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Growth of a Vapour Bubble in a Superheated Liquid of Variable Surface Tension and Viscosity Between Two-phase Flow

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Abstract: The growth of a vapour bubble in a superheated liquid of variable surface tension and viscosity between two finite boundaries is introduced. The problem is solved analytically using the modified method of Plesset and Zwick method. The pressure difference is described in terms of temperature difference and initial pressure difference. The surface tension, viscosity, and initial and final time of bubble growth are derived in terms of some physical parameters. The growth of bubble radius is proportional to the thermal diffusivity, the initial pressure difference and its coefficient. On contrary the growth is inversely proportional to the initial void fraction and the density ratio. Moreover, better agreements with some experimental data are achieved rather than some of previous theoretical efforts.

Keywords: vapour bubble growth, void fraction, variable surface tension, variable viscosity, superheated liquid

1 Introduction

The importance of studying the bubble dynamics comes from its plentiful applications which can be appeared in many branches of sciences and technology like in biomedical applications as damaging of cells [7], explaining some symptoms of some diseases like DCS (Decompression Sickness) [15], explaining volcanic eruptions [4], laser applications [26], thermal ink-jet printer [1], actuation [9], mixing [23], pumping [25] and many others.

The theory of the growth of a vapour bubble in a superheated liquid has been considered by several authors. The equation of motion of a spherical bubble growth (or collapse) was first presented by Rayleigh [20] under the effect of the stage of inertia controlled. Many theoretical and numerical studies were followed to explain the other growth-controlled stages as the asymptotic solution that presented by Plesset and Zwick [19]; which was under consideration of the thermal diffusion controlled growth, neglecting liquid inertia and thin thermal boundary layer. The approximated solution for the bubble wall temperature [19] has good agreement with the experimental data of Dergarabedian [2] for moderate superheats up to 6°C. The problem was presented in a new treatment by Scriven [21] by solving the heat equation without assuming a thin thermal boundary layer. The solution for moderate superheat was identical to that of Plesset and Zwick [19]. The inertia and thermal diffusion controlled growth were combined by Mikic et al. [12] using the Clausius-Clapeyron equation. The result was a generalized expression which was valid over the entire growth range, and has good agreement with the experimental data of Lien [10]. Olek et al. [17] introduced a new model for bubble growth in a uniformly superheated liquid which is valid for both inertia and heat diffusion controlled growth. Their model gives better agreements with the experimental data of Hooper and Abdelmessih [6]. Mohammadein and Gad-Elrab [13], and Mohammadein and Gouda [14] introduced some modifications to the solutions of Plesset and Zwick [19], and Scriven [21] respectively. The results give the previous solutions as special cases.

Different formulae of the pressure difference $\Delta P = (P_v(T) - P_\infty)$, were one of main differences

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between the previous analytical solutions. While Rayleigh [20] assumed it to be constant, Plesset-Zwick [19] considered it as a linear function in temperature difference $\Delta P = A(T_R - T_{sat})$, Mikic et al. [12] used Clausius-Clapeyron equation to express it in the form $\Delta P = \frac{\rho_v L}{T_{sat}}(T_R - T_{sat})$. The primary objective of the present study is to introduce an analytical solution to the problem of the growth of a bubble in a viscous superheated liquid with variable surface tension and viscosity adjacent to the bubble boundary. This variation of surface tension and viscosity around the growing bubble comes from the fact that: While the bubble is growing the surrounding liquid particles get more closed to each other, consequently the surface tension and viscosity at the bubble boundary are affected.

These increments in surface tension and viscosity at the bubble boundaries may explain the reason of why the bubble collapses after it reaches its most possible radius, because the bubble growth raise the potential of the pressure due to the surface tension and viscosity. When these potentials reach its maximum values at the bubble's maximum radius, the bubble then collapses to reduce those potentials. This also may explain that: why the time of collapse of the bubble is so shorter than the time of its growth. The resultant formula gives a relation between the bubble radius and the time that containing several physical parameters, which used to discuss the effect of these parameters on the growth of the vapour bubble.

