac{Y_n}{n^{1/p}}\right)\log^2 n<\infty. \tag{3.4}$$

In fact,

$$\sum_{n=1}^{\infty}Var\left(\frac{Y_n}{n^{1/p}}\right)\log^2 n\le\sum_{n=1}^{\infty}\frac{EY_n^2}{n^{2/p}}\log^2 n$$

$$\le C\sum_{n=1}^{\infty}P(|X_1|\ge n^{1/p})\log^2 n+C\sum_{n=1}^{\infty}\frac{EX_1^2I(|X_1|<n^{1/p})}{n^{2/p}}\log^2 n$$
$$\le C+C\sum_{n=1}^{\infty}\frac{\log^2 n}{n^{2/p}}\sum_{k=1}^{n}EX_1^2I(k-1\le|X_1|^p<k)$$
$$=C+C\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{\log^2 n}{n^{2/p}}E|X_1|^p|X_1|^{2-p}I(k-1\le|X_1|^p<k)$$
$$\le C+C\sum_{k=1}^{\infty}k^{1-2/p}E|X_1|^pk^{(2-p)/p}I(k-1\le|X_1|^p<k)$$
$$<\infty.$$

Hence (3.2) holds. Next, we will prove (3.3). It will be divided into two cases:

(i) If $p=1$, by $E|X_1|^p<\infty$ and Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}n^{1/p}P(|X_n|\ge n^{1/p})=0, \tag{3.5}$$

$$\lim_{n\to\infty}EX_nI(|X_n|<n^{1/p})$$
$$=\lim_{n\to\infty}\int_\Omega X_1(\omega)I(|X_1(\omega)|<n^{1/p})P(d\omega)$$
$$=EX_1=0. \tag{3.6}$$

Thus,

$$|EY_n|\le n^{1/p}P(|X_n|\ge n^{1/p})$$
$$+|EX_nI(|X_n|<n^{1/p})|\to 0,\ a.s.\ n\to\infty.$$

By the Toeplitz lemma, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}EY_k=0.$$

(ii) If $p\ne 1$, by the Kronecker's lemma, to prove (3.3), it suffices to show that

$$\sum_{n=1}^{\infty}\frac{|EY_n|}{n^{1/p}}<\infty. \tag{3.7}$$

For $0<p<1$,

$$\sum_{n=1}^{\infty}\frac{|EY_n|}{n^{1/p}}\le\sum_{n=1}^{\infty}P(|X_1|\ge n^{1/p})+\sum_{n=1}^{\infty}\frac{E|X_n|I(|X_n|<n^{1/p})}{n^{1/p}}$$
$$\le C+\sum_{n=1}^{\infty}\sum_{j=1}^{n}n^{-1/p}E|X_1|I(j-1\le|X_1|^p<j)$$
$$=C+\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}n^{-1/p}E|X_1|I(j-1\le|X_1|^p<j)$$

$$\le C+C\sum_{j=1}^{\infty}j^{1-1/p}E|X_1|^pj^{(1-p)/p}I(j-1\le|X_1|^p<j)<\infty.$$

For $1\le p<2$, by $EX_n=0$, we can see that

$$\sum_{n=1}^{\infty}\frac{|EY_n|}{n^{1/p}}\le\sum_{n=1}^{\infty}P(|X_1|\ge n^{1/p})+\sum_{n=1}^{\infty}\frac{|EX_nI(|X_n|<n^{1/p})|}{n^{1/p}}$$
$$\le C+\sum_{n=1}^{\infty}n^{-1/p}E|X_n|I(|X_n|\ge n^{1/p})$$
$$=C+\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}n^{-1/p}E|X_1|I(j\le|X_1|^p<j+1)$$
$$=C+\sum_{j=1}^{\infty}\sum_{n=1}^{j}n^{-1/p}E|X_1|I(j\le|X_1|^p<j+1)$$
$$\le C+C\sum_{j=1}^{\infty}j^{1-1/p}E|X_1|^pj^{(1-p)/p}I(j\le|X_1|^p<j+1)$$
$$<\infty.$$

Thus (3.7) holds, which implies (3.3) by Kronecker's lemma. We get the desired result.

4 Strong stability for weighted sums of NOD sequences

In this section, we will study the strong stability for weighted sums of NOD random variables. Firstly, we will give some definitions as follows:

Definition 4.1. A random variable sequence $\{X_n,n\ge 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant C, such that

$$P(|X_n|>x)\le CP(|X|>x) \tag{4.1}$$

for all $x\ge 0$ and $n\ge 1$.