In this study, a linear form of the pressure difference is introduced, $\Delta P = A(T_R - T_{sat}) + b\Delta P_0$, that includes beside the effect of temperature difference a multiple term of the initial pressure difference, which affects also on the growth process. This improvement was first introduced by Mohammadein and Mohamed [16] , which gave good results in mass diffusion problem. Beside the analytical treatments and experimental investigations to the problem, several numerical computations were carried out by coupling various forms of equation of motion with various special forms of the energy equation like [8, 9, 22]. Under the influence of heat transfer growth, for a vapour bubble in a viscous, superheated liquid of variable surface tension and viscosity in the region adjacent to the bubble's interface, the growth problem is solved analytically by using Plesset and Zwick method [19] after modifying the pressure difference to include beside the temperature difference a multiple term of the initial pressure difference. The solution technique is performed by solving the equation of motion derived by Scriven [21], which contains the effect of density ratio of the vapour to the liquid, in pair of the approximated solution of the heat equation that presented by Plesset and Zwick [18] to obtain a relation between the bubble radius and the time that containing the effect of several physical parameters that include thermal diffusivity, Jacob Number, density ratio, void fraction and other parameters.

That is after introducing the surface tension and viscosity in variable forms [3, 16] in the momentum equation. At the end of the solution, two equations are introduced that represent the evolution of the surface tension and viscosity throughout the growth process. After obtaining the solution formula, a numerical implementation is performed and some graphs are presented to discuss the effect of these parameters on the growth of vapour bubble and comparison between this work and some of the previous theoretical and experimental work is performed. The resultant formula gives a good agreement with these previous works at some initial superheats.

2 Analysis

A single vapour bubble is considered to grow inside a superheated, incompressible liquid of variable surface tension and viscosity adjacent to the bubble boundary. The dependent surface tension in terms of fluid properties is considered [16]. Moreover, the dependent viscosity is reformulated [3]. The bubble radius is considered to grow between two finite radius boundaries R_0 and R_m. The growth is affected by some parameters such as the pressure difference ΔP between the bubble pressure $P_v(R(t),t)$ and the ambient pressure $P_\infty(t)$, density ratio, void fraction and other physical parameters.

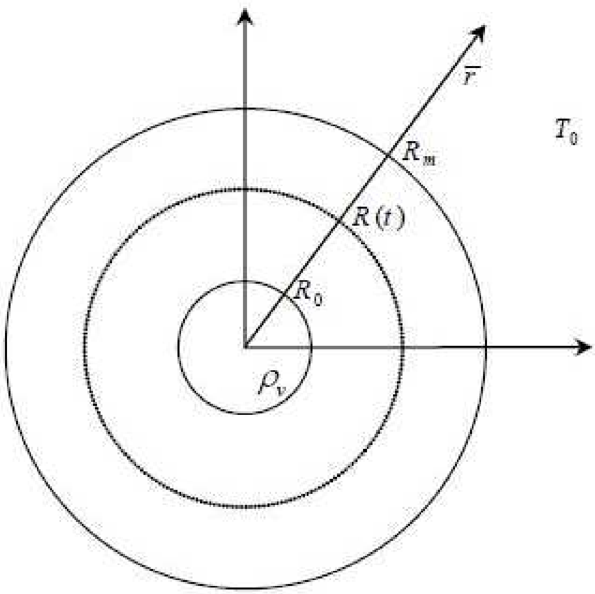

Fig. 1: The Problem Sketch.

Taking into account the following assumptions

–The bubble is assumed to have a spherical geometry.
–Pressure and heat distribution inside the bubble is assumed to be uniform.
–Temperature variations are appreciable only in a thin boundary layer surrounding the growing bubble.

The mathematical model describing this problem consists of three equations: mass, momentum and heat equations.

–***Mass equation*** The mass equation for incompressible fluid has the form

$$\nabla .\, \bar{u}(r,t) = 0. \tag{1}$$

The liquid velocity inside the mixture under the continuity equation of the liquid and spherical symmetry of the spherical bubble is

$$u(r,t) = \frac{\varepsilon R^2}{r^2}\dot{R}, \tag{2}$$

where, $\varepsilon = 1 - \rho_v/\rho_l$, is the density ratio.

–***Momentum equation*** The momentum equation for the growth of the bubble in an incompressible liquid derived by Scriven [21], which is one of the forms of the extended Rayleigh-Plesset equations, is

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{\Delta P - 2\sigma(t)/R}{\tilde{\rho}} - \frac{4\eta\dot{R}}{\rho_l R}, \tag{3}$$

where, $\tilde{\rho} = \varepsilon\rho_l$.

–***Heat equation***

$$\frac{\partial T_l}{\partial t} + \frac{\varepsilon R^2\dot{R}}{r^2}\frac{\partial T_l}{\partial r} = \frac{a_l}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_l}{\partial r}\right). \tag{4}$$

By using Plesset and Zwick method [19] after modifying formula of the linear function that represents the pressure difference $\Delta P$to include the effect of the initial pressure difference ΔP_0, under the proposed initial conditions

$$R(0) = R_0,\ \dot{R}(0) = \dot{R}_0 \ and\ \ddot{R}(0) = 0, \tag{5}$$

where $R_0 = (1+\delta)R_{cr}$ and $R_{cr} = \frac{2\sigma}{P_v - P_\infty} = \frac{2\sigma}{\Delta P_0}$ is the critical bubble radius (unstable equilibrium radius) [9] In this case, the surface tension and viscosity are proposed to be some functions of time as in the following form [3, 16]:

$$\sigma(t) = Bg\rho_l R^2(t), \tag{6}$$

and

$$\eta(t) = CR(t)/\dot{R}(t), \tag{7}$$

where

$$B = \frac{\sigma_0}{g\rho_l R_0^2} \ and\ C = \frac{\eta_0\dot{R}_0}{R_0}. \tag{8}$$

These relations were formulated in the sense of physical dependence between the bubble radius and other physical parameters by using the theory of dimensional analysis. The pressure difference in Eq. (3) is modified to take the form

$$\Delta P = A(T_R - T_{sat}) + b\Delta P_0, \tag{9}$$

where $\Delta P_0 = \frac{2\sigma_0}{R_{cr}} = 2B\rho_l g R_0^2/R_{cr}$. The equation of motion (3) in this case will take the form

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{A(T_R - T_{sat}) + b\Delta P_0 - 2Bg\rho_l R}{\tilde{\rho}} - \frac{4C}{\rho_l}, \tag{10}$$

After getting the constant $A$by applying the initial conditions (5) into Eq. (10), The Momentum Eq. (10) will take the form

$$\frac{1}{2R^2\dot{R}}\frac{d}{dt}\left(R^3\dot{R}^2\right) = \frac{\beta}{\tilde{\rho}}\frac{\Delta T_R^*}{\Delta T_0} + \frac{3}{2}\dot{R}_0^2 - \frac{2Bg}{\varepsilon}(R - R_0), \tag{11}$$

where

$$\beta = \frac{3}{2}\tilde{\rho}\dot{R}_0^2 - b\Delta P_0 + 2B\rho_l g R_0 + 4\varepsilon C.\Delta T_0 = T_0 - T_{sat}, \tag{12}$$

and

$$\Delta T_R^* = T_R - T_0 = -\left(\frac{a_l}{\pi}\right)^{1/2}\int_0^t\left(\frac{R^2(x)\left(\frac{\partial T}{\partial r}\right)_{r=R(x)}}{\left(\int_x^t R^4(y)\,dy\right)^{1/2}}\right)dx. \tag{13}$$

Eq. (13) is a solution of the Heat equation (4), that achieved by Plesset and Zwick [18] under the assumption that the radial motion of the bubble is sufficiently rapid that any translational motion may be neglected which leads to vanish the convection term in the Eq. (4) and using the thin layer approximation around the growing bubble with error less than 10%. [9, 18, 27]. Introducing new dimensionless variables ψ and τ in place of $R$and t respectively, where

$$\psi = \left(\frac{R}{R_0}\right)^3 \ and\ \tau = \frac{\gamma}{R_0^4}\int_0^t R^4(x)\,dx, \tag{14}$$

where γ is a constant given by

$$\gamma = \left(\frac{2\sigma_0}{\rho_l R_0^3}\right)^{1/2}, \sigma_0 = Bg\rho_l R_0^2.$$

After expressing the old variables with the new dimensionless variables, The momentum Eq. (11) becomes

$$\frac{\varepsilon}{6\psi'}\frac{d}{d\tau}\left(\psi^{7/3}\psi'^2\right) = \psi^{1/3} - 1 + \frac{3\varepsilon\dot{R}_0^2}{4BgR_0} - \lambda\int_0^\tau \frac{\psi'(v)}{(\tau - v)^{1/2}}\,dv, \tag{15}$$

where

$$\lambda = \frac{\beta}{\Delta T_0}\left(\frac{\gamma a_l}{9\pi}\right)^{1/2}\frac{\rho_v L}{2B\rho_l g k_l}. \tag{16}$$