Definition 4.2. A random variable sequence $\{Y_n,n\ge 1\}$ is said to be strongly stable if there exist two constant sequences $\{b_n,n\ge 1\}$ and $\{d_n,n\ge 1\}$ with $0<b_n\uparrow\infty$, such that

$$b_n^{-1}Y_n-d_n\to 0\ \ a.s. \tag{4.2}$$

Lemma 4.1. Let $\{X_n,n\ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha>0$ and $b>0$, the following statement holds:

$$E|X_n|^\alpha I(|X_n|\le b)\le C\{E|X|^\alpha I(|X|\le b)+b^\alpha P(|X|>b)\}.$$

Where C is a positive constant.

Theorem 4.1. Let $\{a_n,n\ge 1\}$ and $\{b_n,n\ge 1\}$ be two sequences of positive numbers with $c_1=b_1/a_1$, $c_n=b_n/(a_n\log n)$ for $n\ge 2$ and $b_n\uparrow\infty$. Let $\{X_n,n\ge 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X. Define $N(x)=Card\{n:c_n\le x\}$, $R(x)=\int_x^\infty N(y)y^{-3}dy$, $x>0$.

If the following conditions are satisfied:

(i) $N(x)<\infty$ for any $x>0$,

(ii) $R(1)=\int_1^\infty N(y)y^{-3}dy<\infty$,

(iii) $EX^2R(|X|)<\infty$,

then there exist $d_n\in R$, $n=1,2,\cdots$ such that

$$b_n^{-1}\sum_{i=1}^{n}a_iX_i-d_n\to 0\ \ a.s. \tag{4.3}$$

Proof. Denote

$X_k^{c_k} = -c_k I(X_k < -c_k) + X_k I(|X_k| \le c_k) + c_k I(X_k > c_k),\ k \ge 1,$

then $\{X_k^{c_k}, k \ge 1\}$ and $\{a_k X_k^{c_k}/b_k, k \ge 1\}$ are still NOD from Lemma 1.1.

Since $N(x)$ is nondecreasing, then for any $x > 0$

$$R(x) \ge N(x)\int_x^\infty y^{-3}dy = \frac{1}{2}x^{-2}N(x), \quad (4.4)$$

which implies that

$$EN(|X|) \le 2EX^2R(|X|) < \infty.$$

Therefore

$$\sum_{i=1}^\infty P\left(X_i \ne X_i^{c_i}\right) = \sum_{i=1}^\infty P(|X_i| > c_i)$$

$$\le C\sum_{i=1}^\infty P(|X| > c_i) \le CEN(|X|) < \infty. \quad (4.5)$$

By Borel-Cantelli lemma for any sequence $\{d_n, n \ge 1\} \subset R$, the sequences $\{b_n^{-1}\sum_{i=1}^n a_iX_i - d_n\}$ and $\{b_n^{-1}\sum_{i=1}^n a_iX_i^{c_i} - d_n\}$ converge to the same limit on the same set.

We will show that $b_n^{-1}\sum_{i=1}^n a_i(X_i^{c_i} - EX_i^{c_i}) \to 0\ a.s.$, which gives the theorem with $d_n = b_n^{-1}\sum_{i=1}^n a_iEX_i^{c_i}$.

It follows from Lemma 4.1 that

$$\sum_{n=1}^\infty \frac{Var\left(a_nX_n^{c_n}\right)}{b_n^2}\log^2 n \le \sum_{n=1}^\infty c_n^{-2}E\left(X_n^{c_n}\right)^2$$

$$\le 3\sum_{n=1}^\infty P(|X_n| > c_n) + 3\sum_{n=1}^\infty c_n^{-2}EX_n^2 I(|X_n| \le c_n)$$

$$\le C\sum_{n=1}^\infty P(|X| > c_n) + C\sum_{n=1}^\infty c_n^{-2}EX^2I(|X| \le c_n)$$

$$\le CEN(|X|) + C\sum_{n=1}^\infty c_n^{-2}EX^2I(|X| \le c_n). \quad (4.6)$$

$$\sum_{n=1}^\infty c_n^{-2}EX^2I(|X| \le c_n) = \sum_{n:c_n\le 1} c_n^{-2}EX^2I(|X| \le c_n)$$

$$+ \sum_{n:c_n>1} c_n^{-2}EX^2I(|X| \le c_n) \doteq I_1 + I_2. \quad (4.7)$$

Since $N(1) = Card\{n : c_n \le 1\} \le 2R(1) < \infty$ from (4.4) and condition (ii), then $I_1 < \infty$.

$$I_2 = \sum_{n:c_n>1} c_n^{-2}EX^2I(|X| \le c_n)$$

$$= \sum_{k=2}^\infty \sum_{k-1<c_n\le k} c_n^{-2}EX^2I(|X| \le c_n)$$

$$\le \sum_{k=2}^\infty (N(k) - N(k-1))(k-1)^{-2}EX^2I(|X| \le k)$$

$$\le \sum_{k=2}^\infty (N(k) - N(k-1))(k-1)^{-2}EX^2I(|X| \le 1)$$

$$+ \sum_{k=2}^\infty (N(k) - N(k-1))(k-1)^{-2}EX^2I(1 < |X| \le k)$$

$$\doteq I_{21} + I_{22}.$$

$$I_{21} \le C\sum_{k=2}^\infty (N(k) - N(k-1))\sum_{j=k-1}^\infty j^{-3}$$

$$= C\sum_{j=1}^\infty j^{-3}\sum_{k=2}^{j+1}(N(k) - N(k-1))$$

$$\le C\sum_{j=1}^\infty (j+1)^{-3}N(j+1) \le C\int_1^\infty y^{-3}N(y)dy < \infty.$$