At the complete growth of the bubble

$$t \to t_m, R(t_m) \to R_m,\ \dot{R}(t_m) \to 0 \ and\ \psi \to \psi_m, \tag{17}$$

and Eq. (15) becomes

$$\psi_m^{1/3} - 1 + \frac{3\varepsilon\dot{R}_0^2}{4BgR_0} - \lambda \int_0^\tau \frac{\psi'(v)}{(\tau - v)^{1/2}}\, dv \to 0,$$

therefore

$$\int_0^\tau \frac{\psi'(v)}{(\tau - v)^{1/2}}\, dv = \lambda^{-1}\left(\psi_m^{1/3} - 1 + \frac{3\varepsilon\dot{R}_0^2}{4BgR_0}\right). \tag{18}$$

After using the transformation $v = \xi\tau$, we get the following expression

$$\psi(\tau) = \frac{2}{\pi\lambda}\left(\varphi_0^{-1/3} - 1 + \frac{3\varepsilon\dot{R}_0^2}{4BgR_0}\right)\tau^{1/2}, \tag{19}$$

which leads, by Eqs.(14) and (16), to express the bubble radius as

$$R(t) = \left(\frac{12}{\pi}\right)^{1/2} \frac{2BR_0\rho_l g}{\beta}\left(\varphi_0^{-1/3} - 1 + \frac{3\varepsilon\dot{R}_0^2}{4BgR_0}\right) \times \frac{k_l \Delta T_0}{\rho_v L}\left(\frac{t}{a_l}\right)^{1/2}. \tag{20}$$

Substituting from Eq. (12) into Eq. (20)

$$R(t) = \left(\frac{12}{\pi}\right)^{1/2} \frac{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0\left(\varphi_0^{-1/3} - 1\right)}{3\tilde{\rho}\dot{R}_0^2 - 2b\Delta P_0 + 4B\rho_l gR_0 + 8\varepsilon C}\tilde{J}_a (a_l t)^{1/2}. \tag{21}$$

provided that

$$b < \frac{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0 + 8\varepsilon C}{2\Delta P_0}\; for\ \varphi_0 > \left(1 - \frac{3\tilde{\rho}\dot{R}_0^2}{4B\rho_l gR_0}\right)^{-3},$$

and

$$b > \frac{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0 + 8\varepsilon C}{2\Delta P_0}\; for\ \varphi_0 < \left(1 - \frac{3\tilde{\rho}\dot{R}_0^2}{4B\rho_l gR_0}\right)^{-3}. \tag{22}$$

where $a_l = \frac{k_l}{\rho_l C_{pl}}$, $J_a = \frac{\rho_l C_{pl}}{\rho_v L}\Delta T_0$and $\varphi_0 = (R_0/R_m)^3$ is the initial void fraction, $0 < \varphi_0 < 1$.

The following equations give expressions for the initial and final times of the growth and a relation between the ratio between the initial and final times to the initial and final bubble radii or to the initial void fractionφ_0.

$$t_0 = \frac{\pi}{12a_l}\left(\left(\frac{3\tilde{\rho}\dot{R}_0^2 - 2b\Delta P_0 + 4B\rho_l gR_0 + 8\varepsilon C}{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0\left(\varphi_0^{-1/3} - 1\right)}\right)\frac{R_0}{\tilde{J}_a}\right)^2, \tag{23}$$

$$t_m = \frac{\pi}{12a_l}\left(\left(\frac{3\tilde{\rho}\dot{R}_0^2 - 2b\Delta P_0 + 4B\rho_l gR_0 + 8\varepsilon C}{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0\left(\varphi_0^{-1/3} - 1\right)}\right)\frac{R_m}{\tilde{J}_a}\right)^2, \tag{24}$$

$$\frac{t_0}{t_m} = \left(\frac{R_0}{R_m}\right)^2 = \varphi_0^{2/3}. \tag{25}$$

The resultant formula, Eq. (21), can be reduced to the Plesset-Zwick formula [19]if

$$b = \frac{2B\rho_l gR_0\left(2 - \varphi_0^{-1/3}\right) + 4\varepsilon C}{\Delta P_0}, \tag{26}$$

The Eq. (25) show that the relation between the final and initial bubble radius is

$$R_m^2 = \frac{t_m}{t_0}R_0^2, \tag{27}$$

which represents, physically, a straight line its slope is$(t_m/t_0)^{1/2}$. We can deduct also that the growth time period is

$$t_m - t_0 = \frac{\pi}{12a_l}\left(\frac{3\tilde{\rho}\dot{R}_0^2 - 2b\Delta P_0 + 4B\rho_l gR_0 + 8\varepsilon C}{3\tilde{\rho}\dot{R}_0^2 + 4B\rho_l gR_0\left(\varphi_0^{-1/3} - 1\right)}\right)^2 \times \frac{R_m^2 - R_0^2}{\tilde{J}_a^2}. \tag{28}$$

The results were implemented to the data given in Table (1), to give the collection of graphs (2-6) which explain the effect of the given physical parameters on the growth process.

3 Evolution of surface tension and viscosity around the growing bubble

From Eqs. (8) and (21) into Eqs. (6) and (7) we get the following formulae of the evolution of the surface tension and viscosity throughout the growth process of the vapour bubble.

$$\sigma(t) = \frac{12\sigma_0 a_l}{\pi R_0^2}\left(\frac{3\tilde{\rho}R_0\dot{R}_0^2 + 4\sigma_0\left(\varphi_0^{-1/3} - 1\right)}{3\tilde{\rho}R_0\dot{R}_0^2 - 2b_2R_0\Delta P_0 + 4\sigma_0 + 8\varepsilon\eta_0\dot{R}_0}\tilde{J}_a\right)^2 t, \tag{29}$$

and

$$\eta(t) = \frac{2\eta_0\dot{R}_0}{R_0}t. \tag{30}$$

4 Implementation

In the following Table, the initial values and physical parameters of the water were taken at temperature 102^oat pressure 1.0877 *bar* [5]. By using the data in Table (1), we get the following graphs that demonstrate the effect of the physical parameters on the growth of the gas bubble.

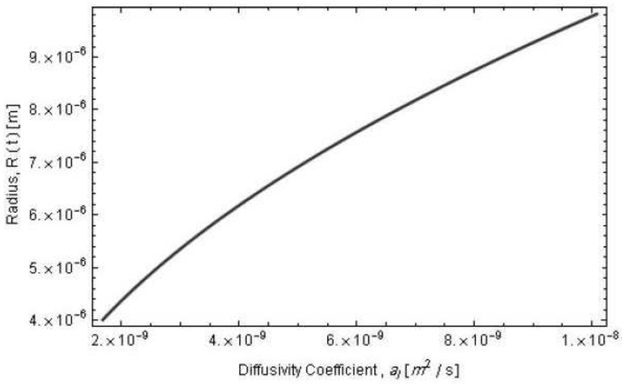

Fig. 2: The relation between the bubble radius R and the thermal diffusivity a_l for the indicated range of a_l.

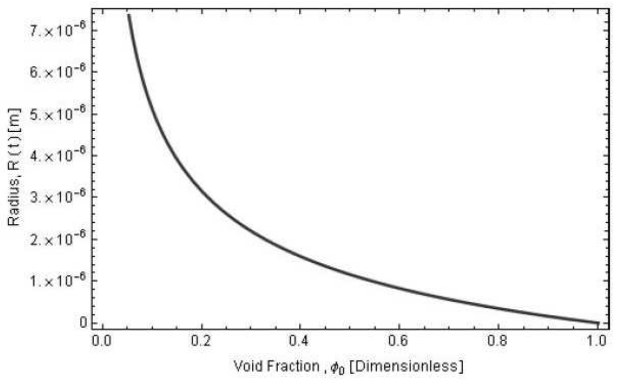

Fig. 3: The relation between the bubble radius R and the initial void fraction φ_0 through its range, $0 < \varphi_0 < 1$.

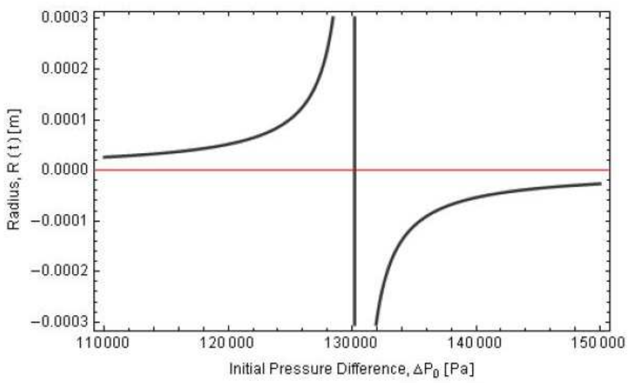

Fig. 4: The relation between the bubble radius R and the initial pressure difference ΔP_0 around the stability-limit value of ΔP_0.