Since $N(x)$ is nondecreasing and $R(x)$ is nonincreasing, then

$$I_{22} = \sum_{k=2}^\infty (N(k) - N(k-1))(k-1)^{-2}EX^2I(1 < |X| \le k)$$

$$= \sum_{k=2}^\infty (N(k) - N(k-1))(k-1)^{-2}\sum_{m=2}^k EX^2I(m-1 < |X| \le m)$$

$$= \sum_{m=2}^\infty EX^2I(m-1 < |X| \le m)\sum_{k=m}^\infty (N(k) - N(k-1))(k-1)^{-2}$$

$$\le \sum_{m=2}^\infty EX^2I(m-1 < |X| \le m)\sum_{k=m}^\infty N(k)\left((k-1)^{-2} - k^{-2}\right)$$

$$\le C\sum_{m=2}^\infty EX^2I(m-1 < |X| \le m)\sum_{k=m}^\infty \int_k^{k+1} N(x)x^{-3}dx$$

$$= C\sum_{m=2}^\infty R(m)EX^2I(m-1 < |X| \le m)$$

$$\le C\sum_{m=2}^\infty EX^2R(|X|)I(m-1 < |X| \le m) \le CEX^2R(|X|) < \infty.$$

Therefore

$$\sum_{n=1}^\infty \frac{Var\left(a_nX_n^{c_n}\right)}{b_n^2}\log^2 n < \infty \quad (4.8)$$

following from the above statements. By Corollary 1.1 and Kronecker's Lemma, it follows that

$$b_n^{-1}\sum_{i=1}^n a_i\left(X_i^{c_i} - EX_i^{c_i}\right) \to 0\ \ a.s. \quad (4.9)$$

Taking

$d_n = b_n^{-1}\sum_{i=1}^n a_iEX_i^{c_i},\ n \ge 1$, then $b_n^{-1}\sum_{i=1}^n a_iX_i^{c_i} - d_n \to 0\ a.s.$

We complete the proof of the theorem.

Acknowledgement

This work is supported by Foundation of Anhui Educational Committee (KJ2013Z225). The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] Bozorgnia, A., Patterson, R. F., Taylor, R. L., Limit theorems for dependent random variables. World Congress Nonlinear Analysts, **92**, 1639–1650 (1996).

[2] Christofides, T. C., Maximal inequalities for demimartingales and a strong law of large numbers. Statist. Probab. Lett., **50**, 357–363 (2000).

[3] Fazekas, I., Klesov, O., A general approach to the strong law of large numbers. Theory Probab. Appl., **45**, 436–449 (2001).

[4] Gan, S. X., The Hájek–Rényi inequality for Banach space valued martingales and the p smoothness of Banach space. Statist. Probab. Lett., **32**, 245–248 (1997).

[5] Hájek, J., Rényi, A., A generalization of an inequality of Kolmogorov. Acta Math. Acad. Sci. Hungar., **6**, 281–284 (1955).

[6] Hu, S. H., Chen, G. J., Wang, X. J., On extending the Brunk-Prokhorov strong law of large numbers for martingale differences. Statist. Probab. Lett., **78**, 3187–3194 (2008).
[7] Hu, S. H., Wang, X. J., Yang, W. Z., Zhao, T., The Hájek-Rényi-type inequality for associated random variables. Statist. Probab. Lett., **79**, 884–888 (2009).
[8] Joag-Dev, K., Proschan, F., Negative association of random variables with applications. Ann. Statist., **11**, 286–295 (1983).
[9] Kim, H. C., The Hájek-Rényi inequality for weighted sums of negatively orthant dependent random variables. Int. J. Contemp. Math. Sci., **1**, 297–303 (2006).
[10] Liu, J. J., Gan, S. X., Chen, P. Y., The Hájek-Rényi inequality for NA random variables and its application. Statist. Probab. Lett., **43**, 99–105 (1999).
[11] Prakasa Rao, B. L. S., Hájek-Rényi-type inequality for associated sequences. Statist. Probab. Lett., **57**, 139–143 (2002).
[12] Sung, H. S., A note on the Hájek-Rényi inequality for associated random variables. Statist. Probab. Lett., **78**, 885–889 (2008).

Ma Songlin received the MS degree in Statistics and Probability from Anhui University in 2007. He is currently a teacher in Chaohu University. His research interests are in the areas of Probability limit theorems.