Table 1: The values of the parameters.

	Value		Value
R_0	$1.0 \times 10^{-6} m$	ΔT_0	$2.0\ K$
δ	1.0×10^{-6}	k_l	$0.6786\ W.m^{-1}.K^{-1}$ [24]
$\dot{R}_0$	$0.1 m.s^{-1}$	C_{pl}	$4219.2\ J.kg^{-1}.K^{-1}$ [24]
σ_0	$0.058525\ N.m^{-1}$ [24]	L	$2251 \times 10^3 J.kg^{-1}$
η_0	$0.2758 \times 10^{-3} Pa.s$ [24]	g	$9.81 m.s^{-2}$
ρ_l	$956.95\ kg.m^{-3}$ [5]	φ_0	1.0×10^{-3}
ρ_v	$0.6385\ kg.m^{-3}$ [5]	b	0.9

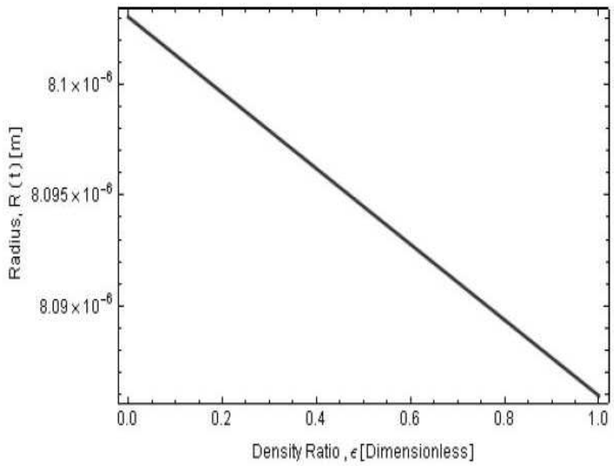

Fig. 5: The relation between the bubble radius R and the density ratio ε through its range, $0 < \varepsilon < 1$.

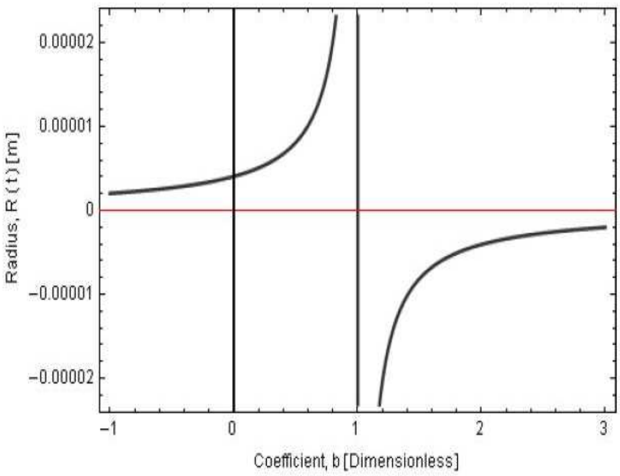

Fig. 6: The relation between the bubble radius R and the coefficient b around the stability-limit value of b.

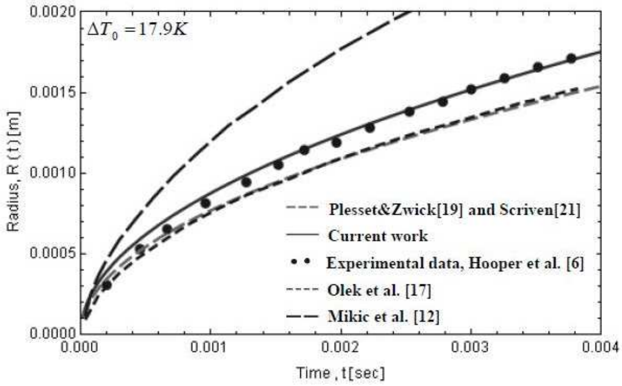

Fig. 7: Comparison between the present work and some of previous works for $\sigma_0 = 0.05539\ N.m^{-1}$, $\eta_0 = 2.362 \times 10^{-4}\ Pa.s$, $\rho_l = 944.76\ kg.m^{-3}$, $\rho_v = 1.0559\ kg.m^{-3} b_2 = 1 \times 10^{-2}$, $k_l = 0.6832\ W.m^{-1}.K^{-1}$, $C_{pl} = 4243\ J.kg^{-1}.K^{-1}$, $\varphi_0 = 1.437 \times 10^{-4}$, $\Delta T_0 = 17.9\ K$.

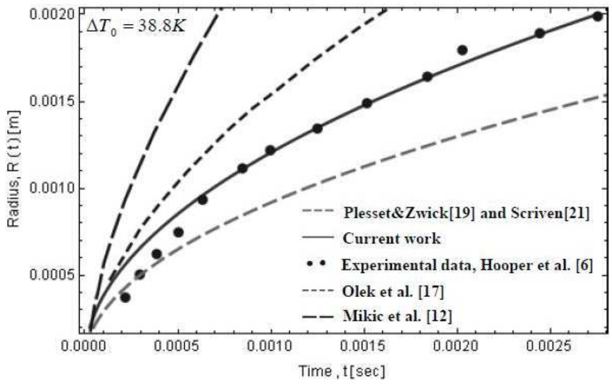

Fig. 8: Comparison between the present work and some of previous works for $\sigma_0 = 0.0511\ N.m^{-1}$, $\eta_0 = 1.997 \times 10^{-4}\ Pa.s$, $\rho_l = 927.07\ kg.m^{-3}$, $\rho_v = 1.913\ kg.m^{-3}$, $b_2 = 1 \times 10^{-2}$, $k_l = 0.6849\ W.m^{-1}.K^{-1}$, $C_{pl} = 4281.6\ J.kg^{-1}.K^{-1}$, $\varphi_0 = 1.0 \times 10^{-3}$, $\Delta T_0 = 38.8\ K$.

5 Results and Discussion

The momentum Eq. (3), which includes the effect of the density ratio ε, is solved analytically using the Plesset-Zwick method [19] after modifying the pressure difference to be expressed as a linear function in which the initial pressure difference appears beside the temperature difference, given by Eq. (9). The surface tension and viscosity are obtained as functions of some physical parameters.

The Eq. (21) represents the growth of bubble radius and the effect of the physical parameters on the growth process, Eqs. (22) give the necessary conditions for the solution to be stable, that is by expressing the suitable values of the coefficient $b$according to the given values of the void fraction φ_0. Eqs. (25) give mathematical formulae to estimate the initial and final time of the growth process and relation between these two times in terms of the given physical parameters.

The figures (2-6) explain the relation between the bubble radius at some instant $\left(t = 1.0 \times 10^{-8} s\right)$ for some intervals of the parameters a_l, φ_0, ΔP_0, ε and b respectively, while the figures (7-8) explain comparisons between the current work and some of previous theoretical and experimental works, for two different cases of the initial superheat ΔT_0.

The figures (2-6) show that, the growth of the bubble radius is proportional to the thermal diffusivity a_l, the initial pressure difference $\Delta P_0$and its coefficient b, while it's inversely proportional to the initial void fraction$\varphi_0$and the density ratio ε.

Moreover, Figs. 4 and 6 explain the stability domain of the initial pressure difference ΔP_0 and its coefficient b; and show that, the growth of the bubble radius, for the given data, is only possible in the interval $b < 1.001$, which agrees with the conditions (22) and for ΔP_0 is only possible in the interval $\Delta P_0 < 130.19$ kPa. It can be observed that, the growth is noticeably affected by small changes of the parameters: void fraction $\varphi_0$for small values near zero, the initial pressure difference ΔP_0 and its coefficient b near their possible upper limits, which are $\Delta P_0 < 130.19$ kPa and $b < 1.001$ respectively. It can also be noticeable from Figs. (3) and (5) that, at the given instant, the change of the bubble radius is affected by changing the parameter $\varphi_0$more than changing of the parameter ε . This can be seen from the wide range of R values in Fig. (3), which is more than its range in Fig. (5) over the range of φ_0 and ε respectively which lies between 0 and 1.

The Figs. (7-8) explain comparisons between the current work and some of previous theoretical works for Plesset and Zwick [19], Scriven [21], Mikic et al. [12], Olek et al. [17] and the experimental work of Hooper and Abdelmessih [6], for two different cases of the initial superheat ΔT_0 at 17.9 K and 38.8 K.

For the value of the coefficient $b = 1 \times 10^{-2}$, better agreements is achieved by this work rather than the previous works to the experimental data of Hooper and Abdelmessih [6] for the two plotted-cases for the initial superheats at 17.9 K and 38.8 K.

This model fits the experimental data than others, that's because this model contains the effect of some physical parameters which didn't included to the mentioned, previous studies like surface tension, viscosity and initial pressure difference, while these parameters is important and its effect must be taken in consideration. And we know that the more the effects are taken under consideration, the more the accurate formula could be achieved.In other words, this solution manipulates the physical problem deeply by taking into account the effect of some omitted parameters' effects.

6 Conclusion

Growth of a vapour bubble is solved analytically under the effect of variable surface tension and viscosity, and other physical parameters, taking into account the effect of density ratio ε.The pressure difference is proposed to include the initial pressure difference ΔP_0, its coefficient gives the necessary condition for the growth Eqs. (22)

The values of physical parameters are given by Table (1). The discussion of results and figures concluded the following remarks:

1. The growth of bubble radius is proportional with thermal diffusivity a_l, the initial pressure difference ΔP_0 and its coefficient b.

2. The growth of bubble radius is proportional inversely with initial void fraction$\varphi_0$and the density ratio ε.

3. The current growth formula Eq. (21) gives better agreements when $b = 1 \times 10^{-2}$, to the experimental data of Hooper and Abdelmessih [6] than the previous theoretical works that presented by Plesset and Zwick [19], Scriven [21], Mikic et al. [12] and Olek et al. [17], for two different cases of the initial superheat ΔT_0 at 17.9 K and 38.8 K.

The above concluded remarks prove the validity of the proposed model, and how to extend the present model in more properties of fluid and flow.

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A List of Symbols:

Nomenclature:

a_l *Thermal diffusivity of the liquid* $\left(m^2.s^{-1}\right)$
b *Coefficient of the initial pressure difference, defined by Eq. (9)*
B *Coefficient of surface tension, defined by Eq. (8)*
C *Coefficient of viscosity, defined by Eq. (8)* (Pa)
C_{pl} *Heat capacity of the liquid at constant pressure* $\left(J.(kg.K)^{-1}\right)$
g *Acceleration of the gravity* $\left(m.s^{-2}\right)$
J_a *Jacob number, defined by Eq. (21)*
k_l *Thermal conductivity of the liquid* $\left(J.(s.m.K)^{-1}\right)$
L *Latent heat of vapourization* $\left(J.kg^{-1}\right)$
ΔP *The difference of pressure inside and at great distance from the bubble* $\left(N.m^{-2}\right)$
ΔP_0 *The initial difference of pressure inside and at great distance from the bubble* $\left(N.m^{-2}\right)$
r *The distance from the origin of the bubble* (m)
R_0 *Initial bubble wall radius* (m)
R_{cr} *Critical bubble wall radius, sometimes called Blake's critical threshold* (m)
R *Instantaneous bubble wall radius* (m)
$\dot{R}$ *Instantaneous bubble wall velocity* $\left(m.s^{-1}\right)$
$\ddot{R}$ *Instantaneous bubble wall acceleration* $\left(m.s^{-2}\right)$
t *Time elapsed* (s)
T_0 *The ambient temperature in the liquid at great distance from the bubble boundaries* (K)
ΔT_0 *Initial Temperature Difference (Initial Superheat), defined by Eq.(12)* (K)
T_R *The temperature of the bubble boundary* (K)
ΔT_R^* *The temperature difference defined by Eq. (13)* (K)
$u(r)$ *Velocity of liquid at distance r from the bubble origin* $\left(m.s^{-1}\right)$

Greek symbols:

ε *Density ratio, A fraction equals the two phase densities ratio subtracted from one, defined by Eq.(2)*
$\tilde{\rho}$ *The two phase density difference* $(\tilde{\rho} = \varepsilon\rho_l = \rho_l - \rho_g)$ $\left(kg.m^{-3}\right)$
ρ_v *Density of the vapour inside the bubble* $\left(kg.m^{-3}\right)$
ρ_l *Density of the liquid surrounding the bubble* $\left(kg.m^{-3}\right)$
σ *The surface tension of liquid surrounding the bubble* $\left(N.m^{-1}\right)$
τ *Dimensionless variable defined by Eq. (14)*
φ_0 *Initial void fraction defined by Eq. (21)* $(Dimensionles)$
ψ *Dimensionless volume variable (instantaneous bubble volume to its initial volume) defined by Eq. (14)*

Subscripts:

0 *Initial value*
l *Liquid*
m *Maximum value*
sat *Saturation*
v *Vapour bubble*

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